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AN APPROXIMATION OF FUZZY NUMBERS BASED ON POLYNOMIAL FORM FUZZY NUMBERS

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Abstract. In this paper, we approximate an arbitrary fuzzy number by a polynomial fuzzy number through minimizing the distance between them. Throughout this work, we used a distance that is a meter on the set of all fuzzy numbers with continuous left and right spread functions. To support our claims analytically, we have proven some theorems and given supplementary corollaries.

1. Introduction

Comparison of fuzzy numbers is an indispensable part of most systems using such numbers. To this end, many researchers active in the Fuzzy Theory domain have tried to make fuzzy numbers comparable. Some authors have approximated a fuzzy number by a single crisp number. This method which is called ranking suffers from loss of some useful information.

Some authors such as [8] convert a given fuzzy number into an interval and solve an interval arithmetic problem instead of a more complicated fuzzy computation. However, the fuzzy central concept fades here. Finding the nearest triangular or trapezoidal fuzzy number associated to an arbitrary given fuzzy number is another method on which some authors such as [2], [4], [6], [10] and [11] have concentrated. However, this method fails to guarantee the same modal value (or interval). Also, some authors such as [12] and [13] have made a considerable contribution to the coefficients of polynomial, the concept that we have used in this research.

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In this paper, we propose two methods for approximating a given arbitrary fuzzy number with a polynomial fuzzy number to a great degree of accuracy. The first method splits the approximation problem into two sub-problems and solves them separately whereas the second one solves the problem in a general form.

2. Basic Concepts

In this section, the basic concepts used throughout the paper are given. Let $\mathcal{F}(\mathbb{R})$ be the set of all fuzzy numbers (the set of all normal and convex fuzzy sets) on the real line.

Definition 2.1. *A generalized LR fuzzy number $\tilde{u}$ with the membership function $\mu_{\tilde{u}}(x)$, $x \in \mathbb{R}$ can be defined as [1]:*

$$\mu_{\tilde{u}}(x) = \begin{cases} L_{\tilde{u}}(x), & a \le x \le b, \\ 1, & b \le x \le c, \\ R_{\tilde{u}}(x), & c \le x \le d, \\ 0, & otherwise, \end{cases} \tag{2.1}$$

where $L_{\tilde{u}}$ is the left membership function and $R_{\tilde{u}}$ is the right membership function. It is assumed that $L_{\tilde{u}}$ is increasing in $[a, b]$ and $R_{\tilde{u}}$ is decreasing in $[a, b]$, and that $L_{\tilde{u}}(a) = R_{\tilde{u}}(d) = 0$ and $L_{\tilde{u}}(b) = R_{\tilde{u}}(c) = 1$. In addition, if $L_{\tilde{u}}$ and $R_{\tilde{u}}$ are linear, then $\tilde{u}$ is a trapezoidal fuzzy number, which is denoted by $\tilde{u} = (a, b, c, d)$. If $b = c$, we denoted it by $\tilde{u} = (a, c, d)$, which is a triangular fuzzy number.

The parametric form of a fuzzy number is given by $\tilde{u} = (\underline{u}, \overline{u})$, where $\underline{u}$ and $\overline{u}$ are functions defined over $[0, 1]$ and satisfy the following requirements:

(1) $\underline{u}$ is a monotonically increasing left continuous function.
(2) $\overline{u}$ is a monotonically decreasing left continuous function.
(3) $\underline{u} \le \overline{u}$, in $[0, 1]$.

We name $\underline{u}$ and $\overline{u}$, left and right spread functions, respectively. If a is a crisp number, then $\underline{u}(r) = \overline{u}(r) = a$, for $\forall r \in [0, 1]$.

Definition 2.2. *We say that a fuzzy number $\tilde{v}$ has an $m-$degree polynomial form, if there exist two polynomials p and q of degree at most m such that $\tilde{v} = (p, q)$ [3].*

Let $\tilde{v} \in \mathcal{F}_m(\mathbb{R})$ be the set of all $m-$degree polynomial form fuzzy numbers. For $0 < \alpha \le 1$, α-cut of a fuzzy number $\tilde{u}$ is defined by [5] as follows:

$$[\tilde{u}]^{\alpha} = \{t \in \mathbb{R} \mid \mu_{\tilde{u}}(t) \ge \alpha\}. \tag{2.2}$$

The core of a fuzzy number is defined by [5] as follows:

$$core(\tilde{u}) = \{t \in \mathbb{R} \mid \mu_{\tilde{u}}(t) = 1\}. \tag{2.3}$$

Let $\mathcal{F}_c(\mathbb{R})$ be the set of all fuzzy numbers with continuous left and right spread functions and let $\mathcal{F}_m(\mathbb{R})$ be the set of all $m-$degree polynomial form fuzzy numbers [3]. We also consider Π_m as the set of all polynomials of degree at most m.

