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ON NONCLASSICAL IMPULSIVE ORDINARY DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. Results on mild solutions of nonclassical differential equations with impulsive and nonlocal conditions are extended to a case when the nonlocal conditions are necessarily non Lipschitz and non compact.

1. Introduction

We study the following quantum stochastic differential equation (QSDE) with impulsive nonlocal conditions introduced in [1];

$$\begin{aligned} dz(t) &= A(t)z(t) + E(t,(z(t))d\wedge_{\pi}(t) + F(t,z(t))dA_f(t) \\ &\quad + G(t,z(t))dA_g^+(t) + H(t,z(t))dt, \\ &\quad \text{almost all } \ t \in I, t \neq t_k, k = 1,...,m \qquad (1) \\ &\quad \Delta z(t_k) = J_k(z(t_k^-)), k = 1,...,m \\ &\quad z(t_0) = z_0 + g(z), \quad t \in [0,T] \end{aligned}$$

where

(i) A is a family of semigroup defined in [1]

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(ii) E, F, G, H are stochastic processes.

(iii) $J_k \in C(\tilde{\mathcal{B}}, \tilde{\mathcal{B}}), k = 1, 2, ..., m$ and $\Delta z(t_k)$ is the difference between $z(t^{+_k})$ and $z(t_k^-)$.

(v) $g : \tilde{\mathcal{B}} \to PC(I, sesq(I\!\!D\underline{\otimes}I\!\!E))$ is a nonlocal condition that is not Lipschitz and not compact.

(vi) $z \in \tilde{\mathcal{B}}$ is a stochastic process and $\eta, \xi \in I\!\!D\underline{\otimes}I\!\!E$ is arbitrary.

Problems with nonlocal conditions have been an area of interest, mostly because of the advantage they have over initial value problems. Existence of solution of nonlocal problems for different types of differential equations were extensively discussed in the literature by using various methods (See [1, 2, 3-14] and the references therein). The motivation for this study, is that nonlocal problems occur naturally when modeling physical problems. In [2], impulsive quantum stochastic differential equations (IQSDE) with initial value conditions were studied. The multivalued maps are lower respectively upper semicontinuous. In [1], existence results for Eq.(1) with nonlocal conditions that are completely continuous were established. We showed that the function g which constitute the nonlocal condition is compact and Lipschitz continuous. Several interesting results on nonlocal impulsive differential equations satisfying some Lipschitz and compactness conditions have been established in [6-9]. In this study, existence of solution of Eq. (1) is established with nonlocal conditions that are not necessarily Lipschitz and compact. We adopt the most suitable fixed point method to establish this result. Impulsive QSDEs have found applications in quantum continuous measurements, especially when the mean number of photons up to time t_i is momentary giving rise to impulses on the counting stochastic processes associated with the observables $x(t_i)$. See [1, 2] and the references therein.

2. Preliminaries

The definitions of the following spaces $L^2_{loc}(\tilde{\mathcal{B}})_{mvs}$, $\tilde{\mathcal{B}}$, $PC(I, \tilde{\mathcal{B}})$, $PC'(I, \tilde{\mathcal{B}})$, $PC(I, sesq(I\!\!D\underline{\otimes}I\!\!E))$ and $PC'(I, sesq(I\!\!D\underline{\otimes}I\!\!E))$ are adopted from [1, 2]. The spaces $\tilde{\mathcal{B}}$ and $PC(I, sesq(I\!\!D\underline{\otimes}I\!\!E))$ denote the locally convex and Banach spaces respectively. The Hausdorff distance, $\rho(A, B)$ is defined as:

$$\rho(A, B) = \max(\delta(A, B), \delta(B, A)), A, B \in clos(\mathbb{C})$$

and

$$d(z, B) = Inf_{y \in B}|z - y|, \quad \delta(A, B) = Sup_{z \in A} d(z, B)$$

where $x \in \mathbb{C}$ is as defined in [1] and ρ is a metric.

