International Journal of Analysis and Applications
Volume 18, Number 5 (2020), 819-837
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-18-2020-819

UNION SOFT SET THEORY APPLIED TO ORDERED SEMIGROUPS

RAEES KHAN[1,*], ASGHAR KHAN[2], MUHAMMAD UZAIR KHAN[2,3], NASIR KHAN[1]

[1] *Department of Mathematics, FATA University, Darra Adam Khel, N.M.D. Kohat, KP, Pakistan*

[2] *Department of Mathematics, Abdul Wali Khan University Mardan, Pakistan*

[3] *Department of Mathematics & Statistics, Bacha Khan University Charsadda, KP, Pakistan*

[*] *Corresponding author: raeeskhan@fu.edu.pk*

Abstract. The uni-soft type of bi-ideals in ordered semigroup is considered. The notion of a uni-soft bi-ideal is introduced and the related properties are investigated. The concept of $\delta-$exclusive set is given and the relations between uni-soft bi-ideals and $\delta-$exclusive set are discussed. The concepts of two types of prime uni-soft bi-ideals of an ordered semigroup S are given and it is proved that, a non-constant uni-soft bi-ideal of S is prime in the second sense if and only if each of its proper $\delta-$exclusive set is a prime bi-ideal of S. The characterizations of left and right simple ordered semigroups are considered. Using the notion of uni-soft bi-ideals, some semilattices of left and right simple semigroups are provided. By using the properties of uni-soft bi-ideals, the characterization of a regular ordered semigroup is provided. In the last section of this paper, the characterizations of both regular and intra-regular ordered semigroups are provided.

1. Introduction

The notion of soft set was introduced in 1999 by Molodtsov [20] as a new mathematical tool for dealing with uncertainties. Due to its importance, it has received much attention in the mean of algebraic structures such as groups [9], semirings [11], rings [1], ordered semigroups [15] and hemirings [19, 22] and so on. Feng et al. discussed soft relations in semigroups (see [12, 13]) and explored decomposition of fuzzy soft sets with finite value spaces. Also, Feng and Li [14] considered soft product operations. Jun *et al.*, [15] applied

Received December 12^{th}, 2017; accepted February 5^{th}, 2018; published July 23^{rd}, 2020.

2010 *Mathematics Subject Classification.* 06D72, 20M99, 20M12.

Key words and phrases. ordered semigroup; left/right regular and completely regular ordered semigroup; regular and intra-regular ordered semigroup; left and right simple subsemigroup; uni-soft bi-ideal; uni-soft left/right ideal; uni-soft quasi-ideal.

the concept of soft set theory to ordered semigroups. They applied the notion of soft sets by Molodtsov to ordered semigroups and introduced the notions of (trivial, whole) soft ordered semigroups, soft ordered subsemigroups, soft r-ideals, soft l-ideals, and r-idealistic and l-idealistic soft ordered semigroups [16]. They investigated various related properties by using these notions. In [6–8] Khan *et al.*, characterized different classes of ordered semigroups by using uni-soft quasi-ideals and uni-soft ideals.

In this paper, we introduce the notion of a uni-soft bi-ideal in ordered semigroups. The concept of $\delta-$exclusive set is given and the relations between uni-soft bi-ideals and $\delta-$exclusive set are discussed. We also define two types of prime uni-soft bi-ideals of an ordered semigroup S and prove that, a non-constant uni-soft bi-ideal of S is prime in the second sense if and only if each of its proper $\delta-$exclusive set is a prime bi-ideal of S. The characterizations of some classes of ordered semigroups are given. By using the notion of uni-soft bi-ideals, some semilattices of left and right simple subsemigroups are discussed. Using the concept of an $\delta-$exclusive set, it is proved that a soft set of S over U is a uni-soft bi-ideal if and only if the non-empty $\delta-$exclusive set is a bi-ideal. Regular ordered semigroups are chracterized by the properties of uni-soft bi-ideals and it is shown that every uni-soft bi-ideal of S over U is idempotent if and only if the ordered semigroup is regular. In the last section of this paper, the characterizations of both regular and intra-regular ordered semigroups are discussed.

2. Preliminaries

In this section, we give some basic definitions and results, which are necessary for the subsequent sections.

By an *ordered semigroup* we mean a structure $(S, \cdot, \leq)$ such that:

(OS1) $(S, \cdot)$ is a semigroup.

(OS2) $(S, \leq)$ is a poset.

(OS3) $(\forall a, b, x \in S)$ $(a \leq b \Longrightarrow ax \leq bx$ and $xa \leq xb)$.

For $A \subseteq S$, we denote

$$(A] := \{t \in S : t \leq h \text{ for some } h \in A\}.$$

For $A, B \subseteq S$, we have $AB := \{ab : a \in A, b \in B\}$. A nonempty subset A of an ordered semigroup S is called a *subsemigroup* of S if $A^2 \subseteq A$. A nonempty subset A of S is called a *left* (resp. *right*) ideal of S if: (1) $SA \subseteq A$(resp. $AS \subseteq A$) and (2) $a \in A$, $S \ni b \leq a$, implies $b \in A$. By a *two-sided ideal* or simply an *ideal* of S we mean a non-empty subset of S which is both a left and a right ideal of S. A nonempty subset A of an ordered semigroup S is called a *bi-ideal* of S if: (1) $A^2 \subseteq A$, (2) $ASA \subseteq A$ and (3) $a \in A, S \ni b \leq a$, implies $b \in A$. Let S be an ordered semigroup and $\emptyset \neq A \subseteq S$. Then the set $(A \cup A^2 \cup ASA]$ is the bi-ideal of S generated by A. In particular, if $A = \{x\}$ $(x \in S)$, then we write $(x \cup x^2 \cup xSx]$, instead of $(\{x\} \cup \{x^2\} \cup \{x\}S\{x\}]$ (see [2]).

3. Basic operations of soft sets

From now on, U is an initial universe, E is a set of parameters, $P(U)$ is the power set of U and $A, B, C... \subseteq E$.

Definition 3.1. *A soft set F_A over U is defined as $F_A : E \longrightarrow P(U)$ such that $F_A(x) = \emptyset$ if $x \notin A$. Hence F_A is also called an approximation function.*

A soft set F_A over U can be represented by the set of ordered pairs $F_A = \{(x, F_A(x)) | x \in E, F_A(x) \in P(U)\}$. It is clear from Definition 3.1, that a soft set is a *parameterized family* of subsets of U. Note that the set of all soft sets over U will be denoted $S(U)$.

Definition 3.2. *(i) Let $F_A, F_B \in S(U)$. Then F_A is called a soft subset of F_B, denoted by $F_A \widetilde{\subseteq} F_B$ if $F_A(x) \subseteq F_B(x)$ for all $x \in E$.*

(ii) Let $F_A, F_B \in S(U)$. Then the soft union of F_A and F_B, denoted by $F_A \widetilde{\cup} F_B = F_{A \cup B}$, is defined by $\left(F_A \widetilde{\cup} F_B\right)(x) = F_A(x) \cup F_B(x)$ for all $x \in E$.

(iii) Let $F_A, F_B \in S(U)$. Then the soft intersection of F_A and F_B, denoted by $F_A \widetilde{\cap} F_B = F_{A \cap B}$, is defined by $\left(F_A \widetilde{\cap} F_B\right)(x) = F_A(x) \cap F_B(x)$ for all $x \in E$.

For $x \in S$, we define

$$A_x = \{(y, z) \in S \times S | x \leq yz\}.$$

Definition 3.3. *Let (F_S, S) and (G_S, S) be two soft sets over U. Then, the soft intersection-union product, denoted by $f_S \Diamond g_S$, is defined by*

$$F_S \Diamond G_S : S \longrightarrow P(U), x \longmapsto \begin{cases} \bigcap\limits_{(y,z) \in A_x} \{F_S(y) \cup G_S(z)\} & \text{if } A_x \neq \emptyset, \\ U & \text{if } A_x = \emptyset, \end{cases}$$

for all $x \in S$. One can easily prove that "$\Diamond$" on $S(U)$ is well defined and the set $\left(S(U), \Diamond, \widetilde{\subseteq}\right)$ forms an ordered semigroup (see [7]).