We can write a fuzzy number $\tilde{u} \in \mathcal{F}_m(\mathbb{R})$ as follows:

$$\tilde{u} = (\underline{u}, \overline{u}), \tag{2.4}$$

where $\underline{u}$, $\overline{u} \in \Pi_m$.

3. A Parametric Distance

In order to measure the distance between two fuzzy numbers, here, we propose a new definition.

Definition 3.1. *For $\tilde{u}, \tilde{v} \in \mathcal{F}(\mathbb{R})$, the distance of $\tilde{u}$ and $\tilde{v}$ is defined by*

$$D_{p,q}(\tilde{u}, \tilde{v}) = \left(\int_0^1 q|\underline{u}(r) - \underline{v}(r)|^p \, \mathrm{d}r + \int_0^1 (1-q)|\overline{u}(r) - \overline{v}(r)|^p \, \mathrm{d}r \right)^{\frac{1}{p}}, \tag{3.1}$$

where $q \in [0, 1]$ and $p > 0$.

Theorem 3.1. *$D_{p,q}$ is a metric on $\mathcal{F}_c(\mathbb{R})$.*

Proof. It can be found in [7].

□

As the q changes in (3.1), the distance $D_{p,q}$ gets biased towards either the left spread function or the right one.

4. The Best Polynomial Fuzzy Number to an Arbitrary Fuzzy Number

Knowing the fact that a fuzzy number can be approximated in terms of an m-degree polynomial, it is now aimed at finding the nearest m-degree polynomial to a given fuzzy number. To this end, the proposed parametric distance defined in Section 3 is used.

Assume that $\tilde{u}$ is an arbitrary fuzzy number and $\tilde{v}$ is an approximated m-degree polynomial form fuzzy number. For $p = 2$ in (3.1), the distance becomes as:

$$D_{2,q}(\tilde{u}, \tilde{v}) = \left(\int_0^1 q|\underline{u}(r) - \underline{v}(r)|^2 \, \mathrm{d}r + \int_0^1 (1-q)|\overline{u}(r) - \overline{v}(r)|^2 \, \mathrm{d}r \right)^{\frac{1}{2}}, \tag{4.1}$$

where $q \in [0, 1]$.

Now, the approximation problem becomes as:

$$\begin{cases} \min\limits_{\tilde{v} \in \mathcal{F}_m} D_{2,q}(\tilde{u}, \tilde{v}), \\ \text{s.t.} \\ \tilde{v}(1) = \tilde{u}(1), \end{cases} \tag{4.2}$$

which can be expanded as follows:

$$\begin{cases} \min\limits_{\underline{v}, \overline{v} \in \Pi_m} \int_0^1 q|\underline{u}(r) - \underline{v}(r)|^2 \, \mathrm{d}r + \int_0^1 (1-q)|\overline{u}(r) - \overline{v}(r)|^2 \, \mathrm{d}r, \\ \text{s.t.} \\ \underline{v}(1) = \underline{u}(1), \\ \overline{v}(1) = \overline{u}(1). \end{cases} \tag{4.3}$$

Before solving this problem, let's present the Lemma 4.1 which will come in handy in our approximation method.

Lemma 4.1. *Let f and g be two arbitrary functions defined on a domain $D \subseteq \mathbb{R}$. Then over this domain we have:*

$$\min(f(x) + g(x)) \geq \min f(x) + \min g(x). \tag{4.4}$$

Proof. Straightforward. $\square$

In the following, we propose our two new approximation methods which minimize the distance first based on *splitting the problem* and second based on *general form*.

4.1. **Minimization by splitting the problem.** From Lemma 4.1, it is clear that splitting the problem (4.3) into two sub-problems will lead us to have a less objective value. Since q is constant and both terms of the objective functions in (4.3) are non-negative, by Lemma 4.1 the problem is divided into two independent sub-problems:

$$\begin{cases} \min\limits_{\underline{v} \in \Pi_m} \int_0^1 |\underline{u}(r) - \underline{v}(r)|^2 \, \mathrm{d}r, \\ \text{s.t.} \\ \underline{v}(1) = \underline{u}(1). \end{cases} \tag{4.5}$$

and

$$\begin{cases} \min\limits_{\overline{v} \in \Pi_m} \int_0^1 |\overline{u}(r) - \overline{v}(r)|^2 \, \mathrm{d}r, \\ \text{s.t.} \\ \overline{v}(1) = \overline{u}(1). \end{cases} \tag{4.6}$$

Assume that $\underline{v}(r) = \sum_{j=0}^{m} a_j r^j$, for solving the problem (4.5) with Lagrangian method we define the following function:

$$F(\mathbf{a},\lambda) = \int_0^1 |\underline{u}(r) - \underline{v}(r)|^2 \,\mathrm{d}r - \lambda(\underline{u}(1) - \underline{v}(1)), \tag{4.7}$$

where $\mathbf{a} = (a_0, a_1, ..., a_m)^t$.

