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STUDY OF SOLUTION FOR A PARABOLIC INTEGRODIFFERENTIAL EQUATION WITH THE SECOND KIND INTEGRAL CONDITION

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Abstract. In this paper, we establish sufficient conditions for the existence, uniqueness and numerical solution for a parabolic integrodifferential equation with the second kind integral condition. The existence, uniqueness of a strong solution for the linear problem based on a priori estimate "energy inequality" and transformation of the linear problem to linear first-order ordinary differential equation with second member. Then by using a priori estimate and applying an iterative process based on results obtained for the linear problem, we prove the existence, uniqueness of the weak generalized solution of the integrodifferential problem. Also we have developed an efficient numerical scheme, which uses temporary problems with standard boundary conditions. A suitable combination of the auxiliary solutions defines an approximate solution to the original nonlocal problem, the algebraic matrices obtained after the full discretization are tridiagonal, then the solution is obtained by using the Thomas algorithm. Some numerical results are reported to show the efficiency and accuracy of the scheme.

1. Introduction

The topic of integro-differential equations which are combination of differential and integral has attracted many scientists and researchers due to their applications in many areas; see, for example, [16, 17] . Many mathematical formulation of physical phenomena contain integro-differential equations, and these equations may arise in fluid dynamics, biological models, and chemical kinetics; for more details, see [20, 40] .

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Integro-differential equations are usually difficult to solve analytically, so it is required to obtain an efficient approximate solution.

Nowadays various nonlocal problems for partial differential equations have been actively studied and one can find a lot of papers dealing with them (see [13]- [29] , [12]- [21] and references therein). Afterwards, the nonlocal problems for integro-differential equation with integral conditions was studied by many authors, see A. Merad and A. Bouziani [23] , [26]. Motivated by this we study a parabolic integrodifferential equation with nonlocal second kind integral condition.

2. Preliminaries and functional spaces

In the rectangular domain $\Omega = (0,1) \times (0,T)$, with $T < \infty$,we consider the equation:

$$\pounds u = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \int_0^t a\left(t-s\right) g\left(s,u\right) ds + f(x,t), \tag{2.1}$$

with the initial data

$$\ell u = u(x,0) = \varphi\left(x\right), \quad x \in \left(0,1\right), \tag{2.2}$$

with the Second Kind Integral Conditions

$$u_x\left(0,t\right) = \int_0^1 K_0\left(x,t\right) u\left(x,t\right) dx, \tag{2.3}$$

$$u_x\left(1,t\right) = \int_0^1 K_1\left(x,t\right) u\left(x,t\right) dx, \tag{2.4}$$

where f, φ, K_0, K_1 ansd g are known functions. Note that a is bounded function where

$$\left|a\left(t-s\right)\right| < a_0, \quad a_0 \text{ is a positive constant.}$$

And the function g verify the following inequality

$$\left\|g\left(s,u\right)\right\|_{L^2(\Omega)} \leqslant C_1 \left\|u\right\|_{L^2(\Omega)} + C_2, \quad C_1, C_2 \text{ are positive constants.}$$

We shall assume that the function φ satisfies a compatibility conditions with (2.3) and (2.4), i.e.,

$$\begin{aligned} \varphi_x\left(0\right) &= \int_0^1 K_0\left(x,0\right) \varphi\left(x\right) dx, \\ \varphi_x\left(1\right) &= \int_0^1 K_1\left(x,0\right) \varphi\left(x\right) dx. \end{aligned}$$

Some problems of modern physics and technology can be described in terms of partial differential equations with nonlocal conditions. The integral term of our problem (that is, $\int_0^t a\left(t-s\right) g\left(s,u\right) ds$ appears,because in some fields such as the heat transfer, nuclear reactor dynamics and thermoelasticity, we need to reflect the effects of the memory of the system in model, but describing such a system as a function at a given space and time ignores the effect of past history. Therefore, the way of remedy this difficulty is including an integral term in the basic partial differential equation that leads to a Partial integro-differential equations(

PIDE) [39]. The study of the problem (2.1)-(2.2)with some special types of boundary conditions of the form $u_x(0,t) = \alpha(t)$ and $\int_0^1 u\,(x,t)\,dx = E(t)$ motivated by the works of Dabas and Bahuguna [15],and Guezane-Lakoud et al. [18].

