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ALGEBRAIC STRUCTURE OF GRAPH OPERATIONS IN TERMS OF DEGREE SEQUENCES

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Abstract. In this paper, by means of the degree sequences (DS) of graphs and some graph theoretical and combinatorial methods, we determine the algebraic structure of the set of simple connected graphs according to two graph operations, namely join and Corona product. We shall conclude that in the case of join product, the set of graphs forms an abelian monoid whereas in the case of Corona product, this set is not even associative, it only satisfies two conditions, closeness and identity element. We also give a result on distributive law related to these two operations.

1. Introduction

[1,2] Let $G = (V(G), E(G))$ be a simple and connected graph with $\mid V(G) \mid= n$ vertices and $\mid E(G) \mid= m$ edges. Usually we use the notations V and E instead of $V(G)$ and $E(G)$, respectively. Here, by the word "*simple*", we mean that the graphs we consider do not have loops or multiple edges. Similar studies can be

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done for non-simple graphs as well.

For a vertex $v \in V$, we denote the *degree* of v by $d_G(v)$, which is defined as the number of edges of G meeting at v. A vertex with degree one is called a *pendant vertex.*

The notion of degree of a graph provides us an area to study various structural properties of graphs and hence attracts the attention of many graph theorists and also other scientists including chemists. If $d_i,\ 1 \leq i \leq n$, are the degrees of the vertices v_i of a graph G in any order, then the *degree sequence (DS)* of G is the sequence $\{d_1,\ d_2,\ \cdots,\ d_n\}$. Also, in many papers, the DS is taken to be a non-decreasing sequence, whenever possible.

Conversely, a non-negative sequence $\{d_1,\ d_2,\ \cdots,\ d_n\}$ will be called *realizable* if it is the DS of any graph. It is clear from the definition that for a realizable DS, there is at least one graph having this DS. For example, the completely different two graphs in Figure 1 have the same DS.

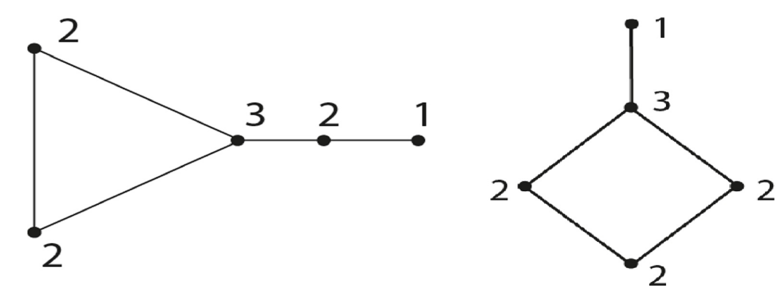

Fig. 1 Graphs with the same DS

For convenience and brevity, we shall denote the DS having repeated degrees with a shorter DS. For example, if the degree d_i of the vertex v_i appears z_i times in the DS of a graph G, then we use $\left\{d_1^{(z_1)},\ d_2^{(z_2)},\ \cdots,\ d_l^{(z_\ell)}\right\}$ instead of $\{d_1,\ d_2,\ \cdots,\ d_n\}$ where $\ell \leq n$. Here the members z_i are called the *frequencies* of the degrees. When $\ell = n$, that is, when all degrees are different, the DS is called *perfect.*

It is an open problem to determine that which DSs are realizable and there are several algorithms to determine that.

As usual, we denote by $P_n,\ C_n,\ S_n,\ K_n,\ T_{r,s}$ and $K_{r,s}$ the *path, cycle, star, complete, complete bipartite and tadpole graphs*, respectively, which are the most used graph examples in literature, see Figure 2.

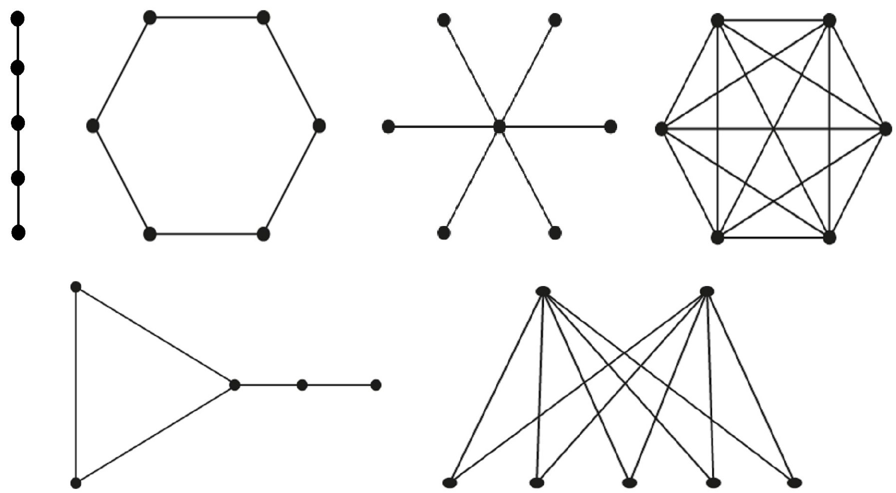

Fig. 2 $P_5,\ C_6,\ S_7,\ K_6,\ T_{3,2},\ K_{2,5}$

The number of vertices and edges of these well-known graph classes are given in Table 1.