The necessary condition to minimize the function F is that the gradient of function should be zero. The gradient of function F can be shown as follows:

$$\bigtriangledown F(\mathbf{a},\lambda) = \begin{bmatrix} 2H\mathbf{a} + \lambda\mathbf{1} - 2\underline{\mathbf{m}} \\ \\ \mathbf{1}^t\mathbf{a} - \underline{u}(1) \end{bmatrix}, \tag{4.8}$$

where $\mathbf{1} = (1, ..., 1)^t$, H is the $m+1$ Hermitian matrix which its elements defined as $H_{ij} = (i+j+1)^{-1}$ and $\underline{\mathbf{m}}$ is the momentum $\underline{u}$:

$$\underline{\mathbf{m}} = \left[\int_0^1 r^i \underline{u}(r)\,\mathrm{d}r\right]^t_{i=0,\dots,m}. \tag{4.9}$$

$\bigtriangledown F = 0$ gives following:

$$\begin{cases} H\mathbf{a} + \dfrac{1}{2}\lambda\mathbf{1} - \underline{\mathbf{m}} = 0, \\ \\ \underline{v}(1) = \underline{u}(1). \end{cases} \tag{4.10}$$

Thus, we define $Q_F : \mathbb{R}^{m+2} \longrightarrow \mathbb{R}^{m+2}$ as follows:

$$Q_F(\mathbf{a},\lambda) = \begin{bmatrix} H\mathbf{a} + \dfrac{1}{2}\lambda\mathbf{1} - \underline{\mathbf{m}} \\ \\ \mathbf{1}^t\mathbf{a} - \underline{u}(1) \end{bmatrix}. \tag{4.11}$$

Hence, we try to solve $Q_F(\underline{\mathbf{x}}) = 0$ which is a linear system such that:

$$\underline{\mathbf{x}} = (\mathbf{a},\lambda)^t. \tag{4.12}$$

To solve this system, we have:

$$Q_F(\mathbf{a},\lambda) = A\,\underline{\mathbf{x}} - \underline{\mathbf{R}}, \tag{4.13}$$

where

$$\underline{\mathbf{R}} = \begin{bmatrix} \underline{\mathbf{m}} \\ \\ \underline{u}(1) \end{bmatrix}, \tag{4.14}$$

And A is as follows:

$$A = \begin{bmatrix} H & \dfrac{1}{2}\mathbf{1} \\ \\ \mathbf{1}^t & 0 \end{bmatrix}. \tag{4.15}$$

Hence, we have:

$$\underline{\mathbf{x}} = A^{-1}\ \underline{\mathbf{R}}, \tag{4.16}$$

Theorem 4.1. *The inverse matrix of A has the following form:*

$$A^{-1} = \begin{bmatrix} H^{-1} - \mathbf{v}\mathbf{v}^t & \dfrac{1}{m+1}\mathbf{v} \\ \\ \dfrac{2}{m+1}\mathbf{v}^t & -\dfrac{2}{(m+1)^2} \end{bmatrix}, \tag{4.17}$$

where $\mathbf{v} = \frac{1}{m+1}H^{-1}\mathbf{1}$.

Proof. It is straightforward. □

We consider the solution of minimization problem (4.7) as $\underline{\mathbf{x}}^* = (\mathbf{a}^*, \lambda^*)^t$ such that:

$$\begin{cases} \mathbf{a}^* = (H^{-1} - \mathbf{v}\mathbf{v}^t)\underline{\mathbf{m}} + \dfrac{\underline{u}(1)}{m+1}\mathbf{v}, \\ \\ \lambda^* = \dfrac{2}{m+1}\mathbf{v}^t\underline{\mathbf{m}} - \dfrac{2\underline{u}(1)}{(m+1)^2}. \end{cases} \tag{4.18}$$

Now, in the same way, we solve the Problem (10).