4. Uni-soft bi-ideals

For an ordered semigroup, the soft sets "$\emptyset_S$" and "$\top_S$" of S over U are defined as follows:

$$\emptyset_S : S \longrightarrow P(U), x \longmapsto \emptyset_S(x) = \emptyset,$$

$$\top_S : S \longrightarrow P(U), x \longmapsto \top_S(x) = U \text{ for all } x \in S.$$

Clearly, the soft set "$\emptyset_S$" (resp. "$\top_S$") of an ordered semigroup S over U is the *least* (resp., the *greatest*) element of the ordered semigroup $(S(U), \Diamond, \widetilde{\subseteq})$. The soft set "$\emptyset_S$" is the *null* element of $(S(U), \Diamond, \widetilde{\subseteq})$ (that is $F_S \Diamond \emptyset_S = \emptyset_S \Diamond F_S = \emptyset_S$ and $\emptyset_S \widetilde{\subseteq} F_S$ for every $F_S \in S(U)$. The soft set $(\top_S, S)$ is called the *whole soft set* over U, where $\top_S(x) = U$ for all $x \in S$.

For a non-empty subset A of S, the *characteristic soft set* (χ_A, A) over U is a soft set defined as follows:

$$\chi_A : S \longrightarrow P(U), x \longmapsto \begin{cases} U & \text{if } x \in A, \\ \emptyset & \text{if } x \in S \backslash A. \end{cases}$$

For the characteristic soft set (χ_A, A) over U, the soft set (χ_A^c, A) over U is given as follows:

$$\chi_A^c : S \longrightarrow P(U), x \longmapsto \begin{cases} \emptyset & \text{if } x \in A, \\ U & \text{if } x \in S \backslash A. \end{cases}$$

Definition 4.1. *(cf. [15]). Let S be an ordered semigroup. A soft set (F_S, S) of S over U is called a union-soft semigroup (briefly, uni-soft semigroup) of S over U if:*

$$F_S(xy) \subseteq F_S(x) \cup F_S(y) \quad \forall x, y \in S.$$

Definition 4.2. *(cf. [15]). Let S be an ordered semigroup. A soft set (F_S, S) of S over U is called a union-soft left (resp. right) ideal (briefly, uni-soft left (resp. right) ideal) of S over U if*

(SU1) $x \leq y \implies F_S(x) \subseteq F_S(y)$,

(SU2) $F_S(xy) \subseteq F_S(y)$ (resp. $F_S(xy) \subseteq F_S(x)$) $\forall x, y \in S$.

If (F_S, S) is both a uni-soft left ideal and a uni-soft right ideal of S over U, then (F_S, S) is called a *uni-soft ideal* of S over U.

Definition 4.3. *(cf. [6]). Let S be an ordered semigroup. A uni-soft semigroup (F_S, S) of S over U is called a union-soft bi-ideal (briefly, uni-soft bi-ideal) of S over U if*

(SU5) $x \leq y \Longrightarrow F_S(x) \subseteq F_S(y)$.

(SU6) $F_S(xyz) \subseteq F_S(x) \cup F_S(z)$ $\forall$ $x, y, z \in S$.

Definition 4.4. *A soft set (F_S, S) of an ordered semigroup $(S, \cdot, \leq)$ over U is called idempotent if $(F_S \Diamond F_S, S) = (F_S, S)$.*

For a soft set (F_A, S) over U and a subset δ of U, the δ-exclusive set of (F_A, S) denoted by $e_A(F_A; \delta)$ is defined by

$$e_A(F_A; \delta) := \{x \in A | F_A(x) \subseteq \delta\}.$$

Theorem 4.1. *A soft set (F_S, S) over U is a uni-soft bi-ideal over U if and only if the nonempty δ-exclusive set of (F_S, S) is a bi-ideal of S for all $\delta \in P(U)$.*

Proof. Assume that (F_S, S) is a uni-soft bi-ideal over U. Let $\delta \in P(U)$ be such that $e_S(F_S;\delta) \neq \emptyset$. Let $x, y \in S$ with $x \leq y$ be such that $y \in e_S(F_S;\delta)$. Then $F_S(y) \subseteq \delta$. By (SU5) we have

$$F_S(x) \subseteq F_S(y) \subseteq \delta$$

and that $x \in e_S(F_S;\delta)$. Let $x, y \in e_S(F_S;\delta)$. Then $F_S(x) \subseteq \delta$ and $F_S(y) \subseteq \delta$. By (SU6) we have

$$F_S(xy) \subseteq F_S(x) \cup F_S(y) \subseteq \delta$$

and so $xy \in e_S(F_S;\delta)$. For $x, z \in e_S(F_S;\delta)$. Then $F_S(x) \subseteq \delta$ and $F_S(z) \subseteq \delta$. By (SU6) we have

$$F_S(xyz) \subseteq F_S(x) \cup F_S(z) \subseteq \delta$$

hence $xyz \in e_S(F_S;\delta)$. Therefore $e_S(F_S;\delta)$ is a bi-ideal of S.

Conversely, suppose that the nonempty δ-exclusive set of (F_S, S) is a bi-ideal of S for all $\delta \in P(U)$. Let $x, y \in S$ with $x \leq y$ be such that $F_S(x) \supset F_S(y) = \delta_y$ then $y \in e_S(F_S;\delta_y)$ but $x \notin e_S(F_S;\delta_y)$. This is a contradiction. Hence $F_S(x) \subseteq F_S(y)$ for all $x \leq y$. If there exist $x, y \in S$ such that

$$F_S(xy) \supset F_S(x) \cup F_S(y) = \delta_x \cup \delta_y = \delta_z$$

then $x \in e_S(F_S;\delta_z)$ and $y \in e_S(F_S;\delta_z)$ but $xy \notin e_S(F_S;\delta_z)$. This is a contradiction. Hence $F_S(xy) \subseteq F_S(x) \cup F_S(y)$ for all $x, y \in S$. If there exist $x, y, z \in S$ such that

$$F_S(xyz) \supset F_S(x) \cup F_S(z) = \delta_x \cup \delta_z = \delta_s$$

then $x \in e_S(F_S;\delta_s)$ and $z \in e_S(F_S;\delta_s)$ but $xyz \notin e_S(F_S;\delta_s)$. This is a contradiction. Hence $F_S(xyz) \subseteq F_S(x) \cup F_S(z)$ for all $x, y \in S$. □

Corollary 4.1. *(cf. [8]). For any nonempty subset B of S, the following are equivalent.*

(1) B is a bi-ideal of S.

(2) The soft set (χ^c_B, S) over U is a uni-soft bi-ideal over U.

A bi-ideal P of an ordered semigroup S is called *prime* if $P \neq S$ and for any bi-ideals A, B of S from $AB \subseteq P$ it follows that $A \subseteq P$ or $B \subseteq P$. By analogy a non-constant uni-soft bi-ideal (F_S, S) of S over U is called *prime* (in the first sense) if for any uni-soft bi-ideals $(G_S, S), (H_S, S)$ of S over U from $(G_S \Diamond H_S, S) \,\tilde{\supseteq}\, (F_S, S)$ it follows that $(G_S, S) \tilde{\supseteq} (F_S, S)$ or $(H_S, S) \tilde{\supseteq} (F_S, S)$.