TABLE 1. The number of vertices and edges of some graphs

G	$\sharp$ vertices	$\sharp$ edges
P_n	n	n-1
C_n	n	n
S_n	n	n-1
K_n	n	$\binom{n}{2}$
$K_{r,s}$	r+s	rs
$T_{r,s}$	r+s	r+s

Another important reason to study the DSs of graphs is topological indices. A topological index (or a graph invariant) is a fixed invariant number for two isomorphic graphs and gives some information about the graph under consideration. These indices are especially useful in the study of molecular graphs. Some of the topological indices are defined by means of the vertex degrees: first and second Zagreb indices, first and second multiplicative Zagreb indices, atom-bond connectivity index, Narumi-Katayama index, geometric-arithmetic index, harmonic index and sum-connectivity index etc. Therefore to know about the DS of the graph will help to obtain information about, e.g., the chemical properties of the graph. There are many papers on degree based topological indices, see e.g. [2]- [3].

The modern study of DSs started in 1981 by Bollobas, [1]. The same year, Tyshkevich et.al. established a correspondence between DS of a graph and some structural properties of this graph, [8]. In 1987, Tychkevich et.al. written a survey on the same correspondence, [9]. In [10], the authors gave a new version of the Erdös-Gallai theorem on the realizability of a given DS. In 2008, a new criterion on the same problem is given by Triphati and Tyagi, [7]. The same year, Kim et.al. gave a necessary and sufficient condition for the same problem, [5]. Ivanyi et.al, [4], gave an enumeration of DSs of simple graphs. Miller, [6] also gave a criteria for the realizability of given DSs.

There are several *graph operations* used in calculating some chemical invariants of graphs. Amongst these the join, cartesian, Corona product, union, disjunction, and symmetric difference are well-known. In this paper, after recalling two of these operations, join and Corona product, we shall determine the DS of these new product graphs and by means of these calculations, we shall study the algebraic properties of the join and Corona product of two graphs.

Let G_1 and G_2 be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. The *join* $G_1 \vee G_2$ of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$. Thus, for example, $K_p \vee K_q = K_{p,q}$, the complete bipartite graph. We have $|V(G_1 \vee G_2)| = n_1 + n_2$ and $|E(G_1 \vee G_2)| = m_1 + m_2 + n_1 n_2$.

The *Corona product* $G_1 \circ G_2$ of two graphs G_1 and G_2 is defined to be the graph Γ obtained by taking one copy of G_1 (which has n_1 vertices) and n_1 copies of G_2, and then joining the $i-$th vertex of G_1 to every vertex in the $i-$th copy of G_2, for $i = 1,\ 2,\ \cdots,\ n_1$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V(G_1) = \{u_1,\ u_2,\ \cdots,\ u_{n_1}\},\ |E(G_1)| = m_1$ and $V(G_2) = \{v_1,\ v_2,\ \cdots v_{n_2}\}$, $|E(G_2)| = m_2$. Then it follows from the definition of the Corona product that $G_1 \circ G_2$ has $n_1(1 + n_2)$ vertices and $m_1 + n_1 m_2 + n_1 n_2$ edges, where

$$V(G_1 \circ G_2) = \{(u_i, v_j), i = 1, 2, ..., n_1;\ j = 0, 1, 2, ..., n_2\}$$

and

$$\begin{aligned} E(G_1 \circ G_2) = &\{((u_i, v_0), (u_k, v_0)), (u_i, u_k) \in E(G_1)\} \\ &\cup \{((u_i, v_j), (u_i, v_\ell)), (v_j, v_\ell) \in E(G_2),\ i = 1, 2, ..., n_1\} \\ &\cup \{((u_i, v_0), (u_i, v_\ell)), \ell = 1, 2, ..., n_2,\ i = 1, 2, ..., n_1\}. \end{aligned}$$

It is clear that if G_1 is connected, then $G_1 \circ G_2$ is connected, and in general $G_1 \circ G_2$ is not isomorphic to $G_2 \circ G_1$.

2. Algebraic Properties of Join

In this section, we deal with some algebraic properties of the join of two graphs. We shall try to determine the abstract algebraic structure of this new graph and also give the DS of the join graph $G_1 \vee G_2$ of two graphs G_1 and G_2 where G_1 and G_2 are choosen from $P_n,\ C_n,\ S_n,\ K_n,\ T_{r,s}$ and $K_{r,s}$. In particular, in the case of join operation, the set of graphs forms an abelian monoid whereas in the case of Corona product, the set of graphs is not even associative, it only satisfies two conditions, closedness and identity element. It is clear to see that there are no zero divisors for both products.

Theorem 2.1. *The DSs of all possible joins of the path, cycle, star, complete, tadpole and complete bipartite graphs are given in Table 2.*

Proof. We make the proof only for $P_r \vee P_s$ and $S_r \vee C_s$. Let $P_r = \left\{1^{(2)},\ 2^{(r-2)}\right\}$ and $P_s = \left\{1^{(2)},\ 2^{(s-2)}\right\}$. To visualize the situation, see Figure 3.

There are two types of vertices in each of P_r and P_s. Therefore there are $2+2=4$ types of vertices in $P_r \vee P_s$. The first type is the two end vertices of P_r (red ones) which are connected with the next green vertex in P_r and s vertices in P_s. Each of these two vertices add $s+1$ to the DS of $P_r \vee P_s$. Therefore they add $(s+1)^{(2)}$.

The second type of vertices are the mid ones in P_r (green ones) each of which is connected to two neighboring vertices in P_r and s vertices in P_s. Each of these $r-2$ vertices adds $s+2$ to the DS of $P_r \vee P_s$. Therefore $(s+2)^{(r-2)}$ is added.