Let $\overline{v}(r) = \sum_{j=0}^{m} b_j r^j$, $\mathbf{b} = (b_0, b_1, ..., b_m)^t$. For solving with Lagrangian method , we continue by defining G as follows:

$$G(\mathbf{b}, \mu) = \int_0^1 |\overline{u}(r) - \overline{v}(r)|^2\, \mathrm{d}r - \mu(\overline{u}(1) - \overline{v}(1)). \tag{4.19}$$

Let define the momentum vector of $\overline{u}$ as:

$$\overline{\mathbf{m}} = \left[\int_0^1 r^i \overline{u}(r)\, \mathrm{d}r\right]^t_{i=0,\dots,m}, \tag{4.20}$$

Thus, we define $Q_G : \mathbb{R}^{m+2} \longrightarrow \mathbb{R}^{m+2}$ as follows:

$$Q_G(\mathbf{b}, \mu) = \begin{bmatrix} H\mathbf{b} + \dfrac{1}{2}\mu\mathbf{1} - \overline{\mathbf{m}} \\ \\ \mathbf{1}^t\mathbf{b} - \overline{u}(1) \end{bmatrix}. \tag{4.21}$$

as we did it before we have a linear system az follows:

$$Q_G(\mathbf{b}, \mu) = A\ \overline{\mathbf{x}} - \overline{\mathbf{R}}, \tag{4.22}$$

where

$$\overline{\mathbf{x}} = (\mathbf{b}, \mu)^t, \tag{4.23}$$

thus:

$$\overline{\mathbf{x}} = A^{-1}\ \overline{\mathbf{R}}, \tag{4.24}$$

Considering the solution of minimization problem (4.7) as $\overline{\mathbf{x}}^* = (\mathbf{b}^*, \mu^*)^t$ such that:

$$\begin{cases} \mathbf{b}^* = (H^{-1} - \mathbf{v}\mathbf{v}^t)\overline{\mathbf{m}} + \dfrac{\overline{u}(1)}{m+1}\mathbf{v}, \\ \\ \mu^* = \dfrac{2}{m+1}\mathbf{v}^t\overline{\mathbf{m}} - \dfrac{2\overline{u}(1)}{(m+1)^2}. \end{cases} \tag{4.25}$$

In summary, assuming $\tilde{u} \in \mathcal{F}(\mathbb{R})$ be an arbitrary fuzzy number, we find the best approximation of $\tilde{u}$ out of $\mathcal{F}_m$ for a fixed integer m. In this case, $\tilde{u}_m^*$ is the best approximation of $\tilde{u}$, such that:
$\underline{u}^*(r) = \sum\limits_{j=0}^{m} a_j^* r^j$ and $\overline{u}^*(r) = \sum\limits_{j=0}^{m} b_j^* r^j$,
where $\mathbf{a}^* = (a_0^*, a_1^*, ..., a_m^*)^t$ and $\mathbf{b}^* = (b_0^*, b_1^*, ..., b_m^*)^t$.

We denote the best approximation of $\tilde{u} \in \mathcal{F}$ out of $\mathcal{F}_m$ by $\tilde{u}_m^*$. In following theorem we show that the best approximation of an arbitrary polynomial fuzzy number is itself.

Theorem 4.2. *If $\tilde{u} \in \mathcal{F}_m$ then its best approximation $\tilde{u}_m^*$, out of $\mathcal{F}_m(\mathbb{R})$ with respect to distance (3.1) exists and $\tilde{u}_m^* = \tilde{u}$.*

Proof. Straightforward. □

Corollary 4.2. *Best approximation of an arbitrary trapezoidal fuzzy number is itself.*

Proof. It can obtained by Theorem 4.2. □

4.2. **Minimization of the problem in general form.** In this section, we try to solve problem (4.3) in general form. Assume that $\underline{v}(r) = \sum\limits_{j=0}^{m} a_j r^j$ and $\overline{v}(r) = \sum\limits_{j=0}^{m} b_j r^j$. To this end, for solving the problem (4.3) with Lagrangian method we define the following function:

$$\begin{aligned} E(\mathbf{a}, \mathbf{b}, \lambda, \mu) = & \int_0^1 q(\underline{u}(r) - \underline{v}(r))^2 \,\mathrm{d}r + \int_0^1 (1-q)(\overline{u}(r) - \overline{v}(r))^2 \,\mathrm{d}r \\ & - \lambda(\underline{u}(1) - \underline{v}(1)) - \mu(\overline{u}(1) - \overline{v}(1)), \end{aligned} \tag{4.26}$$

where $\mathbf{a} = (a_0, a_1, ..., a_m)^t$, $\mathbf{b} = (b_0, b_1, ..., b_m)^t$.

The necessary condition to minimize the function E is that the gradient of function should be zero. The gradient of function E can be shown as follows:

$$\bigtriangledown E(\mathbf{a}, \mathbf{b}, \lambda, \mu) = \begin{bmatrix} 2qH\mathbf{a} + \lambda\mathbf{1} - 2q\underline{\mathbf{m}} \\ \\ 2(1-q)H\mathbf{b} + \mu\mathbf{1} - 2(1-q)\overline{\mathbf{m}} \\ \\ \mathbf{1}^t\mathbf{a} - \underline{u}(1) \\ \\ \mathbf{1}^t\mathbf{b} - \overline{u}(1) \end{bmatrix}, \tag{4.27}$$

where H is the $m+1$ Hermitian matrix, $\mathbf{1} = (1, ..., 1)^t$, $\underline{\mathbf{m}}$ and $\overline{\mathbf{m}}$ respectively are the momentum vectors of $\underline{u}$ and $\overline{u}$ defined in (4.9) and (4.20).

