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ON INTEGRATED AND DIFFERENTIATED $\mathbb{C}_2$-SEQUENCE SPACES

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Abstract. The integrated and differentiated $\mathbb{C}_2$-sequence spaces are defined and studied by using the norm on the bicomplex space $\mathbb{C}_2$, infinite matrices of the bicomplex number and the Orlicz functions. We also studied some topological properties of the $\mathbb{C}_2$-sequence spaces We define the α-duals of the integrated and differentiated $\mathbb{C}_2$-sequence spaces.

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1. Introduction

The set of bicomplex numbers [8] is denoted by $\mathbb{C}_2$ and sets of real and complex numbers are denoted as $\mathbb{C}_0$ and $\mathbb{C}_1$, respectively.The set of bicomplex number is defined as (cf. [8], [9])

$$\begin{aligned}\mathbb{C}_2 &:= \{a_1 + i_1 a_2 + i_2 a_3 + i_1 i_2 a_4 : a_k \in \mathbb{C}_0,\, 1 \leq k \leq 4\}\\ &:= \{w_1 + i_2 w_2 : w_1, w_2 \in \mathbb{C}_1\}\end{aligned}$$

where $i_1^2 = i_2^2 = -1$, $i_1 i_2 = i_2 i_1$.

The set of bicomplex numbers $\mathbb{C}_2$ have exactly two non-trivial idempotent elements denoted by e_1 and e_2 give as $e_1 = (1 + i_1 i_2)/2$ and $e_2 = (1 - i_1 i_2)/2$. Note that $e_1 + e_2 = 1$ and $e_1.e_2 = 0$. The number $\xi = w_1 + i_2 w_2$ can be uniquely expressed as a complex combination of e_1 and e_2 [8].

$$\xi = w_1 + i_2 w_2 = {}^1\xi e_1 + {}^2\xi e_2, \tag{1.1}$$

where ${}^1\xi = w_1 - i_1 w_2$ and ${}^2\xi = w_1 + i_1 w_2$. The complex coefficients ${}^1\xi$ and ${}^2\xi$ are called the *idempotent components* of ξ, and ${}^1\xi e_1 + {}^2\xi e_2$ is known as *idempotent representation* of bicomplex number ξ.

The auxiliary complex spaces $\mathbb{A}_1$ and $\mathbb{A}_2$ are defined as follows:

$$\mathbb{A}_1 = \left\{{}^1\xi \,:\, \xi \in \mathbb{C}_2\right\} \quad \text{and} \quad \mathbb{A}_2 = \left\{{}^2\xi \,:\, \xi \in \mathbb{C}_2\right\}.$$

The norm in $\mathbb{C}_2$ is defined as follows:

$$||\xi|| \;=\; \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2} = \sqrt{|w_1|^2 + |w_2|^2} = \sqrt{\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}} \tag{1.2}$$

Further, the norm of the product of two bicomplex numbers and the product of their norms are connected by means of the following inequality:

$$||\xi \,.\, \eta|| \leq \sqrt{2}\;\; ||\xi|| \,.\, ||\eta|| \tag{1.3}$$

The inequality given in (1.3) is the best possible relation . For this reason, we call $\mathbb{C}_2$ as *modified complex Banach algebra* [8].

Throughout the paper, the ω_4, c, c_0 and $\ell^\infty_{\mathbb{C}_2}$ denote the space of all bicomplex sequences, convergent sequences, null sequences and all bounded sequences. We denote the zero sequence $(0, 0, 0, \ldots, 0, \ldots)$ by π. Refer the book by Mursaleen [?] for details about summability methods.

The Orlicz function $\mathcal{M}$ is defined as $\mathcal{M} : [0, \infty) \to [0, \infty)$. It is continuous, non-decreasing and $\mathcal{M}(0) = 0, \mathcal{M}(x) > 0$ for $x > 0$. Also, for $\lambda \in (0, 1)$ it satisfies the condition

$$\mathcal{M}(\lambda x + (1 - \lambda) y) \quad \leq \quad \lambda \mathcal{M}(x) + (1 - \lambda)\mathcal{M}(y) \tag{1.4}$$

and If the condition of convexity of the Orlicz function $\mathcal{M}$ is replaced by $\mathcal{M}(x + y) \leq \mathcal{M}(x) + \mathcal{M}(y)$, then the function $\mathcal{M}$ is called the modulus function.