The third type of vertices are the two end vertices of P_s (black ones) each of which is connected to one vertex in P_s and r vertices in P_r. They add $(r+1)^2$ to the DS of $P_r \vee P_s$. The fourth and last type of vertices are the mid-vertices (blue ones) in P_s. Their number is $s-2$ and each of which similarly adds $r+2$ to the DS of $P_r \vee P_s$. So their contribution is $(r+2)^{(s-2)}$. Therefore the required DS is

$$P_r \vee P_s = \{(s+1)^{(2)}, (s+2)^{(r-2)}, (r+1)^{(2)}, (r+2)^{(s-2)}\}.$$

Now we recall that $S_r = \{1^{(r-1)}, r-1\}$ and $C_s = \left\{2^{(s)}\right\}$. In Figure 4, the join of these two graphs is drawn for $r = s = 5$.

TABLE 2. The DSs of the join of some well-known graph types

G_1	G_2	$G_1 \vee G_2$
P_r	P_s	$\{(s+1)^{(2)},\ (s+2)^{(r-2)},\ (r+1)^{(2)},\ (r+2)^{(s-2)}\}$
P_r	C_s	$\{(s+1)^{(2)}, (s+2)^{(r-2)}, (r+2)^{(s)}\}$
P_r	S_s	$\{(s+1)^{(2)}, (s+2)^{(r-2)}, (r+1)^{(s-1)}, r+s-1\}$
P_r	K_s	$\{(s+1)^{(2)}, (s+2)^{(r-2)}, (r+s-1)^{(s)}\}$
P_r	$T_{s,t}$	$\{(s+t+1)^{(2)}, (s+t+2)^{(r-2)}, r+1, (r+2)^{(s+t-2)}, r+3\}$
P_r	$K_{s,t}$	$\{(s+t+1)^{(2)}, (s+t+2)^{(r-2)}, (r+s)^{(t)}, (r+t)^{(s)}\}$
C_r	P_s	$\{(s+2)^{(r)}, (r+1)^{(2)}, (r+2)^{(s-2)}\}$
C_r	C_s	$\{(s+2)^{(r)}, (r+2)^{(s)}\}$
C_r	S_s	$\{(s+2)^{(r)}, r+s-1, (r+1)^{(s-1)}\}$
C_r	K_s	$\{(s+2)^{(r)}, (r+s-1)^{(s)}\}$
C_r	$T_{s,t}$	$\{(s+t+2)^{(r)}, r+1, (r+2)^{(s+t-2)}, r+3\}$
C_r	$K_{s,t}$	$\{(s+t+2)^{(r)}, (r+s)^{(t)}, (r+t)^{(s)}\}$
S_r	P_s	$\{(s+1)^{(r-1)}, r+s-1, (r+1)^{(2)}, (r+2)^{(s-2)}\}$
S_r	C_s	$\{(s+1)^{(r-1)}, r+s-1, (r+2)^{(s)}\}$
S_r	S_s	$\{(s+1)^{(r-1)}, (r+s-1)^{(2)}, (r+1)^{(s-1)}\}$
S_r	K_s	$\{(s+1)^{(r-1)}, s+r-1, (r+s-1)^{(s)}\}$
S_r	$T_{s,t}$	$\{(s+t+1)^{(r-1)}, r+s+t-1, r+1, (r+2)^{(s+t-2)}, r+3\}$
S_r	$K_{s,t}$	$\{(s+t+1)^{(r-1)}, r+s+t-1, (r+s)^{(t)}, (r+t)^{(s)}\}$
K_r	P_s	$\{(r+s-1)^{(r)}, (r+1)^{(2)}, (r+2)^{(s-2)}\}$
K_r	C_s	$\{(r+s-1)^{(r)}, (r+2)^{(s)}\}$
K_r	S_s	$\{(r+s-1)^{(r)}, (r+1)^{(s-1)}, r+s-1\}$
K_r	K_s	$\{(r+s-1)^{(r+s)}\}$
K_r	$T_{s,t}$	$\{(r+s+t-1)^{(r)}, r+1, (r+2)^{(s+t-2)}, r+3\}$
K_r	$K_{s,t}$	$\{(r+s+t-1)^{(r)}, (r+s)^{(t)}, (r+t)^{(s)}\}$
$T_{r,s}$	P_t	$\{(t+2)^{(r+s-2)}, t+1, t+3, (r+s+1)^{(2)}, (r+s+2)^{(t-2)}\}$
$T_{r,s}$	C_t	$\{(t+2)^{(r+s-2)}, t+1, t+3, (r+s+2)^{(t)}\}$
$T_{r,s}$	S_t	$\{(t+2)^{(r+s-2)}, t+1, t+3, (r+s+1)^{(t-1)}, r+s+t-1\}$
$T_{r,s}$	K_t	$\{(t+2)^{(r+s-2)}, t+1, t+3, (r+s+t-1)^{(t)}\}$
$T_{r,s}$	$T_{t,m}$	$\left\{\begin{array}{c} (t+m+2)^{(r+s-2)}, t+m+1, t+m+3, \\ (r+s+2)^{(t+m-2)}, r+s+1, r+s+3 \end{array}\right\}$
$T_{r,s}$	$K_{t,m}$	$\{(t+m+2)^{(r+s-2)}, t+m+1, t+m+3, (r+s+t)^{(m)}, (r+s+m)^{(t)}\}$
$K_{r,s}$	P_t	$\{(r+t)^{(s)}, (s+t)^{(r)}, (r+s+1)^{(2)}, (r+s+2)^{(t-2)}\}$
$K_{r,s}$	C_t	$\{(r+t)^{(s)}, (s+t)^{(r)}, (r+s+2)^{(t)}\}$
$K_{r,s}$	S_t	$\{(r+t)^{(s)}, (s+t)^{(r)}, (r+s+1)^{(t-1)}, r+s+t-1\}$
$K_{r,s}$	K_t	$\{(r+t)^{(s)}, (s+t)^{(r)}, (r+s+t-1)^{(t)}\}$
$K_{r,s}$	$T_{t,m}$	$\{(r+t+m)^{(s)}, (s+t+m)^{(r)}, (r+s+2)^{(t+m-2)}, r+s+1, r+s+3\}$
$K_{r,s}$	$K_{t,m}$	$\{(r+t+m)^{(s)}, (s+t+m)^{(r)}, (r+s+m)^{(t)}, (r+s+t)^{(m)}\}$