$\bigtriangledown E = 0$ gives following:

$$\begin{cases} qH\mathbf{a} + \frac{1}{2}\lambda_1 \mathbf{1} - q\underline{\mathbf{m}} = 0, \\ \\ (1-q)H\mathbf{b} + \frac{1}{2}\mu_1 \mathbf{1} - (1-q)\overline{\mathbf{m}} = 0, \\ \\ \underline{v}(1) = \underline{u}(1), \\ \\ \overline{v}(1) = \overline{u}(1). \end{cases} \tag{4.28}$$

Now, we define Q_E as follows:

$$Q_E(\mathbf{a}, \lambda, \mathbf{b}, \mu) = \begin{bmatrix} qH\mathbf{a} + \frac{1}{2}\lambda \mathbf{1} - q\underline{\mathbf{m}} \\ \\ \mathbf{1}^t\mathbf{a} - \underline{u}(1) \\ \\ (1-q)H\mathbf{b} + \frac{1}{2}\mu \mathbf{1} - (1-q)\overline{\mathbf{m}} \\ \\ \mathbf{1}^t\mathbf{b} - \overline{u}(1) \end{bmatrix}. \tag{4.29}$$

Hence, we try to solve $Q_E(\mathbf{t}) = 0$ which is a linear system such that:

$$\mathbf{t} = (\mathbf{a}, \lambda, \mathbf{b}, \mu)^t. \tag{4.30}$$

To solve this system, we let $Q_E(\mathbf{t}) = A_E\ t - Z = 0$ where:

$$A_E = \begin{bmatrix} qH & \frac{1}{2}\mathbf{1} & 0 & 0 \\ \mathbf{1}^t & 0 & 0 & 0 \\ 0 & 0 & (1-q)H & \frac{1}{2}\mathbf{1} \\ 0 & 0 & \mathbf{1}^t & 0 \end{bmatrix}_{4\times 4}, \tag{4.31}$$

and

$$Z = \begin{bmatrix} q\underline{\mathbf{m}} \\ \\ \underline{u}(1) \\ \\ (1-q)\overline{\mathbf{m}} \\ \\ \overline{u}(1) \end{bmatrix}. \tag{4.32}$$

Now we countinue with finding the invers of coefficient matrix, A_E. By considering $A_{E,\gamma}$ as

$$A_{E,\gamma} = \begin{bmatrix} \gamma H & \frac{1}{2}\mathbf{1} \\ \\ \mathbf{1}^t & 0 \end{bmatrix}, \tag{4.33}$$

where $\gamma \in [0,1]$ and we have

$$A_E = \begin{bmatrix} A_{E,q} & 0 \\ \\ 0 & A_{E,(1-q)} \end{bmatrix}. \tag{4.34}$$

Lemma 4.3. *$A^{-1}_{E,\gamma}$ has the following form:*

$$A^{-1}_{E,\gamma} = \begin{bmatrix} \frac{1}{\gamma}(H^{-1} - \mathbf{v}\mathbf{v}^t) & \frac{1}{m+1}\mathbf{v} \\ \\ \frac{2}{m+1}\mathbf{v}^t & -\gamma\frac{2}{(m+1)^2} \end{bmatrix}, \tag{4.35}$$

such that $\mathbf{v} = \frac{1}{m+1}H^{-1}\mathbf{1}$.

Proof. Straightforward. □

Theorem 4.3. *The inverse matrix of A_E has the following form:*

$$A^{-1}_E = \begin{bmatrix} A^{-1}_{E,q} & 0 \\ \\ 0 & A^{-1}_{E,(1-q)} \end{bmatrix}, \tag{4.36}$$

Proof. It is straightforward. □

To do this end, with Theorem 4.3 we have:

$$t = A^{-1}_E\ Z, \tag{4.37}$$

We consider the solution of minimization problem (4.26) as $\mathbf{x_E}^* = (\mathbf{a}^*, \lambda^*, \mathbf{b}^*, \mu^*)^t$ such that:

$$\begin{cases} \mathbf{a}^* = (H^{-1} - \mathbf{v}\mathbf{v}^t)\underline{\mathbf{m}} + \dfrac{\underline{u}(1)}{m+1}\mathbf{v}, \\ \\ \mathbf{b}^* = (H^{-1} - \mathbf{v}\mathbf{v}^t)\overline{\mathbf{m}} + \dfrac{\overline{u}(1)}{m+1}\mathbf{v}, \\ \\ \lambda^* = \dfrac{2q}{m+1}\mathbf{v}^t\underline{\mathbf{m}} - \dfrac{2q\underline{u}(1)}{(m+1)^2}, \\ \\ \mu^* = \dfrac{2(1-q)}{m+1}\mathbf{v}^t\overline{\mathbf{m}} - \dfrac{2(1-q)\overline{u}(1)}{(m+1)^2}. \end{cases} \tag{4.38}$$

Analogous to the Theorem 4.2, in following theorem we again show that the best approximation of an arbitrary polynomial fuzzy number is itself.