Definition 1. A stochastic process $z \in PC(I, \tilde{\mathcal{A}})$ is called a solution of Eq. (1) if it satisfies the integral equation

$$
\begin{aligned}
z(t) \quad = \quad & S(t)[z_0 + g(z)] \\
& + \int_0^t S(t-s)(E(s,(z(s))d\Lambda_\pi(s) + F(s,z(s))dA_f(s) \\
& + G(s,z(s))dA_g^+(s) + H(s,z(s))ds) + \sum_{0<t_k<t} S(t-t_k)J_k(z(t_k)) \qquad (2)
\end{aligned}
$$

for $t \in [0,T]$. To establish a mild solution with non Lipschitz, non compact and nonlocal condition, we must assume that z(t) near zero does not affect the nonlocal condition g(z). We use the equivalent form of Eq. (1) given by,

$$\frac{d}{dt}\langle \eta, z(t)\xi \rangle = P(t,z(t))(\eta,\xi). \qquad (3)$$

Remark. Equation (3) is also known as nonclassical ordinary differential equation. See [1] and the references therein. Next, we state the following assumptions:

(H_1) The map P in Eq. (3) is continuous and well defined in [1]. Hence, there exists a function $K^P_{\eta\xi} : [0,T] \to I\!R_+$ so that

$$\|P(t,y) - P(t,z)\|_{\eta\xi} \leq K^P_{\eta\xi}(t)\, \|y-z\|_{\eta\xi}\,,$$

$t \in [0,T]$, $y, z \in \tilde{\mathcal{B}}$.

(H_2) $J_k \in C(\tilde{\mathcal{B}}, \tilde{\mathcal{B}})$ and $T(.)$ are compact operators

(H_3) For each $z_0 \in \tilde{\mathcal{B}}$, we have constants $h_{\eta\xi} > 0$ and $M > 0$ so that

$$\|S(t)\|_{\eta\xi} \leq M, t \geq 0$$

and

$$M\left(\|z_0\|_{\eta\xi} + \sup_{\varphi\in H_h}\|g(\varphi)\|_{\eta\xi} + K^P_{\eta\xi}(t)\sup_{s,t\in[0,T]}\|P(s,\varphi(s))\|_{\eta\xi} + \sup_{\varphi\in Z_h}\sum_{k=1}^m \|J_k(\varphi(t_k))\|_{\eta\xi}\right) \leq h_{\eta\xi}$$

where

$$H_h := \left\{\varphi \in PC([0,T],\tilde{\mathcal{A}}) : \|(\varphi(t))\|_{\eta\xi} \leq h_{\eta\xi}, t \in [0,T]\right\}$$

(H_4) $g : PC([0,T],\tilde{\mathcal{A}}) \to \tilde{\mathcal{A}}$ is continuous and constitute the nonlocal condition. Also $g : H_h \to bd$

Let δ depend on $h_{\eta\xi} \in (0,t_1)$ and $g(\varphi) = g(\phi)$, $\varphi, \phi \in H_h$ where $\varphi(s) = \phi(s), s \in [\delta, T]$ and bd denote a bounded set.

3. Main Result

Theorem 1. Let conditions (H_1)-(H_4) hold. Then for $z_0 \in \tilde{\mathcal{B}}$, problem (1) has at least a solution.

Proof. Let $\delta \in (0, t_1)$, define $H(\delta)$ and $H_h(\delta)$ as ;

$$H(\delta) := PC([\delta, T], \tilde{\mathcal{B}}) \text{ for functions in } PC([0,T], \tilde{\mathcal{B}}) \text{ on } [\delta, T]$$

and

$$H_h(\delta) := \left\{ \varphi \in H(\delta) : \|\varphi(t)\|_{\eta\xi} \leq h_{\eta\xi}, t \in [\delta, T] \right\}.$$

Let $z \in H_h(\delta)$ be fixed. Then define a map Γ_z on H_h by

$$\begin{aligned} \Gamma_z(\varphi)(t)(\eta,\xi) &= \langle \eta, [z_0 + g(\tilde{z})]\xi \rangle \\ &\quad + \int_0^t S(t-s) P(s, (\varphi(s))(\eta,\xi) ds \\ &\quad + \sum_{0<t_k<t} S(t-t_k) J_k(u(t_k)), t \in [0,T], \tilde{z} \in H_h(\delta) \end{aligned}$$

where

$$\tilde{z}(t) = z(t), \text{ if } t \in [\delta, T]$$

and

$$\tilde{z}(t) = z(\delta), \text{ if } t \in [0, \delta]$$

This shows that Γ_z is a continuous mapping from H_h to H_h. Also, we have

$$\begin{aligned} &\|\Gamma_z^n(\varphi)(t) - \Gamma_z^n(\phi)(t)\|_{\eta\xi} \\ &= \left| \int_0^t S(t-s)(P(s,\varphi(s))(\eta,\xi) - P(s,\phi(s))(\eta,\xi)ds) \right| \\ &\leq M \int_0^t \|P(s,\varphi(s)) - P(s,\phi(s))\|_{\eta\xi} ds \\ &\leq M \int_0^t K^P_{\eta\xi}(s) \|\varphi(s) - \phi(s)\|_{\eta\xi} ds \qquad (3) \end{aligned}$$