There are two types of vertices in star graph S_r and only one type in cycle C_s. Therefore there are $2+1=3$ types of vertices in $S_r \vee C_s$. The first type of those is the end vertices (black ones) of the star graph S_r. Each of these is connected to the central vertex (blue coloured) by an edge in S_r and to all s vertices in C_s. Therefore each of these vertices adds $s+1$, and in total, $(s+1)^{(r-1)}$ is added to the DS. The second type is the unique central vertex in S_r which is connected to $r-1$ end vertices in S_r and also to all of s vertices in C_s. It adds a total of $r+s-1$ to the DS. The third and final type of vertices is the s vertices in C_s (green ones). Each of them adds $r+2$ to the DS. Therefore a total of $(r+2)^{(s)}$ is added. Hence the DS of the required join graph is $S_r \vee C_s = \{(s+1)^{(r-1)},\ r+s-1,\ (r+2)^{(s)}\}$. □

Now we can study the algebraic structure of the set of graphs according to join operation:

Theorem 2.2. *Let G be the set of all simple connected graphs. Then G is an abelian monoid with the join operation.*

Proof. Let us have three graphs $G_1 = \{\alpha_{11}^{(\beta_{11})}, \cdots, \alpha_{1\ell}^{(\beta_{1\ell})}\}$, $G_2 = \{\alpha_{21}^{(\beta_{21})}, \cdots, \alpha_{2m}^{(\beta_{2m})}\}$ and $G_3 = \{\alpha_{31}^{(\beta_{31})}, \cdots, \alpha_{3n}^{(\beta_{3n})}\}$ with the number of vertices n_1, n_2, n_3, respectively. We shall show that G with the join operation is closed, associative, commutative, has identity element but no inverse elements:

First of all, the join of two simple connected graphs, by definition, is another simple connected graph, so G is closed. For associativeness, note that

$$\begin{aligned}(G_1 \vee G_2) \vee G_3 &= \{(n_2+\alpha_{11})^{(\beta_{11})},\ \cdots, (n_2+\alpha_{1\ell})^{(\beta_{1\ell})},\\ &\qquad (n_1+\alpha_{21})^{(\beta_{21})},\ \cdots, (n_1+\alpha_{2m})^{(\beta_{2m})}\} \vee \{\alpha_{31}^{(\beta_{31})}, \alpha_{32}^{(\beta_{32})}, \alpha_{33}^{(\beta_{33})}\}\\ &= \{(n_2+n_3+\alpha_{11})^{(\beta_{11})},\ \cdots, (n_2+n_3+\alpha_{1\ell})^{(\beta_{1\ell})},\\ &\qquad (n_1+n_3+\alpha_{21})^{(\beta_{21})},\ \cdots, (n_1+n_3+\alpha_{2m})^{(\beta_{2m})},\\ &\qquad (n_1+n_2+\alpha_{31})^{(\beta_{31})},\ \cdots, (n_1+n_2+\alpha_{3n})^{(\beta_{3n})}\}\end{aligned}$$

and

$$\begin{aligned}G_1 \vee (G_2 \vee G_3) &= \{\alpha_{11}^{(\beta_{11})},\ \cdots, \alpha_{1\ell}^{(\beta_{1\ell})}\} \vee \{(n_3+\alpha_{21})^{(\beta_{21})},\ \cdots, (n_3+\alpha_{2m})^{(\beta_{2m})},\\ &\qquad (n_2+\alpha_{31})^{(\beta_{31})},\ \cdots,\ (n_2+\alpha_{3n})^{(\beta_{3n})}\}\\ &= \{(n_2+n_3+\alpha_{11})^{(\beta_{11})},\ \cdots, (n_2+n_3+\alpha_{1\ell})^{(\beta_{1\ell})},\\ &\qquad (n_1+n_3+\alpha_{21})^{(\beta_{21})},\ \cdots, (n_1+n_3+\alpha_{2m})^{(\beta_{2m})},\\ &\qquad (n_1+n_2+\alpha_{31})^{(\beta_{31})},\ \cdots, (n_1+n_2+\alpha_{3n})^{(\beta_{3n})}\}\end{aligned}$$

$$= \{(n_2+n_3+\alpha_{11})^{(\beta_{11})},\ \cdots,(n_2+n_3+\alpha_{1\ell})^{(\beta_{1\ell})},$$
$$(n_1+n_3+\alpha_{21})^{(\beta_{21})},\ \cdots,(n_1+n_3+\alpha_{2m})^{(\beta_{2m})},$$
$$(n_1+n_2+\alpha_{31})^{(\beta_{31})},\ \cdots,(n_1+n_2+\alpha_{3n})^{(\beta_{3n})}\}$$