Theorem 4.2. *A bi-ideal P of an ordered semigroup S is prime if and only if for all $a, b \in S$ from $(aSb] \subseteq P$ it follows that $a \in P$ or $b \in P$.*

Proof. Assume that P is a prime bi-ideal of S and $(aSb] \subseteq P$ for some $a, b \in S$. Then obviously, the sets $A = (aSa]$ and $B = (bSb]$ are bi-ideals of S, because $(aSa]S(aSa] = (aSa](S](aSa] \subseteq (aSaSaSa] \subseteq (aSa]$ and if $x \in S$ and $x \leq y \in (aSa]$, then $x \in ((aSa]] = (aSa]$. Similarly, $(bSb]$ is a bi-ideal of S. So, $AB \subseteq (AB] = ((aSa](bSb]] \subseteq (aSabSb] \subseteq (aSb] \subseteq P$, and consequently $A \subseteq P$ or $B \subseteq P$. Let $\langle x \rangle$ be the bi-ideal of S generated by $x \in S$. If $A \subseteq P$, then $\langle a \rangle \subseteq (aSa] = A \subseteq P$, whence $a \in P$. If $B \subseteq P$, then $\langle b \rangle \subseteq (bSb] = B \subseteq P$, whence $b \in P$.

The converse part is obvious. □

Corollary 4.2. *A bi-ideal P of a commutative ordered semigroup S with identity is prime if and only if for all $a, b \in S$ from $ab \in P$ it follows $a \in P$ or $b \in P$.*

The result expressed by Corollary 4.2, suggests the following definition of prime uni-soft bi-ideals.

Definition 4.5. *A non-constant uni-soft bi-ideal (F_S, S) of S over U is called prime (in the second sense) if for all $\delta \in P(U)$ and $a, b \in S$, the following condition is satisfied:*

if $F_S(axb) \subseteq \delta$ for every $x \in S$ then $F_S(a) \subseteq \delta$ or $F_S(b) \subseteq \delta$.

In other words, a non-constant uni-soft bi-ideal is prime if from the fact that $axb \in e_S(F_S; \delta)$ for every $x \in S$ it follows $a \in e_S(F_S; \delta)$ or $b \in e_S(F_S; \delta)$. It is clear that any uni-soft bi-ideal which is prime in the first sense is prime in the second sense. The converse is not true.

Example 4.1. *Let $S = \{a, b, c\}$ be an ordered semigroup with the following Cayley table and order relation (see [3]).*

$\cdot$	a	b	c
a	a	a	a
b	a	b	b
c	a	c	c

$$\leq := \{(a, a), (b, b), (c, c), (a, b)\}.$$

Let (F_S, S) be a soft set over $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ in which F_S is given by

$$F_S : S \longrightarrow P(U), x \longmapsto \begin{cases} \{1, 3, 5\} \text{ if } x = c \\ \{1, 2, 3, 5, 6\} \text{ if } x = b \\ \{1, 2, 3, 5, 6, 8\} \text{ if } x = a \end{cases}$$

is a uni-soft bi-ideal over U. It is prime in the second sense but it is not prime in the first sense.

Theorem 4.3. *A non-constant soft set (F_S, S) over U is a prime uni-soft bi-ideal over U in the second sense if and only if the nonempty δ-exclusive set of (F_S, S) is a prime bi-ideal of S for all $\delta \in P(U)$.*

Proof. Let a uni-soft bi-ideal (F_S, S) over U be prime in the second sense and let $e_S(F_S;\delta)$ be its arbitrary proper $\delta-$exclusive set, i.e., $\emptyset \neq e_S(F_S;\delta) \neq S$ (obviously $e_S(F_S;\delta)$ is a bi-ideal of S (Theorem 4.1)). If $(aSb] \subseteq e_S(F_S;\delta)$, then $F_S(axb) \subseteq \delta$ for every $x \in S$. Hence $F_S(a) \subseteq \delta$ or $F_S(b) \subseteq \delta$, i.e., $a \in e_S(F_S;\delta)$ or $b \in e_S(F_S;\delta)$ which means that $e_S(F_S;\delta)$ is prime bi-ideal of S (Corollary 4.2).

For the converse part consider a non-constant uni-soft bi-ideal (F_S, S) over U. If it is not prime in the second sense, then there exist $a, b \in S$ such that $F_S(axb) \subseteq \delta$ for all $x \in S$, but $F_S(a) \supset \delta$ and $F_S(b) \supset \delta$. Thus $(aSb] \subseteq e_S(F_S;\delta)$ but $a \notin e_S(F_S;\delta)$ and $b \notin e_S(F_S;\delta)$. Therefore $e_S(F_S;\delta)$ is not prime. This is a contradiction, which proves that (F_S, S) is prime in the second sense. □

Corollary 4.3. *A soft set (χ^c_B, S) over U is a prime uni-soft bi-ideal over U if and only if B is a prime bi-ideal of S.*

An ordered semigroup $(S, \cdot, \leq)$ is called *regular*, if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$ or equivalently: (1) $(\forall a \in S)\,(a \in (aSa])$ and (2) $(\forall A \subseteq S)\,(A \subseteq (ASA])$.

An ordered semigroup S is called *left* (resp., *right*) *regular*, if for every $a \in S$, there exists $x \in S$ such that $a \leq xa^2$ (resp., $a \leq a^2x$) or equivalently: (1) $(\forall a \in S)\left(a \in (Sa^2]\right)\left(\text{resp., } a \in (a^2S]\right)$ and (2) $(\forall A \subseteq S)\left(A \subseteq (SA^2]\right)\left(\text{resp., } A \subseteq (A^2S]\right)$.

An ordered semigroup is called *completely regular* if it is regular, left regular and right regular.

Lemma 4.1. *(cf. [4]). An ordered semigroup S is completely regular if and only if $A \subseteq (A^2SA^2]$ for every $A \subseteq S$. Equivalently, if $a \in (a^2Sa^2]$ for every $a \in S$.*

Theorem 4.4. *An ordered semigroup $(S, \cdot, \leq)$ is completely regular if and only if for every uni-soft bi-ideal (F_S, S) of S over U, we have*

$$(F_S(a), S) = \left(F_S(a^2), S\right) \text{ for every } a \in S.$$

Proof. $(\Longrightarrow)$. Assume that (F_S, S) is a uni-soft bi-ideal of S over U. Let $a \in S$. Since S is completely regular, then $a \in (a^2Sa^2]$. That is, $a \leq a^2xa^2$ for some $x \in S$. Since (F_S, S) is a uni-soft bi-ideal of S over U, we have

$$\begin{aligned} F_S(a) &\subseteq F_S(a^2xa^2) \subseteq F_S(a^2) \cup F_S(a^2) \\ &= F_S(a^2) \subseteq F_S(a) \cup F_S(a) = F_S(a) \,(\text{since } (F_S, S) \text{ is a uni-soft semigroup}). \end{aligned}$$

Thus, $(F_S(a), S) = \left(F_S(a^2), S\right)$.

$(\Longleftarrow)$. Let $A(a^2)$ be a bi-ideal of S generated by a $(a \in S)$, i.e., the set $A(a^2) = (a^2 \cup a^4 \cup a^2Sa^2]$. By Corollary 4.1, the soft set (χ_A^c, S) defined by

$$\chi_A^c : S \longrightarrow P(U), x \longmapsto \begin{cases} \emptyset \text{ if } x \in A(a^2), \\ U \text{ if } x \in A(a^2), \end{cases}$$

is a uni-soft bi-ideal of S over U. By hypothesis, we have $(\chi_A^c(a), S) = \left(\chi_A^c(a^2), S\right)$. Since $a^2 \in A(a^2)$, we have $\chi_A^c(a^2) = \emptyset$. Then $\chi_A^c(a) = \emptyset$ and hence $a \in A(a^2) = (a^2 \cup a^4 \cup a^2Sa^2]$. Thus $a \leq a^2$ or $a \leq a^4$ or $a \leq a^2xa^2$ for some $x \in S$. If $a \leq a^2$, then

$$a \leq a^2 = a.a \leq a^2.a^2 = a.a^2.a \leq a^2.a^2.a^2 \in a^2Sa^2.$$

Similarly, in other cases we get $a \leq a^2va^2$ for some $v \in S$. Consequently, $a \in (a^2Sa^2]$ and by Lemma 4.1, S is completely regular. □

5. Semilattices of left and right simple semigroups

A subsemigroup F of S is called a *filter* (see [5]) of S if:

(1) $(\forall a, b \in S)$ $(ab \in F \Longrightarrow a \in F$ and $b \in F)$ and

(2) $(\forall a \in S)(\forall b \in F)(a \geq b \Longrightarrow a \in F)$.