Theorem 4.4. *If $\tilde{u} \in \mathcal{F}_m$ then its best approximation $\tilde{u}_m^*$, out of $\mathcal{F}_m(\mathbb{R})$ with respect to distance (3.1) exists and $\tilde{u}_m^* = \tilde{u}$.*

Proof. It can be proved by (4.38). □

Corollary 4.4. *If $\tilde{u} \in \mathcal{F}_l$ where $(l \leq m)$, then $\tilde{u}_m^* = \tilde{u}$.*

Proof. straightforward. □

Note that if the obtained approximated coefficients yield a polynomial form fuzzy number, this polynomial is the best approximation of the given fuzzy number.

5. Convergence of Approximation

In this section, the convergence of the proposed approximation methods are shown.

Lemma 5.1. *Let $m \in \mathbb{N}$*

$$\lim_{m\to\infty} \frac{1}{m}\sum_{j=1}^{m}\frac{1}{j} = 0.$$

Proof. It is trivial. □

Theorem 5.1. *If $\underline{u}$ and $\overline{u}$ are integrable functions in $[0,1]$ and $\tilde{u}_m^*$ is the best approximation of $\tilde{u}$ by splitting the problem in Subsection 4.1 out of $\mathcal{F}_m$, then*

$$\lim_{m\to\infty} \tilde{u}_m^* = \tilde{u}.$$

Proof. From (4.18), we have

$$\begin{aligned}\lim_{m\to\infty}\lambda^* &= \lim_{m\to\infty}\left(\frac{2}{m+1}\mathbf{v}^t\underline{\mathbf{m}} - \frac{2\underline{u}(1)}{(m+1)^2}\right)\\ &= 2\lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^{m}\int_0^1 r^j\underline{u}(r)\,\mathrm{d}r\end{aligned}\tag{5.1}$$

Since r^j is nonnegative in $[0,1]$, according to Midpoint Theorem for integrals there exists $\theta_j \in (0,1)$, such that

$$\int_0^1 r^j\underline{u}(r)\,\mathrm{d}r = \frac{\underline{u}(\theta_j)}{j+1}, \quad j = 0, ..., m \tag{5.2}$$

Therefore, from (5.1) and Lemma 5.1 we have

$$\begin{aligned}\lim_{m\to\infty}\lambda^* &= 2\lim_{m\to\infty}\frac{1}{m+1}\sum_{j=0}^{m}\frac{\underline{u}(\theta_j)}{j+1}\\ &\le \|\underline{u}\|_\infty 2\lim_{m\to\infty}\frac{1}{m+1}\sum_{j=1}^{m+1}\frac{1}{j} = 0.\end{aligned}\tag{5.3}$$

From (4.15), (4.18) and (5.3), when $m \to \infty$, $\mathbf{a}^*$ is the solution of $H\mathbf{a} = \underline{\mathbf{m}}$, where H is a Hermitian matrix. In this case, $\mathbf{a}^*$ is the solution of a common crisp problem and for this solution we have the convergence.

Similarly from (4.25) , we have

$$\lim_{m\to\infty}\mu^* = 0 \tag{5.4}$$

and these claims hold for $\mathbf{b}^*$ in $H\mathbf{b} = \overline{\mathbf{m}}$.

Since $\mathbf{a}^*$ and $\mathbf{b}^*$ are both convergent, therefore, $\underline{u}^*$ and $\overline{u}^*$ are also convergent and this completes the proof. □

Theorem 5.2. *If $\underline{u}$ and $\overline{u}$ are integrable functions in $[0,1]$ and $\tilde{u}^*_m$ is the best approximation of $\tilde{u}$ of general form in Subsection 4.2 out of $\mathcal{F}_m$, then*

$$\lim_{m\to\infty}\tilde{u}^*_m = \tilde{u}.$$

Proof. It was obtained by (4.38) and Lemma 5.1. □

Corollary 5.2. *If $\tilde{u} \in \mathcal{F}_m$, the approximation sequence converges to the exact solution in the first iteration.*

Proof. straightforward. □

Considering (4.18), (4.25) and (4.38) for an arbitrary fuzzy number, the best approximation regarding both methods are identical. According to following lemma we present an explicit formula to approximate an arbitrary fuzzy number with a trapezoidal fuzzy number.