Bouziani and Mechri [8], studied , problem (2.1)-(2.2) with purely nonlocal (integral) conditions $\int_0^1 u(x,t) = E(t)$ and $\int_0^1 xu(x,t) = G(t)$. For other models, we refer the reader, for instance, to [6]- [38], and references therein. Most of the previous studies, the authors used the Rothe method (see [10], [8], [15]), the Laplace transform of the problem and then used numerical technique for the inverse Laplace transform to obtain the numerical solution (see [1]).

It is well known that the classical methods used widely to prove solvability of initial-boundary problems break down when applied to nonlocal problems. Nowadays some methods have been advanced for overcoming difficulties arising from nonlocal conditions. These methods are different and the choice of a concrete one depends on a form of a nonlocal condition. In this article, we focus on spatial nonlocal integral conditions like [30], of which we give three examples:

$$\int_0^1 K\,(x,t)\,u\,(x,t)\,dx = 0, \tag{2.5}$$

$$u_x\,(0,t) = \int_0^1 K\,(x,t)\,u\,(x,t)\,dx, \tag{2.6}$$

$$a\,(t)\,u\,(0,t) = \int_0^1 K\,(x,t)\,u\,(x,t)\,dx, \tag{2.7}$$

Condition (2.5) is a nonlocal first kind condition, (2.6) and (2.7) are second kind nonlocal conditions. The kind of a nonlocal integral condition depends on the presence or lack of a term containing a trace of the required solution or its derivative outside the integral [30] . Problems with nonlocal conditions of the forms (2.5) and (2.7) are investigated in [30], [11],and [36]. We pay attention on the second one, (2.6) which has not been studied so far with this class of integro-differential problems.

This paper is organized as follows. In Section 3, we establish the uniquness of solution by using a priori estimate method or the energy-integral method. In Sect 4, we first establish the existence of solutions of the linear problem by using the density of the range of the operator generated by the abstract formulation of the stated linear problem; secondly reformulating the integro-differential problem to a semi-linear problem, and after that we prove the slovability of semi-linear problem by using a priori estimate and applying an iterative process based on results obtained for the linear problem (see [34]), we prove the existence, uniqueness of the weak generalized solution of the integrodifferential problem.Section 5 is devoted to the construction of approximate solutions of problem (2.1)-(2.4), we discretize the problem by backward Euler in time and finite differences in space. The main numerical difficulty become visible after the discretization,the presence of an integral operator in the boundary conditions gives rise to rows/collumns, which are full. To avoid the problems with special solvers for algebraic systems, we design a very easy numerical algorithm, based on superposition principle, this technique lead to a linear systems have a tridiagonal coefficient matrix, so they can be solved very efficiently by fast Gauss elimination (which is also known as the Thomas algorithm). Finally, in section 6 we presents two numerical examples to illustrate the performance and efficiency of the proposed algorithm.

3. An energy estimate and uniqueness of solution

The method used here is one of the most efficient functional analysis methods and important techniques for solving partial differential equations with integral conditions, which has been successfully used in investigating the existence, uniqueness, and continuous dependence of the solutions of PDE's, the so-called a priori estimate method or the energy-integral method. This method is essentially based on the construction of multiplicators for each specific given problem, which provides the a priori estimate from which it is possible to establish the solvability of the posed problem. More precisely, the proof is based on an energy inequality and the density of the range of the operator generated by the abstract formulation of the stated problem, so to investigated the posed problem, we introduce the needed function spaces. In this paper, we prove the existence and the uniqueness for solution of the problem $(2.1)-(2.4)$ as a solution of the operator equation

$$Lu = \mathcal{F}. \tag{3.1}$$

Where $L = (\mathcal{L},\ell)$, with domain of difinition E consisting of functions $u \in L_2\left(0,T,L_2\left(0,1\right)\right) := L^2\left(\Omega\right)$ such that $u_x \in L^2\left(\Omega\right)$ and u satisfies condition (2.3) and (2.4); the operator L is considered from E to F, where E is the Banach space consisting of all functions $u(x,t)$ having a finite norm