The notations $(X : Y)$ denote the class of all matrices M, such that $M : X \to Y$. Therefore, $M \in (X : Y)$ if and only if $M(x) = \{(Mx)_n\}_{n\in\mathbb{N}} \in Y$.

A sequence $\{\xi_n\}$ in $\mathbb{C}_2$ is said to be M-summable to the bicomplex number ξ if $M(\xi_n)$ converges to ξ which is called M-limit of $\{\xi_n\}$.

In [1], the sequence space bv_p is defined, which have all sequences such that their Δ-transform is in ℓ_p, where Δ denotes the matrix $\Delta = \{\delta_{nm}\}$ as

$$\delta_{nm} := \begin{cases} (-1)^{n-k} & , n-1 \le m \le n \\ 0 & , 0 \le m \le n-1 \text{ or } k > n \end{cases} \tag{1.5}$$

We consider following matrices for our $\mathbb{C}_2$-sequence spaces.

$$\omega_{nm} := \begin{cases} \xi & , 1 \le m \le n\xi \\ 0 & , \xi \prec_{Id} \eta \end{cases} \tag{1.6}$$

$$\gamma_{nm} := \begin{cases} \xi & , m = n \\ -\xi & , n-1 = m \\ 0 & , \text{otherwise} \end{cases} \tag{1.7}$$

$$\pi_{nm} := \begin{cases} \xi^{-1} & , 1 \le m \le n \\ 0 & , m > n \end{cases} \tag{1.8}$$

and

$$\pi_{nm} := \begin{cases} \xi^{-1} & , n = m \\ -\xi^{-1} & , n-1 = m \\ 0 & \text{otherwise} \end{cases} \tag{1.9}$$

Here we must note that ξ^{-1} exists if and only if $\xi \in \mathbb{C}_2/\mathbb{O}_2$.

The integrated and differentiated sequence space were first studied by Goes and Goes [3]. In this paper, we define and study some $\mathbb{C}_2$-sequence space. In the last section we studied the α-dual of these sequence spaces.

2. Bicomplex integrated (*int*) and differentiated (*diff*) $\mathbf{C}_2$-sequence spaces

Goes and Goes [3] has given the concept of the integrated sequence space. In this section we will obtain the matrix domains of the sequence space ℓ_1 by using the bicomplex matrices. We shall show that the integrated and differentiated $\mathbb{C}_2$-sequence spaces are Banach Spaces, BK-spaces, norm isomorphic to ℓ_1, separable these spaces have AK-property. The spaces $\int bv$ and $\int \ell_1$ have monotone norms and therefore the spaces $\int bv$ and $\int \ell_1$ have AK-property. Let ω_4 denote the family of bicomplex sequences.

Now we are giving the definitions of some $\mathbb{C}_2$-sequence spaces as follows:

Definition 2.1 (Integrated $\mathbb{C}_2$-sequence spaces)**.**

$$\overline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|) \quad = \quad \left\{\{\xi_n\} \in \omega_4 : \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|n\xi_n\|}{K}\right) < \infty, \mathit{for\ some}\, K > 0\right\}$$

and

$$\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|) \quad := \quad \left\{\xi = \{\xi_n\} \in \omega_4 \ : \sum_{n=2}^{\infty} \mathcal{M}\left(\frac{\|\Delta(n\,\xi_n)\|}{K}\right) < \infty, \mathit{for\ some}\, K > 0\right\}$$

Definition 2.2 (Differentiated $\mathbb{C}_2$-sequence spaces)**.**

$$\underline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|) \quad := \quad \left\{\xi = \{\xi_n\} \in \omega_4 \ : \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|\xi_n/n\|}{K}\right) < \infty, \mathit{for\ some}\, K > 0\right\}$$

$$\underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|) \quad := \quad \left\{\xi = \{\xi_n\} \in \omega_4 \ : \sum_{n=2}^{\infty} \mathcal{M}\left(\frac{\|\Delta(\xi_n/n)\|}{K}\right) < \infty, \mathit{for\ some}\, K > 0\right\}$$

we can redefine the spaces $\underline{\ell}(\mathbb{C}_2, \mathcal{M}, \|.\|)$, $\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$, $\underline{\ell}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ and $\underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ by

$$(\ell_1)_\Omega = \overline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|), \quad (\ell_1)_\Gamma = \overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|), \quad (\ell_1)_\Pi = \underline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|), \quad (\ell_1)_\lambda = \underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|).$$