therefore G is associative. As

$$\begin{aligned} G_1 \vee G_2 &= \{(n_2+\alpha_{11})^{(\beta_{11})},\ \cdots,(n_2+\alpha_{1\ell})^{(\beta_{1\ell})},(n_1+\alpha_{21})^{(\beta_{21})},\ \cdots,(n_1+\alpha_{2m})^{(\beta_{2m})}\} \\ &= \{(n_1+\alpha_{21})^{(\beta_{21})},\ \cdots,(n_1+\alpha_{2m})^{(\beta_{2m})},(n_2+\alpha_{11})^{(\beta_{11})},\ \cdots,(n_2+\alpha_{1\ell})^{(\beta_{1\ell})}\} \\ &= G_2 \vee G_1, \end{aligned}$$

the operation is commutative. Therefore, to find the identity element, one needs to find a graph Z with the property that $G_1 \vee Z = G_1$. Let $Z = \{a_1^{b_1},\ \cdots,a_k^{b_k}\}$ and let the graph Z have c vertices. Then $\{\alpha_{11}^{(\beta_{11})},\ \cdots,\alpha_{1\ell}^{(\beta_{1\ell})}\} \vee \{a_1^{(b_1)},\ \cdots,a_k^{(b_k)}\} = \{\alpha_{11}^{(\beta_{11})},\ \cdots,\alpha_{1\ell}^{(\beta_{1\ell})}\}$ implies that

$$\{(c+\alpha_{11})^{(\beta_{11})},\ \cdots,(c+\alpha_{1\ell})^{(\beta_{1\ell})},(n_1+a_1)^{(b_1)},\ \cdots,(n_1+a_k)^{(b_k)}\} = \{\alpha_{11}^{(\beta_{11})},\ \cdots,\alpha_{1\ell}^{(\beta_{1\ell})}\}$$

and this is only possible when $c = 0$. For $c = 0$, we have

$$\{\alpha_{11}^{(\beta_{11})},\ \cdots,\alpha_{1\ell}^{(\beta_{1\ell})},\ (n_1+a_1)^{(b_1)},\ \ \cdots,(n_1+a_k)^{(b_k)}\} = \{\alpha_{11}^{(\beta_{11})},\ \cdots,\alpha_{1\ell}^{(\beta_{1\ell})}\}.$$

To have this equality, we must have

$$(n_1+a_1)^{(b_1)} = 0,\ \cdots,(n_1+a_k)^{(b_k)} = 0.$$

Hence we must have no terms $(n_1+a_1)^{(b_1)},\ \cdots,(n_1+a_k)^{(b_k)}$ in the DS of the identity element Z. This implies that $b_1 = \cdots = b_k = 0$. Therefore we can symbolically take $Z = \{1^{(0)}\}$ as the identity element.

Finally, let the inverse element of the graph $G_1 = \{\alpha_{11}^{(\beta_{11})},\ \cdots,\alpha_{1\ell}^{(\beta_{1\ell})}\}$ be denoted by $\{c_1^{(d_1)},...,c_k^{(d_k)}\}$. Let the number of vertices of the inverse element be e. Then

$$G_1 \vee \{c_1^{(d_1)},\ \cdots,c_k^{(d_k)}\} = Z = \{1^{(0)}\}$$

implies that

$$\{\alpha_{11}^{(\beta_{11})},\ \cdots,\alpha_{1\ell}^{(\beta_{1\ell})}\} \vee \{c_1^{(d_1)},\ \cdots,c_k^{(d_k)}\} = \{1^{(0)}\}$$

and therefore

$$\{(e+\alpha_{11})^{(\beta_{11})},\ \cdots,(e+\alpha_{1\ell})^{(\beta_{1\ell})},(n_1+c_1)^{(d_1)},\ \cdots,(n_1+c_k)^{(d_k)}\} = \{1^{(0)}\}.$$

As there is no solution to that equation, we conclude that there is no inverse element for the join operation. Therefore the result follows. $\square$

3. Algebraic Properties of Corona Product

Theorem 3.1. *The DSs of all possible Corona products of the path, cycle, star, complete, tadpole and complete bipartite graphs are given in Table 3.*

Theorem 3.2. *Let G be the set of all simple connected graphs. Then G with Corona product operation is closed with identity.*

Proof. First, by the definition of the operation, G is closed. Secondly, for associativeness, we should note that

$$\begin{aligned}(G_1 \circ G_2) \circ G_3 = \{&(n_2+\alpha_{11})^{(\beta_{11})}, \cdots, (n_2+\alpha_{1\ell})^{(\beta_{1\ell})},\\ &(1+\alpha_{21})^{(n_1\beta_{21})}, \cdots, (1+\alpha_{2m})^{(n_1\beta_{2m})}\} \circ \{\alpha_{31}^{(\beta_{31})}, \cdots, \alpha_{3n}^{(\beta_{3n})}\}\\ = \{&(n_2+n_3+\alpha_{11})^{(\beta_{11})}, \cdots, (n_2+n_3+\alpha_{1\ell})^{(\beta_{1\ell})},\\ &(1+n_3+\alpha_{21})^{(n_1\beta_{21})}, \cdots, (1+n_3+\alpha_{2m})^{(n_1\beta_{2m})},\\ &(1+\alpha_{31})^{(n_1(n_1+n_2)\beta_{31})}, \cdots, (1+\alpha_{3n})^{(n_1(n_1+n_2)\beta_{3n})}\}\end{aligned}$$