Since the map $s \to \sup_{s\in[0,T]} \|\varphi(s) - \phi(s)\|_{\eta\xi}$ is continuous, we let

$$R_{\eta\xi} = \sup_{s\in[0,t]} \|\varphi(s) - \phi(s)\|_{\eta\xi}$$

and

$$N_{\eta\xi}(t) = \int_0^t K^P_{\eta\xi}(s) ds.$$

Then from (3), we get

$$\|\Gamma_z^n(\varphi)(t) - \Gamma_z^n(\phi)(t)\|_{\eta\xi} \leq \frac{R_{\eta\xi}(MN_{\eta\xi}(t))^n}{n!}, n = 1, 2, ... \qquad (4)$$

This can also be proved by induction, see [7] in [1].

So we get

$$\begin{aligned}\|\Gamma_z^m(\varphi)(t) - \Gamma_z^m(\phi)(t)\|_{\eta\xi} &= \left\| \sum_{m=k+1}^{n} (\Gamma_z^m(\varphi)(t) - \Gamma_z^m(\phi)(t)) \right\|_{\eta\xi} \\ &\leq \sum_{m=k+1}^{n} \|(\Gamma_z^m(\varphi)(t) - \Gamma_z^m(\phi)(t))\|_{\eta\xi} \\ &\leq \sum_{m=k+1}^{n} \frac{R_{\eta\xi}(MN_{\eta\xi}(T))^m}{m!}\end{aligned}$$

where $t \in [0,T],\ \varphi, \phi \in H_h, m = 1, 2, ...$

Now by considering large m, we see that Γ_z^m is a contraction operator on H_h and hence Γ_z has a unique fixed point given by

$$\begin{aligned}\varphi_z(t)(\eta,\xi) &= \langle \eta, [z_0 + g(\tilde{z})]\xi \rangle \\ &+ \int_0^t S(t-s)P(s,\varphi_z(s))(\eta,\xi)ds \\ &+ \sum_{0<t_k<t} S(t-t_k)J_k(z(t_k)) \in H_h,\ t \in [0,T]\end{aligned}$$

From the above, define a map $\Upsilon : H_h(\delta) \to H_h(\delta)$ by

$$\begin{aligned}\Upsilon(z)(t)(\eta,\xi) = \varphi_z(t)(\eta,\xi) &= \langle \eta, [z_0 + g(\tilde{z})]\xi \rangle \\ &+ \int_0^t S(t-s)P(s,\varphi_z(s))(\eta,\xi)ds \\ &\leq \sum_{0<t_k<t} S(t-t_k)J_k(z(t_k)), t \in [\delta,T] \qquad (5)\end{aligned}$$

The next following steps show that the map Υ has a fixed point:

By applying Lebesgue dominated convergence theorem and Arzela-Ascoli theorem we show that; (1) the operator Υ is continuous;

(2) It maps bd sets into bd sets in $PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E}))$;

(3) Υ maps bd sets into equicontinuous sets in $PC(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E}))$ and by applying the Schauder-Tychonov's fixed point theorem we get a $z^* \in H_h(\delta)$, which is a fixed point.

By replacing $\Upsilon(z)$ with z, $g(\tilde{z})$ with $g(\tilde{z}^*)$ in (5), we get

$$\begin{aligned}z(t)(\eta,\xi) &= \langle \eta, [z_0 + g(\tilde{z}^*)]\xi \rangle \\ &+ \int_0^t S(t-s)P(s,z(s))(\eta,\xi)ds \\ &\leq \sum_{0<t_k<t} S(t-t_k)J_k(z^*(t_k)), t \in [0,T] \qquad (6)\end{aligned}$$

But

$$g(\tilde{z}^*) = g(\tilde{z}),\ \tilde{z}^*(t_k) = \tilde{z}(t_k)$$

and by the definition of the map Υ,

$$\tilde{z}^*(t) = \Upsilon(\tilde{z}^*)(t) = \varphi_{\tilde{z}^*}(t) = \tilde{z}(t).$$

So that by (6), we conclude that $z(t)$ is the required solution of Eq. (1) which is the desired result.

Conclusion

Using the equivalent form of QSDE (1) given by Eq. (3) and having satisfied the conditions of the appropriate fixed point theorem, we conclude that a fixed point exists and it is a solution of the problem.

Conflict of Interest

The authors declare that there is no conflict of interests.

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