$$\left\|u\right\|_E^2 = \left\|u\right\|_{L^2(\Omega)}^2 + \left\|u_x\right\|_{L^2(\Omega)}^2 ,$$

and F is the Hilbert space consisting of all elements $\mathcal{F} = (f,\varphi)$ for which the norm

$$\left\|\mathcal{F}\right\|_F^2 = \left\|f\right\|_{L^2(\Omega)}^2 + \left\|\varphi\right\|_{L^2(0,1)}^2$$

is finite.

Theorem 3.1. *If $\varepsilon > 0$, where $\varepsilon << \frac{1}{2}$. Then for any function $u \in E$ and we have the inequality*

$$\left\|u\right\|_E \leq c\left\|Lu\right\|_F \tag{3.2}$$

where c is a positive constant independent of u.

Proof. Assume that a solution of the problem (2.1) − (2.4) exists. We multiply the equation (2.1) by u and integrating over Ω^{τ}, where $\Omega^{\tau} = (0,1) \times (0,\tau)$, we get

$$\begin{aligned} \int_{\Omega^{\tau}} u \cdot Mu \; dxdt &= \int_{\Omega^{\tau}} u_t \cdot u \; dxdt - \int_{\Omega^{\tau}} u_{xx} \cdot u \; dxdt \\ &= \int_{\Omega^{\tau}} \left[\int_0^1 a\left(t-s\right) g\left(s,u\right) \right] \cdot u \; dxdt + \int_{\Omega^{\tau}} f\left(x,t\right) \cdot u \; dxdt \qquad (3.3) \end{aligned}$$

integrating by parts each term of the left-hand side of (3.3) over Ω^{τ}, $0 < \tau < T$, we obtain

$$\begin{aligned} &\frac{1}{2} \int_0^1 u\left(x,\tau\right)^2 \; dx + \int_{\Omega^{\tau}} u_x^2 \; dxdt \\ = &\int_0^{\tau} u_x\left(1,t\right) u\left(1,t\right) \; dt - \int_0^{\tau} u_x\left(0,t\right) u\left(0,t\right) \; dt + \frac{1}{2} \int_0^1 \varphi^2 \; dx \\ &+ \int_{\Omega^{\tau}} \left[\int_0^1 a\left(t-s\right) g\left(s,u\right) \right] \cdot u \; dxdt + \int_{\Omega^{\tau}} f \cdot u \; dxdt \qquad (3.4) \end{aligned}$$

Our next aim is to derive estimates of the right-hand side part of (3.4).

By using the Cauchy inequality with ε; we have

$$\int_0^{\tau} u_x\left(1,t\right) u\left(1,t\right) dt \; < \; \frac{\varepsilon}{2} \int_0^{\tau} u^2\left(1,t\right) \; dt + \frac{1}{2\varepsilon} \int_0^{\tau} u_x^2\left(1,t\right) dt. \qquad (3.5)$$

$$\int_0^{\tau} u_x\left(0,t\right) u\left(0,t\right) dt \; < \; \frac{\varepsilon}{2} \int_0^{\tau} u^2\left(0,t\right) \; dt + \frac{1}{2\varepsilon} \int_0^{\tau} u_x^2\left(0,t\right) dt. \qquad (3.6)$$

To obtain the estimate, we need the inequalities

$$u^2\left(\xi,t\right) \leqslant 2 \int_x^{\xi} u_x^2 \; dx + 2u^2$$

which easily follow from the equalities

$$u\left(\xi,t\right) = \int_x^{\xi} u_x\left(x,t\right) \; dx + u\left(x,t\right) \qquad \xi = 0 \text{ or } 1.$$