and

$$\begin{aligned}G_1 \circ (G_2 \circ G_3) = &\left\{\alpha_{11}^{(\beta_{11})}, \cdots, \alpha_{1\ell}^{(\beta_{1\ell})}\right\} \circ \{(n_3+\alpha_{21})^{(\beta_{21})}, \cdots, (n_3+\alpha_{2m})^{\beta_{2m}},\\ &(1+\alpha_{31})^{(n_2\beta_{31})}, \cdots, (1+\alpha_{3n})^{(n_2\beta_{3n})}\}\\ = \{&(n_2+n_2n_3+\alpha_{11})^{(\beta_{11})}, \cdots, (n_2+n_2n_3+\alpha_{1\ell})^{(\beta_{1\ell})},\\ &(1+n_3+\alpha_{21})^{(n_1\beta_{21})}, \cdots, (1+n_3+\alpha_{2m})^{(n_1\beta_{2m})},\\ &(2+\alpha_{31})^{(n_1n_2\beta_{31})}, \cdots, (2+\alpha_{3n})^{(n_1n_2\beta_{3n})}\}.\end{aligned}$$

That is, G is not associative.

For the identity element, we should find a graph Z such that $G_1 \circ Z = G_1$. Let $Z = \{a_1^{(b_1)}, \cdots, a_k^{(b_k)}\}$ and let the number of vertices of Z be c. We have

$$\{\alpha_{11}^{(\beta_{11})}, \cdots, \alpha_{1\ell}^{(\beta_{1\ell})}\} \circ \{a_1^{(b_1)}, \cdots, a_k^{(b_k)}\} = \{\alpha_{11}^{(\beta_{11})}, \cdots, \alpha_{1\ell}^{(\beta_{1\ell})}\}$$

and hence, we get

$$\{(c+\alpha_{11})^{(\beta_{11})}, \cdots, (c+\alpha_{1\ell})^{(\beta_{1\ell})}, (1+a_1)^{(n_1b_1)}, \cdots, (1+a_k)^{(n_1b_k)}\} = \{\alpha_{11}^{(\beta_{11})}, \cdots, \alpha_{1\ell}^{(\beta_{1\ell})}\}.$$

For this equation to have a solution, we must have $c = 0$ and also $b_1, \cdots, b_k = 0$. For the sake of brevity, if we take 1 instead of $1 + a_1, \cdots, 1 + a_k$, we conclude that $Z = \{1^{(0)}\}$ is the required identity element. Let

Table 3. The DSs of the Corona product of well-known graph types

G_1	G_2	$G_1 \circ G_2$
G_1	G_2	$G_1 \circ G_2$
P_r	P_s	$\{2^{(2r)}, 3^{(r(s-2))}, (s+1)^{(2)}, (s+2)^{(r-2)}\}$
P_r	C_s	$\{3^{(rs)}, (s+1)^{(2)}, (s+2)^{(r-2)}\}$
P_r	S_s	$\{2^{(r(s-1))}, s^{(r)}, (s+1)^{(2)}, (s+2)^{(r-2)}\}$
P_r	K_s	$\{s^{(rs)}, (s+1)^{(2)}, (s+2)^{(r-2)}\}$
P_r	$T_{s,t}$	$\{2^{(r)}, 3^{(r(s+t-2))}, 4^{(r)}, (s+t+1)^{(2)}, (s+t+2)^{(r-2)}\}$
P_r	$K_{s,t}$	$\{(s+1)^{(rt)}, (t+1)^{(rs)}, (s+t+1)^{(2)}, (s+t+2)^{(r-2)}\}$
C_r	P_s	$\{2^{(2r)}, 3^{(r(s-2))}, (s+2)^{(r)}\}$
C_r	C_s	$\{3^{(rs)}, (s+2)^{(r)}\}$
C_r	S_s	$\{2^{(r(s-1))}, s^{(r)}, (s+2)^{(r)}\}$
C_r	K_s	$\{s^{(rs)}, (s+2)^{(r)}\}$
C_r	$T_{s,t}$	$\{2^{(r)}, 3^{(r(s+t-2))}, 4^{(r)}, (s+t+2)^{(r)}\}$
C_r	$K_{s,t}$	$\{(s+1)^{(rt)}, (t+1)^{(rs)}, (s+t+2)^{(r)}\}$
S_r	P_s	$\{2^{(2r)}, 3^{(r(s-2))}, (s+1)^{(r-1)}, r+s-1\}$
S_r	C_s	$\{3^{(rs)}, (s+1)^{(r-1)}, r+s-1\}$
S_r	S_s	$\{2^{(r(s-1))}, s^{(r)}, (s+1)^{(r-1)}, r+s-1\}$
S_r	K_s	$\{s^{(rs)}, (s+1)^{(r-1)}, r+s-1\}$
S_r	$T_{s,t}$	$\{2^{(r)}, 3^{(r(s+t-2))}, 4^{(r)}, (s+t+1)^{(r-1)}, r+s+t-1\}$
S_r	$K_{s,t}$	$\{(s+1)^{(rt)}, (t+1)^{(rs)}, (s+t+1)^{(r-1)}, r+s+t-1\}$
K_r	P_s	$\{2^{(2r)}, 3^{(r(s-2))}, (r+s-1)^{(r)}\}$
K_r	C_s	$\{3^{(rs)}, (r+s-1)^{(r)}\}$
K_r	S_s	$\{2^{(r(s-1))}, s^{(r)}, (r+s-1)^{(r)}\}$
K_r	K_s	$\{s^{(rs)}, (r+s-1)^{(r)}\}$
K_r	$T_{s,t}$	$\{2^{(r)}, 3^{(r(s+t-2))}, 4^{(r)}, (r+s+t-1)^{(r)}\}$
K_r	$K_{s,t}$	$\{(s+1)^{(rt)}, (t+1)^{(rs)}, (r+s+t-1)^{(r)}\}$
$T_{r,s}$	P_t	$\{2^{(2(r+s))}, 3^{((r+s)(t-2))}, t+1, (t+2)^{(r+s-2)}, t+3\}$
$T_{r,s}$	C_t	$\{3^{((r+s)t)}, t+1, (t+2)^{(r+s-2)}, t+3\}$
$T_{r,s}$	S_t	$\{2^{((r+s)(t-1))}, t^{(r+s)}, t+1, (t+2)^{(r+s-2)}, t+3\}$
$T_{r,s}$	K_t	$\{t^{((r+s)t)}, t+1, (t+2)^{(r+s-2)}, t+3\}$
$T_{r,s}$	$T_{t,m}$	$\left\{\begin{array}{c} 2^{(r+s)}, 3^{((r+s)(t+m-2))}, 4^{(r+s)}, t+m+1, \\ (t+m+2)^{(r+s-2)}, t+m+3 \end{array}\right\}$
$T_{r,s}$	$K_{t,m}$	$\left\{\begin{array}{c} (t+1)^{((r+s)m)}, (m+1)^{((r+s)t)}, t+m+1, \\ (t+m+2)^{(r+s-2)}, t+m+3 \end{array}\right\}$
$K_{r,s}$	P_t	$\{2^{(2(r+s))}, 3^{((r+s)(t-2))}, (r+t)^{(s)}, (s+t)^{(r)}\}$
$K_{r,s}$	C_t	$\{3^{((r+s)t)}, (r+t)^{(s)}, (s+t)^{(r)}\}$
$K_{r,s}$	S_t	$\{2^{((r+s)(t-1))}, t^{(r+s)}, (r+t)^{(s)}, (s+t)^{(r)}\}$
$K_{r,s}$	K_t	$\{t^{((r+s)t)}, (r+t)^{(s)}, (s+t)^{(r)}\}$
$K_{r,s}$	$T_{t,m}$	$\{2^{(r+s)}, 3^{((r+s)(t+m-2))}, 4^{(r+s)}, (r+t+m)^{(s)}, (s+t+m)^{(r)}\}$
$K_{r,s}$	$K_{t,m}$	$\{(t+1)^{((r+s)m)}, (m+1)^{((r+s)t)}, (r+t+m)^{(s)}, (s+t+m)^{(r)}\}$