For $x \in S$, we denote by $N(x)$, the least filter of S generated x $(x \in S)$. By $\mathcal{N}$ we mean the equivalence relation on S defined by $\mathcal{N} := \{(x, y) \in S \times S | N(x) = N(y)\}$ (see [4]). An equivalence relation σ on S is called *congruence* if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* on S, if $(a, a^2) \in \sigma$ and $(ab, ba) \in \sigma$ for each $a, b \in S$ (see [5]). If σ is a semilattice congruence on S then the σ-class $(x)_\sigma$ of S containing x is a subsemigroup of S for every $x \in S$ (see [4]). An ordered semigroup S is called a *semilattice of left and right simple semigroups* if there exists a semilattice congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x is a left and right simple subsemigroup of S for every $x \in S$. Equivalently, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left and right simple subsemigroups of S such that:

(i) $S_\alpha \cap S_\beta = \emptyset$, $\forall \alpha, \beta \in Y$, $\alpha \neq \beta$,

(ii) $S = \bigcup_{\alpha \in Y} S_\alpha$,

(iii) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$, $\alpha, \beta \in Y$.

The semilattice congruences in ordered semigroups are defined exactly as in semigroups without ordered−so the two definitions are equivalent (see [4, 5]).

Lemma 5.1. *(cf. [2]). An ordered semigroup $(S, \cdot, \leq)$ is a semilattice of left and right simple semigroups if and only if for all bi-ideals A, B of S, we have*

$$(A^2] = A \text{ and } (AB] = (BA].$$

Theorem 5.1. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:*

(1) S is regular, left and right simple.

(2) Every uni-soft bi-ideal (F_S, S) of S over U is a constant mapping.

Proof. (1) $\Longrightarrow$ (2). Assume that S is regular and left and right simple. Let (F_S, S) be a uni-soft bi-ideal over U. We consider the set

$$E_S := \{e \in S | e^2 \geq e\}.$$

Since S is regular, so for every $a \in S$, there exists $x \in S$ such that $a \leq axa$ and we have

$$(ax)^2 = (axa)x \geq ax,$$

hence $ax \in E_S$. Thus $E_S \neq \emptyset$.

(A) Let $t \in E_S$, we prove that (F_S, S) is constant mapping on E_S. That is, for every $e \in E_S$, we have $F_S(e) = F_S(t)$. Since $t \in S$ and S is left and right simple, we have $S = (tS]$ and $S = (St]$. Since $e \in S$, we have $e \in (tS]$ and $e \in (St]$, then $e \leq ts$ and $e \leq zt$ for some $s, z \in S$. Hence $e^2 \leq t(sz)t$. Since (F_S, S) is a uni-soft bi-ideal over U, we have

$$F_S(e^2) \subseteq F_S(t(sz)t) \subseteq F_S(t) \cup F_S(t) = F_S(t).$$

On the other hand, since $e \in E_S$, we have $e^2 \geq e$ and hence $F_S(e) \subseteq F_S(e^2)$. Thus, $F_S(e) \subseteq F_S(e^2) \subseteq F_S(t)$. In a similar way, we can prove that $F_S(t) \subseteq F_S(e)$. Thus, $F_S(t) = F_S(e)$.

(B) Now we prove that (F_S, S) is constant mapping on S. That is, $F_S(t) = F_S(e)$ for every $a \in S$. Since S is regular, so for every $a \in S$, there exists $x \in S$ such that $a \leq axa$. Then

$$(ax)^2 = (axa)x \geq xa \text{ and } (xa)^2 = x(axa) \geq xa.$$

Hence $ax, xa \in E_S$. Thus by (A) we have $F_S(ax) = F_S(t)$ and $F_S(xa) = F_S(t)$. Since $(ax)a(xa) = (axa)xa \geq axa \geq a$, we have

$$F_S(a) \subseteq F_S((ax)a(xa)) \subseteq F_S(ax) \cup F_S(xa) = F_S(xa) = F_S(t).$$

Since S is left and right simple and $a \in S$, we have $S = (aS]$ and $S = (aS]$. Since $t \in S$, we have $t \in (aS]$ and $t \in (Sa]$, then $t \leq as_1$ and $t \leq z_1a$ for some $s_1, z_1 \in S$. Now, $t^2 \leq a(s_1z_1)a$ and (F_S, S) is a uni-soft bi-ideal of S over U, we have

$$F_S(t^2) \subseteq F_S(a(s_1z_1)a) \subseteq F_S(a) \cup F_S(a) = F_S(a).$$

Since $t \in E_S$, we have $t^2 \geq t$, then $F_S(t) \subseteq F_S(t^2) \subseteq F_S(a)$. Therefore, $F_S(t) = F_S(a)$.

(2) $\Longrightarrow$ (1). Let $a \in S$. Then (i) $(aS]S(aS] = (aS](S](aS] \subseteq (aSSaS] \subseteq (aS]$, (ii) $(aS](aS] \subseteq (aSaS] \subseteq (aS]$ and (iii) If $x \leq y$ such that $S \ni x \leq y \in (aS]$, then $x \in ((aS]] = (aS]$. Therefore, $(aS]$ is a bi-ideal of S. By Corollary 4.1, the soft set $\left(\chi^c_{(aS]}, S\right)$ defined by

$$\chi^c_{(aS]} : S \longrightarrow P(U), x \longmapsto \begin{cases} \emptyset \text{ if } x \in (aS], \\ U \text{ if } x \notin (aS], \end{cases}$$

is a uni-soft bi-ideal of S over U. By hypothesis, $\left(\chi^c_{(aS]}, S\right)$ is a constant mapping, that is, for every $x \in S$, there exists a subset $\delta \subseteq U$ such that

$$\chi^c_{(aS]}(x) = \delta$$

Let $(aS] \subset S$ and $t \in S$ be such that $t \notin (aS]$. Then $\chi^c_{(aS]}(t) = U$. Since $a^2 \in (aS]$, we have $\chi^c_{(aS]}(a^2) = \emptyset$. This is a contradiction. Thus, $S = (aS]$. Similarly, we can prove that $S = (Sa]$ and therefore, S is left and right simple. Since $S = (aS] = (Sa]$, we have $a \in S = (aS] = (a(Sa]] = (aSa]$ and hence S is regular. This completes the proof. □

Lemma 5.2. *(cf. [2]). Let $(S, \cdot, \leq)$ be an ordered semigroup and $B(x)$ and $B(y)$ be the bi-ideals of S generated by x and y, respectively. Then $B(x)SB(y) \subseteq (xSy]$.*

Theorem 5.2. *An ordered semigroup $(S, \cdot, \leq)$ is a semilattice of left and right simple semigroups if and only if for every uni-soft bi-ideal (F_S, S) of S over U, we have*

$$(F_S(a), S) = \left(F_S(a^2), S\right) \text{ and } (F_S(ab), S) = (F_S(ba), S) \text{ for all } a, b \in S.$$

Proof. ($\Longrightarrow$) (A) Suppose that (F_S, S) be a uni-soft bi-ideal of S over U and let S be a semilattice of left and right simple semigroups. Then by hypothesis, there exists a semilattice Y and a family $\{S_i : i \in Y\}$ of left and right simple subsemigroups of S such that

(i) $S_i \cap S_j = \emptyset \quad \forall i, j \in Y$ and $i \neq j$,

(ii) $S = \bigcup_{i \in Y} S_i$,

(iii) $S_i S_j \subseteq S_{ij}$.