Also by (2.3) and (2.4), we obtain

$$\begin{aligned} &\int_0^{\tau} u_x\left(1,t\right) u\left(1,t\right) dt - \int_0^{\tau} u_x\left(0,t\right) u\left(0,t\right) dt \\ \leqslant \; &\frac{\varepsilon}{2} \int_0^{\tau} u^2\left(1,t\right) dt + \frac{1}{2\varepsilon} \int_0^{\tau} u_x^2\left(1,t\right) dt \\ &+ \frac{\varepsilon}{2} \int_0^{\tau} u^2\left(0,t\right) dt + \frac{1}{2\varepsilon} \int_0^{\tau} u_x^2\left(0,t\right) dt \\ \leqslant \; &\frac{\varepsilon}{2} \int_0^{\tau} \left[2 \int_x^0 u_x^2 \; dx + 2u^2 \right] dt + \frac{1}{2\varepsilon} \int_0^{\tau} \left[\int_0^1 K_0\left(x,t\right) u\left(x,t\right) dx \right]^2 dt \\ &+ \frac{\varepsilon}{2} \int_0^{\tau} \left[2 \int_x^1 u_x^2 \; dx + 2u^2 \right] dt + \frac{1}{2\varepsilon} \int_0^{\tau} \left[\int_0^1 K_1\left(x,t\right) u\left(x,t\right) dx \right]^2 dt \end{aligned}$$

So, by using Holder inequality, we have

$$\begin{aligned} & \int_0^\tau u_x(1,t)\, u(1,t)\, dt - \int_0^\tau u_x(0,t)\, u(0,t)\, dt \\ \leqslant\ & 2\varepsilon \int_{\Omega^\tau} u_x^2\ dxdt + 2\varepsilon \int_0^\tau u^2\ dt + \frac{K}{\varepsilon} \int_{\Omega^\tau} u^2\ dxdt; \end{aligned} \tag{3.7}$$

where the constant $K = \max_{i=\overline{0,1}} \int_{\Omega^\tau} K_i^2(x,t)\ dxdt$.

Now, we estimate $\int_{\Omega^\tau} \left[\int_0^1 a(t-s)\, g(s,u) \right] \cdot u\ dxdt$,

first we can find an constant C verify

$$\|g(s,u)\|_{L^2(\Omega)} \leqslant C_1 \|u\|_{L^2(\Omega)} + C_2 < C \|u\|_{L^2(\Omega)},\ C > 0.$$

Then, we get

$$\begin{aligned} & \int_{\Omega^\tau} \left[\int_0^1 a(t-s)\, g(s,u) \right] \cdot u\ dxdt \\ \leqslant\ & \frac{\varepsilon}{2} \int_{\Omega^\tau} u^2 dxdt + \frac{Ta_0}{2\varepsilon} \int_{\Omega^\tau} g^2 dxdt \\ \leqslant\ & \left(\frac{\varepsilon}{2} + \frac{TCa_0}{2\varepsilon} \right) \int_{\Omega^\tau} u^2 dxdt. \end{aligned} \tag{3.8}$$

Remains apply the inequality the Cauchy inequality with ε to the end terms of the right-hand side part of (3.4) and using (3.7) and (3.8), we get

$$\begin{aligned} & \frac{1}{2} \int_0^1 u(x,\tau)^2\ dx + \int_{\Omega^\tau} u_x^2\ dxdt \\ =\ & 2\varepsilon \int_{\Omega^\tau} u_x^2\ dxdt + 2\varepsilon \int_0^\tau u^2\ dt + \frac{K}{\varepsilon} \int_{\Omega^\tau} u^2\ dxdt + \frac{1}{2} \int_0^1 \varphi^2\ dx \\ & + \left(\frac{\varepsilon}{2} + \frac{TCa_0}{2\varepsilon} \right) \int_{\Omega^\tau} u^2 dxdt + \frac{\varepsilon}{2} \int_{\Omega^\tau} u^2 dxdt + \frac{1}{2\varepsilon} \int_{\Omega^\tau} f^2 dxdt \end{aligned} \tag{3.9}$$

Then, we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^1 u(x,\tau)^2\ dx + \int_{\Omega^\tau} u_x^2\ dxdt \\ =\ & 2\varepsilon \int_{\Omega^\tau} u_x^2\ dxdt + \left(3\varepsilon + \frac{TCa_0}{2\varepsilon} + \frac{K}{\varepsilon} \right) \int_{\Omega^\tau} u^2 dxdt \\ & + \frac{1}{2\varepsilon} \int_{\Omega^\tau} f^2 dxdt + \frac{1}{2} \int_0^1 \varphi^2\ dx \end{aligned}$$