$T = \{c_1^{(d_1)},\ \cdots,\ c_k^{(d_k)}\}$ be the inverse element of a graph $G_1 = \{\alpha_{11}^{(\beta_{11})},\ \cdots,\ \alpha_{1\ell}^{(\beta_{1\ell})}\}$. Then they must satisfy the equation $G_1 \circ \{c_1^{(d_1)}, \cdots, c_k^{(d_k)}\} = Z = \{1^{(0)}\}$. If the number of vertices of T is e, then we have

$$\{\alpha_{11}^{(\beta_{11})},\ \cdots, \alpha_{1\ell}^{(\beta_{1\ell})}\} \circ \{c_1^{(d_1)},\ \cdots, c_k^{(d_k)}\} = \{1^{(0)}\}.$$

In this case, this equation cannot be hold, implying that there is no inverse element in G.

Finally, as

$$\begin{aligned}\{\alpha_{11}^{(\beta_{11})}, \cdots, \alpha_{1\ell}^{(\beta_{1\ell})}\} \circ \{\alpha_{21}^{(\beta_{21})}, \cdots, \alpha_{2m}^{\beta_{2m}}\} = \{&(n_2+\alpha_{11})^{(\beta_{11})}, \cdots, (n_2+\alpha_{1\ell})^{(\beta_{1\ell})},\\ &(1+\alpha_{21})^{(n_1\beta_{21})}, \cdots, (1+\alpha_{2m})^{(n_1\beta_{2m})}\}\end{aligned}$$

and

$$\begin{aligned}\{\alpha_{21}^{(\beta_{21})}, \cdots, \alpha_{2m}^{(\beta_{2m})}\} \circ \{\alpha_{11}^{(\beta_{11})}, \cdots, \alpha_{1\ell}^{(\beta_{1\ell})}\} = \{&(n_1+\alpha_{21})^{(\beta_{21})}, \cdots, (n_1+\alpha_{2m})^{(\beta_{2m})},\\ &(1+\alpha_{11})^{(n_2\beta_{11})}, \cdots, (1+\alpha_{1\ell})^{(n_2\beta_{1\ell})}\},\end{aligned}$$

G can only be commutative when $G_1 = G_2$. In general, G is not commutative. □

Finally, we check whether the distributive law holds when we have join and Corona in place of $\cdot$ and $+$, or vice versa:

Theorem 3.3. *Neither join nor Corona operation is not distributive on each other. That is*

(i) $G_1 \vee (G_2 \circ G_3) \neq (G_1 \vee G_2) \circ (G_1 \vee G_3)$,

(ii) $G_1 \circ (G_2 \vee G_3) \neq (G_1 \circ G_2) \vee (G_1 \circ G_3)$.