Let $a \in S$. Since $S = \bigcup_{i \in Y} S_i$, then there exists $i \in Y$ such that $a \in S_i$. Since S_i is left and right simple for every $i \in Y$. Then

$$S_i = (aS_ia] = \{t \in S : t \leq axa \text{ for some } t \in S_i\},$$

Then $a \leq axa$ for some $x \in S_i$. Since S_i is left and right simple, we have $x \leq aya$ for some $y \in S_i$. Then we have $a \leq axa \leq a(aya)a = a^2ya^2 \in a^2Sa^2$ and so $a \in (a^2Sa^2]$. Thus S is completely regular. Since (F_S, S) is a uni-soft bi-ideal of S over U. By Theorem 4.4, we have $(F_S(a), S) = \left(F_S(a^2), S\right)$.

(B) Let $a, b \in S$. Then by (A), we have $F_S(ab) = F_S((ab)^2) = F_S((ab)^4)$. On the other hand,

$$\begin{aligned}
(ab)^4 &= (aba)(babab) \in B(aba)B(babab) \subseteq (B(aba)B(babab)] \\
&= (B(babab)B(aba)] \text{ (Lemma 5.1)} \\
&= (B(babab)(B(aba)^2]] \text{ (Lemma 5.1)} \\
&= ((B(babab)](B(aba)B(aba)]] \\
&\subseteq ((B(babab)B(aba)B(aba)]] \ \text{ (since } (A](B] \subseteq (AB]) \\
&= (B(babab)B(aba)B(aba)] \ \text{ (since } ((A]] = (A]) \\
&\subseteq (B(babab)SB(aba)] \\
&\subseteq (((babab)\, S\, (aba)]] \ \text{ (Lemma 5.2)} \\
&= ((babab)\, S\, (aba)] \ \text{ (since } ((A]] = (A]) .
\end{aligned}$$

Then $(ab)^4 \leq (babab)\, z\, (aba)$ for some $z \in S$. Since (F_S, S) is a uni-soft bi-ideal of S over U, we have

$$\begin{aligned}
F_S\left((ab)^4\right) \subseteq F_S\left((babab)\, z\, (aba)\right) &= F_S(ba(babza)ba) \\
&\subseteq F_S(ba) \cup F_S(ba) = F_S(ba).
\end{aligned}$$

Thus, $F_S(ab) \subseteq F_S(ba)$. By symmetry we can prove that $F_S(ba) \subseteq F_S(ab)$. Therefore, $(F_S(ab), S) = (F_S(ba), S)$.

($\Longleftarrow$) Assume that $(F_S(a^2), S) = (F_S(a), S)$ and $(F_S(ab), S) = (F_S(ba), S)$ hold for every uni-soft bi-ideal (F_S, S) of S over U. Since $(F_S(a^2), S) = (F_S(a), S)$ so by condition (1) and Theorem 4.4, it follows that S is completely. Let A be a bi-ideal of S and let $a \in A$. Since $a \in S$ and S is completely regular, by Lemma 4.1, we have

$$a \leq a^2xa^2 = a(axa)a \in A(ASA)A \subseteq AAA \subseteq AA = A^2.$$

Then $A \subseteq A^2$ and hence $(A] \subseteq (A^2] \Longrightarrow A \subseteq (A^2]$ (since A is a bi-ideal). On the other hand, since A is a subsemigroup of S, we have $A^2 \subseteq A$ then $(A^2] \subseteq (A] = A$. Therefore, $A = (A^2]$. Let A and B be bi-ideals of S and let $x \in (AB]$. Then $x \leq ab$ for some $a \in A$ and $b \in B$. We consider $B(ab) = (ab \cup abab \cup abSab]$, the bi-ideal of S generated by ab $(a, b \in S)$. By Corollary 4.1, the soft set $\left(\chi^c_{(ab\cup abab\cup abSab]}, S\right)$ defined by

$$\chi^c_{(ab\cup abab\cup abSab]}(x) = \begin{cases} \emptyset \text{ if } x \in (ab \cup abab \cup abSab] \\ U \text{ if } x \notin (ab \cup abab \cup abSab] \end{cases}$$

is a uni-soft bi-ideal of S over U. By hypothesis, $\chi^c_{(ab\cup abab\cup abSab]}(ab) = \chi^c_{(ab\cup abab\cup abSab]}(ba)$. Since $ab \in (ab \cup abababSab]$, we have $\chi^c_{(ab\cup abab\cup abSab]}(ab) = \emptyset$. Thus, $\chi^c_{(ab\cup abab\cup abSab]}(ba) = \emptyset$ and $ba \in (ab \cup abab \cup abSab]$ and we have $ba \leq ab$ or $ba \leq abab$ or $ba \leq (ab)x(ab)$ for some $x \in S$. If $ba \leq ab$, then $x \leq ab \in AB$ and

$x \in (AB]$. Thus $(BA] \subseteq (AB]$. Similarly in other cases we get $(BA] \subseteq (AB]$. By symmetry, we can prove that $(AB] \subseteq (BA]$. Therefore, $(AB] = (BA]$ and by Lemma 5.1, S is a semilattice of left and right simple subsemigroups. This completes the proof. □

6. Regular ordered semigroups

In this section, we characterize regular ordered semigroups in terms of uni-soft bi-ideals and prove that an ordered semigroup is regular if and only if for every uni-soft bi-ideal (F_S, S) we have $(F_S \Diamond \emptyset_S \Diamond F_S, S) = (F_S, S)$.

Lemma 6.1. *Let $(S, \cdot, \leq)$ be an ordered semigroup and (F_S, S) a soft set over U. If (F_S, S) is a uni-soft semigroup over U then $(F_S \Diamond F_S, S) \widetilde{\supseteq} (F_S, S)$. Conversely, if $(F_S \Diamond F_S, S) \widetilde{\supseteq} (F_S, S)$ holds for every soft set (F_S, S) over U, then (F_S, S) is a uni-soft semigroup over U.*

Proof. ($\Longrightarrow$) Suppose that (F_S, S) is a uni-soft semigroup over U. Let $x \in S$. If $A_x = \emptyset$, then $(F_S \Diamond F_S)(x) = U \supseteq F_S(x)$. If $A_x \neq \emptyset$, then

$$\begin{aligned}(F_S \Diamond F_S)(x) &= \bigcap_{(b,c) \in A_x} \{F_S(b) \cup F_S(c)\} \\ &\supseteq \bigcap_{(b,c) \in A_x} F_S(bc) \supseteq \bigcap_{(b,c) \in A_x} F_S(x) \text{ (since } x \leq bc) \\ &= F_S(x).\end{aligned}$$

Hence $(F_S \Diamond F_S, S) \widetilde{\supseteq} (F_S, S)$.