Using Lemma 1 of Gronwall in [32] , we have

$$\begin{aligned} & \int_0^1 u(x,\tau)^2\ dx + \int_{\Omega^\tau} u_x^2\ dxdt \\ \leq\ & d \left(\int_\Omega f^2 dxdt + \int_0^1 \varphi^2 dx \right), \end{aligned} \tag{3.10}$$

where

$$d = \frac{1}{2} \exp\left(3\varepsilon + \frac{TCa_0}{2\varepsilon} + \frac{K}{\varepsilon} \right).$$

By integrating the inequality (3.10) over $(0, T)$, we obtain the desired inequality, where $c = (Td)^{\frac{1}{2}}$. So, we get

$$\begin{aligned} & \|u\|^2_{L^2(\Omega)} + \|u_x\|^2_{L^2(\Omega)} \\ \leqslant \quad & c\left(\|f\|^2_{L^2(\Omega)} + \|\varphi\|^2_{L^2(0,1)}\right). \end{aligned} \tag{3.11}$$

□

4. Existence of solution of the integrodifferential problem

This section is consecrated to the proof of the existence of the solution on the data of the problem $(2.1) - (2.4)$. we can reformulating the integro-differential problem to a semi-linear problem by putting

$$\int_0^t a(t-s)\, g(s,u)\, ds + f(x,t) = H(x,t,u)$$

Where exists a positive constant δ such that

$$\begin{aligned} |H(x,t,u_1) - H(x,t,u_2)| \quad &\leq \quad \delta\left(\|u_1 - u_2\|_{L^2(Q)}\right), \quad (\mathbf{C}^*) \\ \forall u_1, u_2 \quad &\in \quad L^2(\Omega), \qquad (x,t) \in \Omega. \end{aligned}$$

Therefore to study the existence of solution of previous problem $(2.1)-(2.4)$, is enough to study the following semi-linear problem:

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = H(x,t,u), \tag{4.1}$$

with the initial data

$$\ell u = u(x,0) = \varphi(x), \quad x \in (0,1), \tag{4.2}$$

with the Second Kind Integral Conditions

$$u_x(0,t) = \int_0^1 K_0(x,t)\, u(x,t)\, dx, \tag{4.3}$$

$$u_x(1,t) = \int_0^1 K_1(x,t)\, u(x,t)\, dx, \tag{4.4}$$

Let us consider the following auxiliary problem with homogeneous equation

$$\mathcal{L}w = \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} = 0, \tag{4.5}$$

$$\ell w = w(x,0) = \varphi(x), \tag{4.6}$$

$$w_x(0,t) = \int_0^1 K_0(x,t)\, w(x,t)\, dx, \tag{4.7}$$

$$w_x(1,t) = \int_0^1 K_1(x,t)\, w(x,t)\, dx, \tag{4.8}$$

If u is a solution of problem $(4.1)-(4.4)$ and w is a solution of problem $(4.5)-(4.8)$, then $y=u-w$ satisfies

$$\mathcal{L}y=\frac{\partial y}{\partial t}-\frac{\partial^2 y}{\partial x^2}=G\left(x,t,y\right), \tag{4.9}$$

$$\ell y=y(x,0)=0, \tag{4.10}$$

$$y_x\left(0,t\right) = 0, \tag{4.11}$$

$$y_x\left(1,t\right) = 0. \tag{4.12}$$

Where $G\left(x,t,y\right)=H\left(x,t,y+w\right)$, As the function H, the function G satisfies the condition $(\mathbf{C}^{*})$, that is there exists a positive constant δ such that

$$\begin{aligned} |G(x,t,y_1)-G(x,t,y_2)| &\leq \delta\left(\|y_1-y_2\|_{L^2(Q)}\right)(\mathbf{C}^{**}) \\ \forall y_1,y_2 &\in L^2(\Omega), \qquad (x,t)\in\Omega. \end{aligned}$$

To show the existence of solutions of the problem $(4.5)-(4.8)$, it is enough to transform the problem to the linear first-order ordinary differential equation with second member.