Theorem 5.3. *If $\tilde{u} = (\underline{u}, \overline{u})$ is an arbitrary fuzzy number, then its best linear approximation $\tilde{u}_1^*$ regarding the distance (3.1) is $\tilde{u}_1^* = (a_0 + a_1 r, b_0 + b_1 r)$ where:*

$$a_0 = \frac{1}{2}(6\int_0^1 \underline{u}(r)\,\mathrm{d}r - 6\int_0^1 r\underline{u}(r)\,\mathrm{d}r - \underline{u}(1)), \tag{5.5}$$

$$a_1 = -\frac{3}{2}(2\int_0^1 \underline{u}(r)\,\mathrm{d}r - 2\int_0^1 r\underline{u}(r)\,\mathrm{d}r - \underline{u}(1)), \tag{5.6}$$

$$b_0 = \frac{1}{2}(6\int_0^1 \overline{u}(r)\,\mathrm{d}r - 6\int_0^1 r\overline{u}(r)\,\mathrm{d}r - \overline{u}(1)), \tag{5.7}$$

$$b_1 = -\frac{3}{2}(2\int_0^1 \overline{u}(r)\,\mathrm{d}r - 2\int_0^1 r\overline{u}(r)\,\mathrm{d}r - \overline{u}(1)). \tag{5.8}$$

Proof. straightforward. □

In the following, let's present the Lemma 5.3 which will come in handy in showing our best linear approximation of an arbitrary fuzzy number is a trapezoidal fuzzy number.

Lemma 5.3. *For any arbitrary function g, if g is a monotonically increasing left continuous function then:*

$$\int_0^1 xg(x)\,\mathrm{d}x - \int_0^1 g(x)\,\mathrm{d}x + \frac{1}{2}g(1) \geq 0,$$

and if g is a monotonically decreasing left continuous function then:

$$\int_0^1 xg(x)\,\mathrm{d}x - \int_0^1 g(x)\,\mathrm{d}x + \frac{1}{2}g(1) \leq 0,$$

Proof. Straightforward. □

Lemma 5.4. *The best linear approximation of an arbitrary fuzzy number $\tilde{u} = (\underline{u}, \overline{u})$ regarding the distance (3.1) is a trapezoidal fuzzy number.*

Proof. As regards to distance (3.1) and by Lemma 5.3 and 5.3 it was obtained. □

Due to the Theorem 5.3 and Lemma 5.4 for any arbitrary fuzzy number, the nearest trapezoidal fuzzy number regarding the distance (3.1) can be obtained from equations (5.5) - (5.8).

6. Numerical Examples

In this section we present some examples which have been solved by Mathematica software using 10 decimal digits.

Example 6.1. *Let $\tilde{u} = (2r^2 + 1, 5 - r^2)$. By assuming $m = 2$ and each $q \in [0, 1]$ the best approximation of $\tilde{u}$ is itself. According to Theorem 4.2 it could be foretold.*

Example 6.2. *Let $\tilde{u} = (r^2 + 1, 3 - r^2)$. By assuming $m = 1$ and each $q \in [0, 1]$ the best approximation of $\tilde{u}$ can be found by Lemma 5.3. The best approximation is $(\frac{3}{4} + \frac{5}{4}r, \frac{13}{4} - \frac{5}{4}r)$.*

Example 6.3. *Let $\tilde{u} = (e^r, e^{2-r})$. For $m = 1$ and $m = 3$, the best approximations of $\tilde{u}$ are $\tilde{u}_1^*$ and $\tilde{u}_3^*$, where:*

$$\begin{cases} \underline{u}_1^*(r) = & \frac{1}{2}(-12+5e) + \frac{3}{2}(4-e)r \\ \\ \overline{u}_1^*(r) = & \frac{5}{2}e - \frac{3}{2}er \end{cases}$$

$$\begin{cases} \underline{u}_3^*(r) = \frac{1}{4}(-4560+1679e) - \frac{15}{4}(-3216+1183e)r \\ \qquad + \frac{15}{4}(-7168+2637e)r^2 - \frac{35}{4}(-1824+671e)r^3 \\ \overline{u}_3^*(r) = \frac{1}{4}(3599e-1320e^2) + \frac{15}{4}(-2719e+1000e^2)r \\ \qquad - \frac{15}{4}(-6285e+2312e^2)r^2 + \frac{35}{4}(-1631e+600e^2)r^3 \end{cases}$$

and for $q = 0.5$ the distance (4.1) between $\tilde{u}$ and $\tilde{u}_1^*$ is $D(\tilde{u}, \tilde{u}_1^*) = 0.185451$ and the distance between $\tilde{u}$ and $\tilde{u}_3^*$ is $D(\tilde{u}, \tilde{u}_3^*) = 0.000835893$.