Let $\xi = \{\xi_n\} \in \overline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$. Then the Ω-transform of ξ is defined as

$$\zeta_n := (\Omega(\xi))_n = \sum_{m=1}^{n} \mathcal{M}\left(\frac{\|(m\,\xi_m)\|}{K}\right) \qquad \text{for some}\, K > 0$$

or equivalently,

$${}^1\zeta_n := (\Omega({}^1\xi))_n = \sum_{m=1}^{n} \mathcal{M}\left(\frac{|m\ {}^1\xi_m|}{K}\right) \quad \text{and} \quad {}^2\zeta_n := (\Omega({}^2\xi))_n = \sum_{m=1}^{n} \mathcal{M}\left(\frac{|m\ {}^2\xi_m|}{K}\right)$$

Let $\xi = \{\xi_n\} \in \overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$. The Γ-transform of $\{\xi_n\}$ is defined as

$$\zeta_n := (\Omega(\xi))_n = \begin{cases} \xi_1 & , p = 1 \\ \Delta(p\,\xi_p) & , p \geq 2 \end{cases}$$

Let $\xi = \{\xi_n\} \in \underline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$. The Π-transform of $\{\xi_n\}$ is defined as

$$\zeta_n = (\Pi\,\xi)_n = \sum_{p=1}^{n} \mathcal{M}\left(\frac{\|\xi_p/p\|}{K}\right) \qquad \text{for some}\, K > 0$$

Let $\xi = \{\xi_n\} \in \underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$. The Σ-transform of $\{\xi_n\}$ is defined as

$$\zeta_n := (\Sigma(\xi))_n = \begin{cases} \xi_1 & , p = 1 \\ \Delta(p^{-1}\,\xi_p) & , p \geq 2 \end{cases}$$

For the convenience, we use the following notations.

$$K_1 = \overline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|), \quad K_2 = \overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|), \quad K_3 = \underline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|), \quad K_4 = \underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|).$$

Proposition 2.1. *A sequence $\{\xi_n\}$ is in $X(\mathbb{C}_2, \mathcal{M}, \|.\|)$ if and only if $\{^1\xi_n\} \in S(\mathbb{A}_1, \mathcal{M}, \|.\|)$ and $\{^2\xi_n\} \in S(\mathbb{A}_2, \mathcal{M}, \|.\|)$, where $X = K_1, K_2, K_3$ and K_4.*

Theorem 2.1. *The space $\underline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ is a linear space over $\mathbf{C}_0$.*

Proof. Let $\{\xi_n\}, \{\eta_n\} \in \underline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$. Then there exist $P_1 > 0$ and $P_2 > 0$ such that

$$\sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|n\,\xi_n\|}{P_1}\right) < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} \mathcal{M}\left(\frac{\|n\,\xi_n\|}{P_2}\right) < \infty$$

Now let $\alpha, \beta \in \mathbb{C}_2 \setminus \mathbf{O}_2$ and $P = \max\{2\|\alpha\|P_1, 2\|\beta\|P_2\}$. Then

$$\sum_{k=1}^{\infty} \mathcal{M}\left(\frac{\|\alpha\Delta(k\,\xi_k) + \beta\Delta(k\,\eta_k)\|}{P}\right) \leq \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{\|\alpha\,\Delta(k\,\xi_k)\|}{P_1}\right) + \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{\|\beta\,\Delta(k\,\eta_k)\|}{P_2}\right).$$

Therefore, $\{\alpha\,\xi_n + \beta\,\eta_n\} \in \underline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$. Hence, the space $\underline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ is a linear space over $\mathbb{C}_2 \setminus \mathbf{O}_2$. □