For that we integrate the equation (4.5) over $[0,1]$ and using $(4.7)-(4.8)$, we get

$$\int_0^1\frac{\partial w}{\partial t}dx=\int_0^1\left(K_1\left(x,t\right)-K_0\left(x,t\right)\right)w\left(x,t\right)dx,\ \forall x\in[0,1]\,;$$

then, we obtain

$$\int_0^1\left(\frac{\partial w}{\partial t}-K\left(x,t\right)w\left(x,t\right)\right)dx=0,\ \text{where } K_1\left(x,t\right)-K_0\left(x,t\right)=K\left(x,t\right). \tag{4.13}$$

So, we can prove that there existe a function ψ verify that

$$\frac{\partial w}{\partial t}-K\left(x,t\right)w\left(x,t\right)=\psi\left(x,t\right),\ \text{where } \int_0^1\psi\left(x,t\right)dx=0. \tag{4.14}$$

Clearly, that the solution of (4.5) by using (4.6) is given by

$$w\left(x,t\right)=\frac{\varphi\left(x\right)}{\exp 1}\exp\left(\int_0^t K\left(x,\theta\right)d\theta\right)+\exp\left(\int_0^t K\left(x,\theta\right)d\theta\right)\int_0^t\left[\psi(x,\tau)\exp\left(-\int_0^\tau K\left(x,\theta\right)d\theta\right)\right]d\tau.$$

Therefore, the existence of solution is guaranteed.

According to this results, we deduce that problem $(4.5)-(4.8)$ admits a unique solution. Therefore it remains to solve and prove that the problem $(4.9)-(4.12)$ has a unique weak solution.

Let us construct an iteration sequence in the following way: Starting with $y^{(0)}=0$, the sequence $\left\{y^{(n)}\right\}_{n\in N}$ is defined as follows: given the element $y^{(n-1)}$, then for $n=1,2,...$ solve the problem:

$$\frac{\partial y^{(n)}}{\partial t}-\frac{\partial^2 y^{(n)}}{\partial x^2}=G\left(x,t,y^{(n-1)}\right), \tag{4.15}$$

$$y^{(n)}(x,0)=0, \tag{4.16}$$

$$y_x^{(n)}(0,t) = 0, \tag{4.17}$$

$$y_x^{(n)}(1,t) = 0. \tag{4.18}$$

Clearly, for fixed n, each problem $(4.15)-(4.18)$ has a unique solution $y^{(n)}(x,t)$. If we set $Z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t)$, then we have the new problem

$$\frac{\partial Z^{(n)}}{\partial t} - \frac{\partial^2 Z^{(n)}}{\partial x^2} = P^{(n-1)}(x,t), \tag{4.19}$$

$$Z^{(n)}(x,0) = 0, \tag{4.20}$$

$$Z_x^{(n)}(0,t) = 0, \tag{4.21}$$

$$Z_x^{(n)}(1,t) = 0. \tag{4.22}$$

where

$$P^{(n-1)}(x,t) = G\left(x,t,y^{(n)}\right) - G\left(x,t,y^{(n-1)}\right).$$

Lemma 4.1. *Assume that condition* ($\mathbf{C}^{**}$) *holds, then for the linearized problem* $(4.19)-(4.22)$*, we have the a priori estimate*

$$\left\|Z^{(n)}\right\|_{L^2(0,T;\ H^1(0,1))} \leq M \left\|Z^{(n-1)}\right\|_{L^2(0,T;\ H^1(0,1))}, \tag{4.23}$$

where M is a positive constant given by

$$M = \sqrt{\frac{\frac{T}{\varepsilon}\delta^2}{\min\left(\frac{1-\varepsilon T}{2}, T\right)}}.$$