($\Longleftarrow$) Assume that $(F_S \Diamond F_S, S) \widetilde{\supseteq} (F_S, S)$ holds for every soft set (F_S, S) over U. Let $x, y \in S$, then

$$\begin{aligned}F_S(xy) &\subseteq (F_S \Diamond F_S)(xy) \\ &= \bigcap_{(b,c) \in A_{xy}} \{F_S(b) \cup F_S(c)\} \\ &\subseteq F_S(x) \cup F_S(y).\end{aligned}$$

Hence $F_S(xy) \subseteq F_S(x) \cup F_S(y)$ for all $x, y \in S$ and (F_S, S) is a uni-soft semigroup. □

Proposition 6.1. *Let $(S, \cdot, \leq)$ be an ordered semigroup and (F_S, S) a uni-soft bi-ideal of S over U. Then*

$$(F_S \Diamond \emptyset_S \Diamond F_S, S) \widetilde{\supseteq} (F_S, S).$$

Proof. Let (F_S, S) be a uni-soft bi-ideal of S over U. Let $x \in S$. If $A_x = \emptyset$, then $(F_S \Diamond \emptyset_S \Diamond F_S)(x) = U \supseteq F_S(x)$. If $A_x \neq \emptyset$, then

$$\begin{aligned}
(F_S \Diamond \emptyset_S \Diamond F_S)(x) &= \bigcap_{(b,c)\in A_x} \{(F_S \Diamond \emptyset_S)(b) \cup F_S(c)\} \\
&= \bigcap_{(b,c)\in A_x} \left\{ \bigcap_{(b_1,c_1)\in A_b} \{F_S(b_1) \cup \emptyset_S(c_1)\} \cup F_S(c) \right\} \\
&= \bigcap_{(b,c)\in A_x} \bigcap_{(b_1,c_1)\in A_b} \{F_S(b_1) \cup \emptyset_S(c_1) \cup F_S(c)\} \\
&= \bigcap_{(b,c)\in A_x} \bigcap_{(b_1,c_1)\in A_b} \{F_S(b_1) \cup \emptyset \cup F_S(c)\} \\
&= \bigcap_{(b,c)\in A_x} \bigcap_{(b_1,c_1)\in A_b} \{F_S(b_1) \cup F_S(c)\}.
\end{aligned}$$

Since $x \leq bc$ and $b \leq b_1c_1$, we have $x \leq bc \leq (b_1c_1)c$ and (F_S, S) is a uni-soft bi-ideal of S over U, we have

$$F_S(x) \subseteq F_S((b_1c_1)c) \subseteq F_S(b_1) \cup F_S(c).$$

Hence, $(F_S \Diamond \emptyset_S \Diamond F_S)(x) = \bigcap_{(b,c)\in A_x} \bigcap_{(b_1,c_1)\in A_b} \{F_S(b_1) \cup F_S(c)\} \supseteq \bigcap_{(b,c)\in A_x} \bigcap_{(b_1,c_1)\in A_b} F_S(x) = F_S(x)$. Therefore, $(F_S \Diamond \emptyset_S \Diamond F_S, S) \supseteq (F_S, S)$. □

Lemma 6.2. *(cf. [15]). Let (χ_A^c, S) and (χ_B^c, S) be soft sets over U where A and B are nonempty subsets of S. Then the following properties hold:*

(1) $(\chi_A^c, S) \widetilde{\cup} (\chi_B^c, S) = (\chi_{A\cap B}^c, S)$.
(2) $(\chi_A^c, S) \Diamond (\chi_B^c, S) = \left(\chi_{(AB]}^c, S\right)$.

Lemma 6.3. *(cf. [18]). Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following are equivalent:*

(1) S is regular.
(2) $B = (BSB]$ for every bi-ideal B of S.
(3) $B(a) = (B(a)SB(a)]$ for every $a \in S$.

Theorem 6.1. *An ordered semigroup $(S, \cdot, \leq)$ is regular if and only if for every uni-soft bi-ideal (F_S, S) over U, we have*

$$(F_S, S) = (F_S \Diamond \emptyset_S \Diamond F_S, S).$$

Proof. Suppose that S is regular. Let (F_S, S) be a uni-soft bi-ideal of S over U and let $a \in S$. Since S is regular, there exists $x \in S$ such that $a \leq axa = a(xa)$. Then $(a, xa) \in A_a$ and we have

$$\begin{aligned}(F_S \Diamond \emptyset_S \Diamond F_S)(a) &= \bigcap_{(b,c) \in A_a} \{F_S(b) \cup (\emptyset_S \Diamond F_S)(c)\} \\ &\subseteq F_S(a) \cup (\emptyset_S \Diamond F_S)(xa) \\ &= F_S(a) \cup \bigcap_{(b_1,c_1) \in A_{xa}} \{\emptyset_S(b_1) \cup F_S(c_1)\} \\ &\subseteq F_S(a) \cup \emptyset_S(x) \cup F_S(a) \\ &= F_S(a) \cup \emptyset = F_S(a).\end{aligned}$$

Then $(F_S \Diamond \emptyset_S \Diamond F_S, S) \,\widetilde{\subseteq}\, (F_S, S)$. On the other hand, by Proposition 6.1, we have $(F_S \Diamond \emptyset_S \Diamond F_S, S) \,\widetilde{\supseteq}\, (F_S, S)$, therefore, $(F_S \Diamond \emptyset_S \Diamond F_S, S) = (F_S, S)$.

Conversely, assume that $(F_S \Diamond \emptyset_S \Diamond F_S, S) = (F_S, S)$ holds for every uni-soft bi-ideal (F_S, S) over U. To prove that S is regular, by Lemma 6.2, it is enough to prove that

$$B(a) = (B(a)SB(a)] \quad \forall a \in S.$$

Let $y \in B(a)$. Since $B(a)$ is the bi-ideal of S generated by a ($a \in S$). By Corollary 4.1, $\left(\chi^c_{B(a)}, S\right)$ is a uni-soft bi-ideal of S over U. By hypothesis, we have

$$\left(\chi^c_{B(a)} \Diamond \emptyset_S \Diamond \chi^c_{B(a)}\right)(y) = \chi^c_{B(a)}(y).$$

Since $y \in B(a)$, we have $\chi^c_{B(a)}(y) = \emptyset$ and hence $\left(\chi^c_{B(a)} \Diamond \emptyset_S \Diamond \chi^c_{B(a)}\right)(y) = \emptyset$. But by Lemma 6.2, we have $\chi^c_{B(a)} \Diamond \emptyset_S \Diamond \chi^c_{B(a)} = \chi^c_{(B(a)SB(a)]}$. Thus, $\chi^c_{(B(a)SB(a)]}(y) = \emptyset$ and $y \in (B(a)SB(a)]$. Therefore, $B(a) \subseteq (B(a)SB(a)]$. On the other hand, since $B(a)$ is the bi-ideal of S, we have $(B(a)SB(a)] \subseteq (B(a)] = B(a)$. Thus $B(a) = (B(a)SB(a)]$ and S is regular (Lemma 6.3). □

Lemma 6.4. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Let (F_S, S) and (G_S, S) be uni-soft bi-ideals of S over U. Then the soft product $(F_S \Diamond G_S, S)$ of (F_S, S) and (G_S, S) is again a uni-soft bi-ideal of S over U.*

Proof. Let (F_S, S) and (G_S, S) be uni-soft bi-ideals of S over U. Let $x, y, z \in S$. Then

$$\begin{aligned}(F_S \Diamond G_S)(x) \cup (F_S \Diamond G_S)(z) &= \left[\bigcap_{(p,q) \in A_x} \{F_S(p) \cup G_S(q)\} \cup \bigcap_{(p_1,q_1) \in A_z} \{F_S(p_1) \cup G_S(q_1)\}\right] \\ &= \bigcap_{(p,q) \in A_x} \bigcap_{(p_1,q_1) \in A_z} [\{F_S(p) \cup G_S(q)\} \cup \{F_S(p_1) \cup G_S(q_1)\}] \\ &= \bigcap_{(p,q) \in A_x} \bigcap_{(p_1,q_1) \in A_z} [F_S(p) \cup G_S(q) \cup F_S(p_1) \cup G_S(q_1)] \\ &\supseteq \bigcap_{(p,q) \in A_x} \bigcap_{(p_1,q_1) \in A_z} [F_S(p) \cup G_S(q_1)].\end{aligned}$$