Lemma 2.1. *The functions $\|\xi\|_{\overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)} = \sum_{m=1}^{\infty} \|\omega_{nm}\xi_m\|$ and $\|\xi\|_{\underline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)} = \sum_{m=1}^{\infty} \|\pi_{nm}\xi_m\|$ are norms on the spaces $\overline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ and $\underline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$, respectively.*

Theorem 2.2. *The spaces $\overline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ and $\underline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ are Banach spaces with norms $\|\xi\|_{\overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)} = \sum_{m=1}^{\infty} \|\omega_{nm}\xi_m\|$ and $\|\xi\|_{\underline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)} = \sum_{m=1}^{\infty} \|\pi_{nm}\xi_m\|$, respectively.*

Proof. Let $\{\xi_k^n\}$ be a Cauchy sequence in $\overline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$. Then for given $\epsilon > 0$, $\exists\ m_0 \in \mathbb{N}$ such that

$$\|\xi_k^n - \xi_k^m\| < \epsilon, \quad \forall n, m > m_0 \tag{2.1}$$

Therefore,

$$\sum_k \|\Omega(\xi^m)_k - \Omega(\xi^n)_k\| < \epsilon, \quad \forall n, m > m_0$$

$\Rightarrow \{\Omega(\xi^1)_k, \Omega(\xi^2)_k, \Omega(\xi^3)_k, \ldots, \Omega(\xi^n)_k, \ldots\}$ is a Cauchy Sequence of bicomplex numbers. Since, $\mathbb{C}_2$ is a modified Banach space. Therefore, $\{\Omega(\xi^n)_k\}$ is convergence in $\mathbb{C}_2$. Suppose that

$$\Omega(\xi^n)_k \to \Omega(\xi), \qquad n \to \infty, \forall k$$

Using all these limits, we define a sequence $\{\Omega(\xi)_1, \Omega(\xi)_2, \Omega(\xi)_3, \ldots, \}$.

and from equation (2.1), we have

$$\sum_{k=1}^{p} \|\Omega(\xi^m)_k - \Omega(\xi^n)_k\| < \epsilon \tag{2.2}$$

For any $n > m_0$, by letting $m \to \infty$ and $p \to \infty$, we have

$$\|\xi^n - \xi\|_{\overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)} \leq \epsilon$$

In particular,

$$\|\xi\|_{\overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)} \leq K + \|\xi^n\|_{\overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)}, \quad \text{for some} \quad K \geq \epsilon.$$

Hence, $\xi \in \overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)$. Further, $\xi^n \to \xi$. Therefore, $\overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)$ is complete. □

Corollary 2.1. *The space $\underline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)$ is a Banach space.*

Theorem 2.3. *The spaces $\overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)$ and $\underline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)$ are BK-spaces with norms $\|\xi\|_{\overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)} = \sum_{m=1}^{\infty} \|\omega_{nm}\xi_m\|$ and $\|\xi\|_{\underline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)} = \sum_{m=1}^{\infty} \|\pi_{nm}\xi_m\|$, respectively.*

Proof. Let $\{\xi_n\} \in \overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)$. Define $f_p(\xi_n) = \xi_p, \forall n \in \mathbb{N}$. Then

$$\|\xi_n\|_{\overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)} = \sum \|n\,\xi_n\|$$

So that $\|n\,\xi_n\| \leq \|\xi_n\|_{\overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)} \quad \Rightarrow \|\xi_n\| \leq K\|\xi_n\|_{\overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)} \quad \Rightarrow \|f_n(\xi_p)\| \leq K\|\xi_n\|_{\overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)}$.

Therefore, f_n is a continuous linear functional for each n. So, $\overline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)$ is a BK-space. □

In the similar manner, we can prove that $\underline{\ell_1}(\mathbb{C}_2,\mathcal{M},\|.\|)$ is a BK-space.