Example 6.4. *Let $\tilde{u} = (\ln[(e-1)r+1], 2-\ln[(e-1)r+1])$, $p_i(r) = r^i$ and an arbitrary q . The distances (4.1) between $\tilde{u}$ and $\tilde{u}_m^*$, for $m = 1, \cdots, 7$, are shown in Table 1.*

m	$D(\tilde{u}, \tilde{u}_m^*)$
1	4.85342×10^{-2}
2	7.2679×10^{-3}
3	1.27813×10^{-3}
4	2.44005×10^{-4}
5	4.89405×10^{-5}
6	1.04687×10^{-5}
7	5.88816×10^{-6}

Table 1: Distances for different values of m

Regarding Theorem 5.1, it was predictable that increasing the variable m would reduce the associated error. Since $\tilde{u}$ is a symmetric fuzzy number and by (4.38) the best approximation of it is independent of q, the distance (4.1) between $\tilde{u}$ and $\tilde{u}_m^*$ is independent of q.

Example 6.5. *In this example, we approximate a fuzzy number with $m = 1$ by a trapezoidal one and compare the results from our method with the results obtained from other four methods proposed in [2, 9, 11, 14] in a tabular format in Table 2.*

$\underline{u}(r)$ / $\overline{u}(r)$	(1)	(2)	(3)	(4)	(5)
$1-0.3\sqrt{-\ln r}$ $2+0.7\sqrt{-\ln r}$	0.484391 1 2 3.20309	0.50052 0.96775 2.07526 3.16546	0.06790 1.67725 1.67725 3.2866	0.52195 0.89256 2.10743 2.9722	0.50052 0.96775 2.07526 3.16546
3^r $7-3^r$	0.740495 3 4 6.2595	0.84003 2.80092 4.19908 6.15997	0.4905 3.5 3.5 6.5095	0.48633 2.97779 4.02221 6.51367	0.84003 2.80092 4.19908 6.15997
1 2	1 1 2 2	1 1 2 2	0.75 1.5 1.5 2.25	1 1 1 2	1 1 2 2
$r+1$ $5-3r$	1 2 2 5	1 2 2 5	-0.5 2 2 4.5	1 2 2 2.5	1 2 2 5
1 $3-r$	1 1 2 3	1 1 2 3	0.5 1.75 1.75 3	-0.33333 0.66667 2.33333 2.33333	1 1 2 3

Table 2. Numerical results of examples

As it is obvious, our method yields trapezoidal fuzzy numbers with closer cores in comparison with the ones obtained from the other four methods. While the other four methods fail in approximating the m-degree polynomial form fuzzy numbers, our method can approximate all the trapezoidal, triangular and m-degree polynomial form fuzzy numbers.

Example 6.6. *Let* $\tilde{u} = (2 + e^{r-1}, 4 - \ln[(e-1)r + 1])$, $p_i(r) = r^i$. *The distances between* $\tilde{u}$ *and* $\tilde{u}^*_m$, *for* $m = 0, \cdots, 4$ *and* $q = \{0, 0.25, 0.5, 0.75, 1\}$, *are shown in Table 3.*

$D(\tilde{u}, \tilde{u}^*_m)$	$q=0$	$q=0.25$	$q=0.5$	$q=0.75$	$q=1$
$m=0$	0.504053	0.482261	0.459435	0.435415	0.409989
$m=1$	0.0485342	0.0458563	0.0430119	0.0399657	0.0366673
$m=2$	0.0072679	0.00643446	0.0054756	0.00430839	0.00267248
$m=3$	0.00127813	0.00110962	0.000910436	0.000653097	0.000155493
$m=4$	0.000244005	0.000211315	0.000172538	0.000122003	0.000000000

Table 3: Distances for different values of m and q

This table shows that as the variable m increases, the distance (4.1) between exact given fuzzy number and our approximated polynomial form fuzzy number reduces(it can be predicted by Theorem 5.1). In this example $\tilde{u}$ is not a symmetric fuzzy number. Hence, the distance between $\tilde{u}$ and $\tilde{u}_m^*$ depends on q. As we can see in the table, whenever q increases from 0 to 1, the distance decreases. Considering the distance equation (4.1), it can be deduced that the approximation of right spread is more precise than the approximation of the left one.

7. Conclusion

In this paper, a new distance metric was proposed on the set of all fuzzy numbers with continuous left and right spread functions. Using this metric, a given fuzzy number can be approximated through finding the nearest polynomial form fuzzy number out of the set of all m-degree polynomial form fuzzy numbers.

Hence, two methods were proposed to solve the approximation problem. We showed that both of the methods not only yield the same results, but also are convergent. Finally, we investigated our theorems in some numerical examples.

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