Since $x \leq pq$ and $z \leq p_1q_1$, hence $xyz \leq (pq)y(p_1q_1) = p(qyp_1)q_1$ and $(p(qy)p_1, q_1) \in A_{xyz}$. Thus, $A_{xyz} \neq \emptyset$ and we have

$$\begin{aligned}(F_S \Diamond G_S)(x) \cup (F_S \Diamond G_S)(z) &\supseteq \bigcap_{(p,q)\in A_x} \bigcap_{(p_1,q_1)\in A_z} [F_S(p) \cup F_S(p_1) \cup G_S(q_1)] \\ &\supseteq \bigcap_{(p(qy)p_1,q_1)\in A_{xyz}} [F_S(p(qyp_1)q_1) \cup G_S(q_1)] \\ &= \bigcap_{(p(qy)p_1,q_1)\in A_{xyz}} (F_S \Diamond G_S)(xyz) = (F_S \Diamond G_S)(xyz).\end{aligned}$$

Thus, $(F_S \Diamond G_S)(xyz) \subseteq (F_S \Diamond G_S)(x) \cup (F_S \Diamond G_S)(z)$. Similarly, we can prove that $(F_S \Diamond G_S)(xy) \subseteq (F_S \Diamond G_S)(x) \cup (F_S \Diamond G_S)(y)$. Let $x, y \in S$ be such that $x \leq y$. Then $(F_S \Diamond G_S)(x) \subseteq (F_S \Diamond G_S)(y)$. Infact, if $(p, q) \in A_y$, then $y \leq pq$ and we have $x \leq y \leq pq$ it follows that $(p, q) \in A_x$ and hence $A_y \subseteq A_x$. If $A_x = \emptyset$, then $A_y = \emptyset$ and we have $(F_S \Diamond G_S)(y) = U \supseteq (F_S \Diamond G_S)(x)$. If $A_x \neq \emptyset$, then $A_y \neq \emptyset$ and we have

$$\begin{aligned}(F_S \Diamond G_S)(y) &= \bigcap_{(p,q)\in A_y} \{F_S(p) \cup G_S(q)\} \\ &\supseteq \bigcap_{(p,q)\in A_x} \{F_S(p) \cup G_S(q)\} \\ &= (F_S \Diamond G_S)(x).\end{aligned}$$

Hence in both the cases, we have $(F_S \Diamond G_S)(x) \subseteq (F_S \Diamond G_S)(y)$ for all $x, y \in S$ with $x \leq y$. This completes the proof. □

7. Regular and intra-regular ordered semigroups

In this section, we characterize regular and intra-regular ordered semigroups in terms of uni-soft bi-ideals.

Lemma 7.1. *Let S be an ordered semigroup and (F_S, S) a uni-soft bi-ideal of S over U. Then*

$$(F_S \Diamond F_S, S) \,\widetilde{\supseteq}(F_S, S).$$

Proof. Let (F_S, S) be a uni-soft bi-ideal of S over U and let $a \in S$. If $A_a = \emptyset$, then $(F_S \Diamond F_S)(a) = U \supseteq F_S(a)$. If $A_a \neq \emptyset$, then

$$\begin{aligned}(F_S \Diamond F_S)(a) &= \bigcap_{(p,q)\in A_a} \{F_S(p) \cup F_S(q)\} \\ &\supseteq \bigcap_{(p,q)\in A_a} F_S(pq) \\ &\supseteq \bigcap_{(p,q)\in A_a} F_S(a) \,(\text{since } a \leq pq \Longrightarrow F_S(a) \subseteq F_S(pq)) \\ &= F_S(a).\end{aligned}$$

Therefore, $(F_S \Diamond F_S, S) \,\widetilde{\supseteq}(F_S, S)$. □

7.1. **Lemma.** Let (F_S, S) and (G_S, S) be soft subsets of an ordered semigroup S over U. Then $(F_S \Diamond G_S, S) \,\widetilde{\supseteq}\, (\emptyset_S \Diamond G_S, S)$ $\Big($resp., $(F_S \Diamond G_S, S) \,\widetilde{\supseteq}\, (F_S \Diamond \emptyset_S)\Big)$.

Proof. Straightforward. □

Lemma 7.2. *Let (F_S, S) and (G_S, S) be uni-soft bi-ideals of an ordered semigroup S over U. Then $(F_S \widetilde{\cup} G_S, S)$ is a uni-soft bi-ideal of S over U.*

Proof. Straightforward. □

Theorem 7.1. *Let S be an ordered semigroup. Then the following are equivalent:*

(1) S is both regular and intra-regular.
(2) $(F_S \Diamond F_S, S) = (F_S, S)$ for every uni-soft bi-ideal (F_S, S) over U.
(3) $(F_S \widetilde{\cup} G_S, S) = ((F_S \Diamond G_S) \widetilde{\cup} (G_S \Diamond F_S), S)$ for all uni-soft bi-ideals (F_S, S) and (G_S, S) of S over U.

Proof. (1)$\Longrightarrow$(2). Let (F_S, S) be a uni-soft bi-ideal of S over U and let $a \in S$. Since S is regular and intra-regular, there exist $x, y, z \in S$ such that $a \leq axa \leq axaxa$ and $a \leq ya^2z$. Then

$$a \leq axaxa \leq ax(yaz)xa = (axya)(azxa),$$

and hence $(axya, azxa) \in A_a$. Then

$$\begin{aligned} (F_S \Diamond F_S)(a) &= \bigcap_{(p,q) \in A_a} \{F_S(p) \cup F_S(q)\} \\ &\subseteq F_S(axya) \cup F_S(azxa) \\ &\subseteq \{F_S(a) \cup F_S(a)\} \cup \{F_S(a) \cup F_S(a)\} \\ &= F_S(a). \end{aligned}$$

and hence $(F_S \Diamond F_S, S) \widetilde{\subseteq} (F_S, S)$. On the other hand, by Lemma 7.1, we have $(F_S \Diamond F_S, S) \widetilde{\supseteq} (F_S, S)$. Therefore, $(F_S \Diamond F_S, S) \widetilde{\supseteq} (F_S, S)$.

(2)$\Longrightarrow$(3). Let (F_S, S) and (G_S, S) be uni-soft bi-ideals of S over U. Then $\big(F_S \widetilde{\cup} G_S, S\big)$ is a uni-soft bi-ideal of S over U (Lemma 7.2). By (2), we have

$$\begin{aligned} \big(F_S \widetilde{\cup} G_S, S\big) &= \big(\big(F_S \widetilde{\cup} G_S, S\big) \Diamond \big(F_S \widetilde{\cup} G_S, S\big), S\big) \\ &\widetilde{\supseteq} (F_S \Diamond G_S, S). \end{aligned}$$

Similarly, we can prove that $\big(F_S \widetilde{\cup} G_S, S\big) \,\widetilde{\supseteq}\, (G_S \Diamond F_S, S)$. Therefore,

$$\big(F_S \widetilde{\cup} G_S, S\big) \,\widetilde{\supseteq}\, (F_S \Diamond G_S, S) \,\widetilde{\cup}\, (G_S \Diamond F_S, S).$$

On the other hand, since $(F_S\Diamond G_S, S)$ and $(G_S\Diamond F_S, S)$ are uni-soft bi-ideals of S over U. Again by Lemma 7.2, $(F_S\Diamond G_S, S)\,\widetilde{\cup}\,(G_S\Diamond F_S, S)$ is a uni-soft bi-ideal of S over U. By (2), we have

$$\begin{aligned}
\left((F_S\Diamond G_S, S)\,\widetilde{\cup}\,(G_S\Diamond F_S, S), S\right) &= \begin{pmatrix} \left((F_S\Diamond G_S, S)\,\widetilde{\cup}\,(G_S\Diamond F_S, S)\right)\Diamond \\ \left((F_S\Diamond G_S, S)\,\widetilde{\cup}\,(G_S\Diamond F_S, S)\right), S \end{pmatrix} \\
&\widetilde{\supseteq} \left((F_S\Diamond G_S, S)\Diamond(G_S\Diamond F_S, S), S\right) \\
&= (F_S\Diamond(G_S\Diamond G_S)\Diamond F_S, S) \\
&= (F_S\Diamond G_S\Diamond F_S, S) \ \text{ (as } (G_S\Diamond G_S, S) = (G_S, S) \text{ by (1) above)} \\
&\widetilde{\supseteq} (F_S\Diamond\emptyset_S\Diamond F_S, S) \\
&= (F_S, S) \ \text{ (as } (F_S\Diamond\emptyset_S\Diamond F_S, S) = (F_S, S)\text{)}. \qquad (7.1)
\end{aligned}$$