Theorem 2.4. *The space $\overline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)$ is a BK-space with the norm $\|\xi\|_{\overline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)} = \sum_{m=1}^{\infty} \|\Delta(m\,\xi_m)\|$.*

Proof. As we know, $\underline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|) = (\ell_1)_\Sigma$ is true and ℓ_1 is a BK-space with respect to the norm $\|\xi\|_{\ell_1}$ and also the matrix Σ is a triangular matrix.Then by Wilansky [?], the space $\overline{bv}$ is a BK-space. □

Theorem 2.5. *The function $\|\xi\|_{\overline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)} = \sum_{m=1}^{\infty} \|\Delta(m\,\xi_m)\|$ is a norm on $\overline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)$.*

Theorem 2.6. *The spaces $\overline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)$ and $\underline{bv}$ have AK-property.*

Proof. Let $\{\xi_k^n\} \in \underline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)$ and $[\xi_k^n] = \{\xi_1^n, \xi_2^n, \xi_3^n, \ldots, \ldots, \xi_k^n, 0, 0, 0, \ldots\}$.

$$\xi_k^n - [\xi_k^n] = \{0, 0, 0, \ldots, \xi_{k+1}^n, \xi_{k+2}^n, \ldots, \}.$$

$$\begin{aligned}
\Rightarrow \quad \|\xi_k^n - [\xi_k^n]\|_{\underline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)} &= \|0, 0, 0, \ldots, \xi_{k+1}^n, \xi_{k+2}^n, \ldots, \|_{\underline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)}. \\
&= \sum_{p \geq k+1} \mathcal{M}\left(\frac{\|\xi_p^n/p\|}{K}\right) \to 0, \quad \text{as} \quad p \to 0. \\
\Rightarrow \quad [\xi_k^n] \to \xi_k^n \quad \text{as} \quad k \to \infty &
\end{aligned}$$

Then, the space $\underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ has AK-property. □

Theorem 2.7. *The spaces* $\overline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$, $\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$, $\underline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ *and* $\underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ *are norm isomorphic to* ℓ_1.

Proof. We must show that there is a one-one and onto linear mapping between $\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ and ℓ_1.

Suppose that $T : \overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|) \to \ell_l$ be a mapping defined as $\xi \mapsto T\xi$.

Clearly, for $\xi = \theta \quad \Rightarrow T\xi = \theta$.

Now, let $\eta \in \ell_1$. Define a sequence $\{\xi_k\} \in \overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ by

$$\xi_k = \frac{1}{k}\sum_{p=1}^{k} y_p$$

Then

$$\begin{aligned} \|\xi_k\|_{\overline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)} = \sum_k \Delta(k\,\xi_k) &= \sum_k \left\| \sum_{p=1}^{k} p\,\eta_p - (p-1)\sum_{p=1}^{k-1}\eta_p \right\| \\ &= \sum_k \|\eta_k\| = \|\eta\|_{\ell_1} \end{aligned}$$

Therefore, $\xi_n \in \overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$. Hence, the spaces $\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ and ℓ_1 are isomorphic. □

In the similar way, we can prove the isomorphism of remaining spaces.

Theorem 2.8. *The spaces* $\overline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ *and* $\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ *have monotone norm.*

Proof. Let $\{\xi_n\} \in \overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$.

Define $\|\xi_n\|_{\overline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)} = \sum_{k=1} \Delta(k\xi_k)$

and $\|[\xi_p]\|_{\overline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)} = \sum_{k=1}^{n} \|\Delta(p\,\xi_p)\|, \quad \forall \{\xi_k\} \in \overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$.

Now, suppose $q > p$, then

$$\begin{aligned} \|[\xi_p]\|_{\overline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)} &= \sum_{k=1}^{p} \|\Delta(k\,\xi_k)\| \\ &\leq \sum_{k=1}^{q} \|\Delta(k\,\xi_k)\| \\ &\leq \|[\xi_q]\|_{\overline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)} \end{aligned}$$

Also,

$$\sup \|[\xi_n]\|_{\overline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)} = \sup\left(\sum_{k=1}^{n} \|\Delta(k\,\xi_k)\|\right) = \|\xi_n\|_{\overline{bv}(\mathbb{C}_2,\mathcal{M},\|.\|)}.$$

Therefore, the space $\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ has the monotone norm. □

Remark 2.1. *The spaces $\overline{\ell_1}$ and $\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ have AB-property.*

Theorem 2.9. *The following statements hold for $\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ and $\underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ given as :*