Proof. Both claims follow after the following calculations:

$$\begin{aligned}\textbf{(i)}\ G_1 \vee (G_2 \circ G_3) &= \{\alpha_{11}^{(\beta_{11})}, \cdots, \alpha_{1\ell}^{(\beta_{1\ell})}\} \vee \{(n_3+\alpha_{21})^{(\beta_{21})}, \cdots, (n_3+\alpha_{2m})^{(\beta_{2m})},\\ &\quad (1+\alpha_{31})^{(n_2\beta_{31})}, \cdots, (1+\alpha_{3n})^{(n_2\beta_{3n})}\}\\ &= \{(n_2+n_2n_3+\alpha_{11})^{(\beta_{11})}, \cdots, (n_2+n_2n_3+\alpha_{1\ell})^{(\beta_{1\ell})},\\ &\quad (n_1+n_3+\alpha_{21})^{(\beta_{21})}, \cdots, (n_1+n_3+\alpha_{2m})^{(\beta_{2m})},\\ &\quad (1+n_1+\alpha_{31})^{(n_2\beta_{31})}, \cdots, (1+n_1+\alpha_{3n})^{(n_2\beta_{3n})}\}\end{aligned}$$

and

$$
\begin{aligned}
(G_1 \vee G_2) \circ (G_1 \vee G_3) = \{&(n_2+\alpha_{11})^{(\beta_{11})}, \cdots, (n_2+\alpha_{1\ell})^{(\beta_{1\ell})}, (n_1+\alpha_{21})^{(\beta_{21})}, \cdots, \\
&(n_1+\alpha_{2m})^{(\beta_{2m})}\} \circ \{(n_3+\alpha_{11})^{(\beta_{11})}, \cdots, (n_3+\alpha_{1\ell})^{(\beta_{1\ell})}, \\
&(n_1+\alpha_{31})^{(\beta_{31})}, \ \cdots, \ (n_1+\alpha_{3n})^{(\beta_{3n})}\} \\
= \{&(n_1+n_2+n_3+\alpha_{11})^{(\beta_{11})}, \ \cdots, \ (n_1+n_2+n_3+\alpha_{1\ell})^{(\beta_{1\ell})}, \\
&(2n_1+n_3+\alpha_{21})^{(\beta_{21})}, \ \cdots, \ (2n_1+n_3+\alpha_{2m})^{(\beta_{2m})}, \\
&(1+n_3+\alpha_{11})^{(n_1+n_2)\beta_{11}}, \ \cdots, \ (1+n_3+\alpha_{1\ell})^{((n_1+n_2)\beta_{1\ell})}, \\
&(1+n_1+\alpha_{31})^{((n_1+n_2)\beta_{31})}, \ \cdots, \ (1+n_1+\alpha_{3n})^{((n_1+n_2)\beta_{3n})}\}.
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{ii}) G_1 \circ (G_2 \vee G_3) = \{&\alpha_{11}^{(\beta_{11})}, \ \cdots, \ \alpha_{1\ell}^{(\beta_{1\ell})}\} \circ \{(n_3+\alpha_{21})^{(\beta_{21})}, \ \cdots, \ (n_3+\alpha_{2m})^{(\beta_{2m})}, \\
&(n_2+\alpha_{31})^{(\beta_{31})}, \ \cdots, \ (n_2+\alpha_{3n})^{(\beta_{3n})}\} \\
= \{&(n_2+n_3+\alpha_{11})^{(\beta_{11})}, \ \cdots, \ (n_2+n_3+\alpha_{1\ell})^{(\beta_{1\ell})}, \\
&(1+n_3+\alpha_{21})^{(n_1\beta_{21})}, \ \cdots, \ (1+n_3+\alpha_{2m})^{(n_1\beta_{2m})}, \\
&(1+n_2+\alpha_{31})^{(n_1\beta_{31})}, \ \cdots, \ (1+n_2+\alpha_{3n})^{(n_1\beta_{3n})}\}
\end{aligned}
$$

and also

$$
\begin{aligned}
(G_1 \circ G_2) \vee (G_1 \circ G_3) = \{&(n_2+\alpha_{11})^{(\beta_{11})}, \cdots, (n_2+\alpha_{1\ell})^{(\beta_{1\ell})}, (1+\alpha_{21})^{(n_1\beta_{21})}, \cdots, \\
&(1+\alpha_{2m})^{(n_1\beta_{2m})}\} \vee \{(n_3+\alpha_{11})^{(\beta_{11})}, \cdots, (n_3+\alpha_{1\ell})^{(\beta_{1\ell})}, \\
&(1+\alpha_{31})^{(n_1\beta_{31})}, \cdots, (1+\alpha_{3n})^{(n_1\beta_{3n})}\} \\
= \{&(n_1+n_1n_3+n_2+\alpha_{11})^{\beta_{11}}, \cdots, (n_1+n_1n_3+n_2+\alpha_{1\ell})^{(\beta_{1\ell})}, \\
&(1+n_1+n_1n_3+\alpha_{21})^{(n_1\beta_{21})}, \cdots, (1+n_1+n_1n_3+\alpha_{2m})^{(n_1\beta_{2m})}, \\
&(n_1+n_1n_2+n_3+\alpha_{11})^{\beta_{11}}, \cdots, (n_1+n_1n_2+n_3+\alpha_{1\ell})^{(\beta_{1\ell})}, \\
&(1+n_1+n_1n_2+\alpha_{31})^{(n_1\beta_{31})}, \cdots, (1+n_1+n_1n_2+\alpha_{3n})^{(n_1\beta_{3n})}\}.
\end{aligned}
$$

□

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