Similarly, we can prove that $\left((F_S\Diamond G_S, S)\,\widetilde{\cup}\,(G_S\Diamond F_S, S), S\right)\widetilde{\supseteq}(G_S, S)$. Therefore,

$$\left((F_S\Diamond G_S, S)\,\widetilde{\cup}\,(G_S\Diamond F_S, S), S\right)\widetilde{\supseteq}(F_S\widetilde{\cup}G_S, S).$$

Consequently, we have $\left((F_S\Diamond G_S, S)\,\widetilde{\cup}\,(G_S\Diamond F_S, S), S\right) = (F_S\widetilde{\cup}G_S, S)$.

(3)$\Longrightarrow$(1). To prove that S is both regular and intra-regular, it is enough to prove that

$$P\cap Q = (PQ]\cap(QP] \text{ for every bi-ideal } P \text{ and } Q \text{ of } S.$$

Let $b\in P\cap Q$. By Corollary 4.1, (χ_P^c, S) and (χ_Q^c, S) are uni-soft bi-ideals of S over U. By (3), we have

$$\left(\chi_P^c\widetilde{\cup}\chi_Q^c, S\right)(b) = \left(\left(\chi_P^c\Diamond\chi_Q^c\right)\widetilde{\cup}\left(\chi_Q^c\Diamond\chi_P^c\right), S\right)(b).$$

By Lemma 6.2, $\left(\chi_P^c\Diamond\chi_Q^c\right)\widetilde{\cup}\left(\chi_Q^c\Diamond\chi_P^c\right) = \chi^c_{(PQ]\cap(QP]}$ and $\chi_P^c\widetilde{\cup}\chi_Q^c = \chi^c_{P\cap Q}$, hence we have $\chi^c_{(PQ]\cap(QP]}(b) = \chi^c_{P\cap Q}(b) = \emptyset$. Thus, $\chi^c_{(PQ]\cap(QP]}(b) = \emptyset$ and $b\in(PQ]\cap(QP] \Longrightarrow P\cap Q\subseteq(PQ]\cap(QP]$. On the other hand, if $b\in(PQ]\cap(QP]$, then

$$\begin{aligned}
\emptyset &= \left(\chi^c_{(PQ]\cap(QP]}, S\right)(b) \\
&= \left(\chi^c_{(PQ]}\widetilde{\cup}\chi^c_{(QP]}\right)(b) \\
&= \left(\left(\chi_P^c\Diamond\chi_Q^c\right)\widetilde{\cup}\left(\chi_Q^c\Diamond\chi_P^c\right)\right)(b) \\
&= \left(\chi_P^c\widetilde{\cup}\chi_Q^c, S\right)(b) \ \text{ (by (3))} \\
&= \left(\chi^c_{P\cap Q}, S\right)(b).
\end{aligned}$$

Hence, $b\in P\cap Q$ and $(PQ]\cap(QP]\subseteq PQ$. Therefore, $P\cap Q = (PQ]\cap(QP]$ and S is both regular and intra-regular. □

8. Conclusion

In the present paper, we introduced the notion of uni-soft type of bi-ideals of ordered semigroups. Furthermore The notion of a uni-soft bi-ideal is introduced and their related properties is provided. The concept of $\delta-$exclusive set is given and the relations between uni-soft bi-ideals and $\delta-$exclusive set are discussed. The concepts of two types of prime uni-soft bi-ideals of an ordered semigroup S are given. Using the notion of uni-soft bi-ideals, some semilattices of left and right simple semigroups are provided. In our last section the characterizations of both regular and intra-regular ordered semigroups are provided. In our future study of ordered semigroups, we will apply the above new idea to other algebraic structures for more applications.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] U. Acar, F. Koyuncu, and B. Tanay, Soft sets and soft rings, Computers Math. Appl. 59 (11) (2010), 3458–3463.

[2] N. Kehayopulu and M. Tsingelis, Regular ordered semigroups in terms of fuzzy subsets, Inform. Sci. 176 (24) (2006), 3675-3693.

[3] N. Kehayopulu, On completely regular ordered semigroups, Sci. Math. 1 (1) (1998), 27-32.

[4] N. Kehayopulu, On semilattices of simple poe-semigroups, Math. Japon. 38 (2) (1993), 305–318.

[5] N. Kehayopulu, On weakly prime ideals of ordered semigroups, Math. Japon. 35 (6) (1990), 1051–1056.

[6] A. Khan, Y. B. Jun, S. I. A, Shah and R. Khan, Applications of soft sets in ordered semigroups via uni-soft quasi-ideals, J. Intell. Fuzzy Syst. 30 (2016), 97–107.

[7] A. Khan, R. Khan, Y.B. Jun, Uni-soft structure applied to ordered semigroups, Soft Comput. 21 (2017), 1021–1030.

[8] A. Khan, Y. B. Jun and R. Khan, Characterizations of ordered semigroups in terms of int-soft ideals, submitted.

[9] H. Aktas and N. Cağman, Soft sets and soft groups, Inform. Sci. 177 (13) (2007), 2726–2735.

[10] A. O. Atagun and A. Sezgin, Soft substructures of rings, fields and modules, Computers Math. Appl. 61 (3) (2011), 592–601.

[11] F. Feng, Y. B. Jun, and X. Zhao, Soft semirings, Computers Math. Appl. 56 (10) (2008), 2621–2628.

[12] F. Feng, M. I. Ali, and M. Shabir, Soft relations applied to semigroups, Filomat, 27 (7) (2013), 1183–1196.

[13] F. Feng, H. Fujita, Y. B. Jun, and M. Khan, Decomposition of fuzzy soft sets with finite value spaces, Sci. World J. 2014 (2014), Article ID 902687.

[14] F. Feng and Y.M. Li, Soft subsets and soft product operations, Inform. Sci. 232 (2013), 44–57.

[15] Y. B. Jun, S. Z. Song, and G. Muhiuddin, Concave soft sets, critical soft points, and union-soft ideals of ordered semigroups, Sci. World J. 2014 (2104), Article ID 467968.

[16] Y. B. Jun, K. J. Lee, and A. Khan, Soft ordered semigroups, Math. Log. Q. 56 (1) (2010), 42–50.

[17] N. Cagman and S. Enginoglu, Soft set theory and uni-int decision making, Eur. J. Oper. Res. 207 (2) (2010), 848–855.

[18] M. Shabir and A. Khan, Characterizations of ordered semigroups by their fuzzy ideals, Comput. Math. Appl. 59 (2010), 539–549.

[19] X. Ma and J. Zhan, Characterizations of three kinds of hemirings by fuzzy soft h-ideals, J. Intell. Fuzzy Syst. 24 (2013), 535-548.

[20] D. Molodtsov, Soft set theory—first results, Computers Math. Appl. 37 (4-5) (1999), 19–31.

[21] L. A. Zadeh, Fuzzy sets, Inform. Control 8 (1965), 338-353.

[22] J. Zhan, N. Čağman and A. S. Sezer, Applications of soft union sets to hemirings via *SU*-*h*-ideals, J. Intell. Fuzzy Syst. 26 (2014), 1363-1370.