(1) *If $\zeta^{(m)} = \{\zeta_n^{(m)}\}$ is sequence where $\{\zeta_n^{(m)}\} \in \overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ of elements of $\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$, defined as*

$$\zeta_n^{(m)} := \begin{cases} 1/m & , n \geq m \\ 0 & , n < m \end{cases}$$

This sequence is the basis for the space $\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ and select $B_m = (M\xi)_m$, for all $m \in \mathbb{N}$ and matrix M defined in equation (??), then $\xi \in \overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ has the unique representation of the type:

$$\xi = \sum_m (M\xi)_m \, \zeta_n^{(m)}$$

(2) *Define a sequence $\{\eta_n^m\}$ with $\eta_n^m \in \underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ as*

$$\eta_n^{(m)} := \begin{cases} m & , n \geq m \\ 0 & , n < m \end{cases}$$

Then this sequence $\zeta^{(m)}$ is a basis for the space $\overline{bv}$ and for $E_m = (Ax)_m$, for all $m \in \mathbb{N}$, where the matrix A is defined by $\Gamma = [\gamma_{nm}]$, every sequence $\xi \in \overline{bv}$ have unique representation as

$$\xi = \sum_m E_m \zeta^{(m)}$$

Corollary 2.2. *The spaces $\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ and $\underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$ are separable.*

3. $\alpha-$Duals of the $\mathbb{C}_2$-Sequence Spaces

In this section, we determine the $\alpha-$duals of the spaces K_2 and K_4.

Let $\xi = \{\xi_n\}$ and $\eta = \{\eta_n\}$ be sequences, and A and B be two subsets of ω_4. Now let $M = (a_{mk})$ be an infinite matrix of bicomplex numbers. Define $\xi\eta = (\xi_n \eta_n)$,

$\xi_n^{-1} \star B = \{\zeta \in \omega_4 : \zeta\, \xi \in B\}$. $N(A, B) = \cap_{\xi \in A} \xi^{-1} \star B = \{\zeta \in \omega_4 : \zeta\, \xi \in B, \text{for } \xi \in A\}$. In particular, for $B = \ell_1, cs$ or bs, We have $\xi^\alpha = \xi^{-1} \star \ell_1$, $\xi^\beta = \xi^1 \star cs$ and $\xi^\gamma = \xi^{-1} \star bs$. The $\alpha-$ dual of A are given by $A^\alpha = M(A, \ell_1)$.

Suppose that $M_m = (a_{mk})_{k=0}^\infty$ denotes the m-th row of the matrix M. Let $M_m(\xi) = \sum_{k=0}^\infty a_{mk}\xi_k$, $\forall n = 0, 1, 2, \ldots$, and $M(\xi) = [M_m(\xi)]_{m=0}^\infty$, where $M_m \in \xi^\beta$

Lemma 3.1. *[?] Let A_1, A_2 be to BK-spaces, and $M = [\eta_{nm}]$ be a triangular matrix where $\xi_{nm} \in \mathbb{C}_2/\mathbb{O}_2$, then for matrix $S_{A_1}^M = [\xi_{nm}]$ defined with $\nu = \{\nu_m\} \in A_1$ as*

$$\xi_{nm} = \sum_{i=1}^{n} \nu_i \, \eta_{nm} \, \mu_{im}$$

Then $A_2 A_1(M) \subset A_1(M)$ holds if and only if the matrix $S^M_{A_1} = M D_\nu M^{-1} \in (A_1 : A_1)$, where D_ν is a diagonal matrix such that $[D_\nu]_{nn} = \nu_n$, $\forall n \in \mathbb{N}$.

Lemma 3.2. *[?] Let $\{\gamma_k\}$ be a sequence in ω_4 and $M = [\eta_{nm}]$ be an invertible triangular matrix. Define a matrix $S^M_{A_1} = [\xi_{nm}]$ as*

$$\xi_{nm} = \sum_{i=m}^{n} \eta_i \, \mu_{im}$$

Then

$$A_1^{\beta}(M) = \{\eta_m \in \omega_4 : S(M) \in (A_1 : c)\}$$

and

$$A_1^{\gamma}(M) = \{\eta_m \in \omega_4 : S(M) \in (A_1 : \ell_\infty)\}$$

Lemma 3.3. *Let $M = [\xi_{nm}]$ be an infinite matrix of bicomplex numbers. Then*

(1) $M \in (\ell_1 : \ell_1) \iff \sup \sum_{k \in \mathbb{N}} \|\xi_{nm}\| < \infty.$

(2) $M \in (\ell_1 : \ell_\infty) \iff \sup_{k,n \in \mathbb{N}} \|\xi_{nm}\| < \infty$

(3) $M \in (\ell_1, c) \iff \sup_{k,n \in \mathbb{N}} \|\xi_{nm}\| < \infty$ *and for some sequence $\{\kappa_m\}$ such that* $\lim_{n \to \infty} \xi_{nm} = \kappa_m$

Theorem 3.1. *For the space $\underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$, we have*

$$\underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)^{\alpha} = \underline{\alpha}_1$$

where

$$\underline{\alpha}_1 = \left\{ \xi = \{\xi_n\} \in \omega_4 \, : \, \sum_k \left\| \sum_{m=1}^{n} \mathcal{M}\left(\frac{\|\Delta(\xi_m/m)\|}{K} \right) \eta_k \right\| < \infty, (\eta_k) \in \underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|) \text{ for some } K > 0 \right\}$$

Proof. $\{\xi_n\}$ be any sequence in ω_4. Assume the following relation

$$\xi_n \eta_n = \sum_{k=1}^{n} \mathcal{M}\left(\frac{\|\Delta(\xi_n/n)\|}{K} \right) \eta_k = (E\eta)_k$$

where $E = \{e_{nk}\}$ is defined by

$$e_{nm} = \begin{cases} \mathcal{M}\left(\frac{\|\Delta(\xi_n/n)\|}{K} \right) & , 1 \leq m \leq n \\ 0 & , n < m \end{cases} \tag{3.1}$$

Therefore, from the equation (3.1) and the Lemma (3.3) we have

$\left\{ \mathcal{M}\left(\frac{\|\Delta(\xi_n/n)\|}{K} \right) \zeta_n \right\} \in \ell_1$ if and only if $E\eta \in \ell_1$, whenever $\eta \in \ell_1$.

So, $\xi = \{\xi_n s\} \in \overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)^{\alpha}$ if and only if $E \in (\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|) : \ell_1)$.

Hence proved. □

Analogously, we can prove the following theorems.

Theorem 3.2. *For the space* $\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)$

$$\overline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|)^{\alpha} = \underline{\alpha}_2$$

where

$$\underline{\alpha}_1 = \left\{ \xi = \{\xi_n\} \in \omega_4 \, : \, \sum_k \left\| \sum_{m=1}^{n} \mathcal{M}\left(\frac{\|(\xi_m/m)\|}{K} \right) \eta_k \right\| < \infty, (\eta_k) \in \underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|) \ \textit{for some} \, K > 0 \right\}$$

Theorem 3.3. *For the space* $\overline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$

$$\overline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)^{\alpha} = \overline{\alpha}1$$

where

$$\underline{\alpha}_1 = \left\{ \xi = \{\xi_n\} \in \omega_4 \, : \, \sum_k \left\| \sum_{m=1}^{n} \mathcal{M}\left(\frac{\|(m\,\xi_m)\|}{K} \right) \eta_k \right\| < \infty, (\eta_k) \in \underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|) \ \textit{for some} \, K > 0 \right\}$$

Theorem 3.4. *For the space* $\underline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)$

$$\underline{\ell_1}(\mathbb{C}_2, \mathcal{M}, \|.\|)^{\alpha} = \overline{\alpha}_2$$

where

$$\underline{\alpha}_1 = \left\{ \xi = \{\xi_n\} \in \omega_4 \, : \, \sum_k \left\| \sum_{m=1}^{n} \mathcal{M}\left(\frac{\|(\Delta(m\,\xi_m))\|}{K} \right) \eta_k \right\| < \infty, (\eta_k) \in \underline{bv}(\mathbb{C}_2, \mathcal{M}, \|.\|) \ \textit{for some} \, K > 0 \right\}$$

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