Proof. Multiplying the equation (4.19) by $Z^{(n)}$ and integrating over Ω^τ, where $\Omega^\tau = (0,1)\times(0,\tau)$, we get

$$\int_{\Omega^\tau} \frac{\partial Z^{(n)}}{\partial t}\cdot Z^{(n)} dxdt - \int_{\Omega^\tau} \frac{\partial^2 Z^{(n)}}{\partial x^2}\cdot Z^{(n)} dxdt = \int_{\Omega^\tau} P^{(n-1)}\cdot Z^{(n)} dxdt. \tag{4.24}$$

Integrating by parts the second term of the left-hand side in (4.24) and taking into account conditions $(4.20), (4.21)$ and (4.22), we obtain

$$\frac{1}{2}\int_0^1 \left(Z^{(n)}(x,\tau)\right)^2 dx + \int_{\Omega^\tau}\left(\frac{\partial Z^{(n)}}{\partial x}\right)^2 dxdt = \int_{\Omega^\tau} P^{(n-1)}\cdot Z^{(n)} dxdt. \tag{4.25}$$

Using the Cauchy inequality to the right-hand side of (4.25), we get

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left(Z^{(n)}(x,\tau)\right)^2 dx + \int_{\Omega^\tau}\left(\frac{\partial Z^{(n)}}{\partial x}\right)^2 dxdt \\ \leq\ &\frac{1}{2}\int_{\Omega^\tau}\left(P^{(n-1)}\right)^2 dxdt + \frac{1}{2}\int_{\Omega^\tau}\left(Z^{(n)}\right)^2 dxdt. \end{aligned}$$

Using Lemma of Gronwall, we obtain

$$\begin{aligned} &\int_0^1 \left(Z^{(n)}(x,\tau)\right)^2 dx + \int_{\Omega^\tau}\left(\frac{\partial Z^{(n)}}{\partial x}\right)^2 dxdt \\ \leq\ &\exp(T)\int_{\Omega^\tau}\left(P^{(n-1)}\right)^2 dxdt. \end{aligned} \tag{4.26}$$

On the other hand., by virtue of condition $(\mathbf{C}^{**})$, we have

$$\begin{aligned} & \left(\int_{\Omega^{\tau}}\left(P^{(n-1)}\right)^{2}dxdt\right) \qquad (4.27)\\ \leq\ & \delta^{2}\int_{\Omega^{\tau}}\left(\left|Z^{(n-1)}(x,t)\right|+\left|\frac{\partial Z^{(n-1)}(x,t)}{\partial x}\right|\right)^{2}dxdt\\ \leq\ & 2\delta^{2}\int_{0}^{\tau}\left(\left\|Z^{(n-1)}(\bullet,t)\right\|_{L^{2}(0,1)}^{2}+\left\|\frac{\partial Z^{(n-1)}(\bullet,t)}{\partial x}\right\|_{L^{2}(0,1)}^{2}\right)dt. \end{aligned}$$

Substituting (4.27) into (4.26), we get

$$\begin{aligned} & \int_{0}^{1}\left(Z^{(n)}(x,\tau)\right)^{2}dx+\int_{\Omega^{\tau}}\left(\frac{\partial Z^{(n)}}{\partial x}\right)^{2}dxdt\\ \leq\ & 2\delta^{2}\exp(T)\int_{0}^{T}\left(\left\|Z^{(n-1)}(\bullet,t)\right\|_{L^{2}(0,1)}^{2}+\left\|\frac{\partial Z^{(n-1)}(\bullet,t)}{\partial x}\right\|_{L^{2}(0,1)}^{2}\right)dt. \end{aligned}$$

The right hand side here is independent of τ; hence, replacing the left hand side by the upper bound with respect to τ, we obtain

$$\begin{aligned} & \int_{0}^{1}\left(Z^{(n)}(x,\tau)\right)^{2}dx+\int_{\Omega^{T}}\left(\frac{\partial Z^{(n)}}{\partial x}\right)^{2}dxdt\\ \leq\ & 2\delta^{2}\exp(T)\int_{0}^{T}\left(\left\|Z^{(n-1)}(\bullet,t)\right\|_{L^{2}(0,1)}^{2}+\left\|\frac{\partial Z^{(n-1)}(\bullet,t)}{\partial x}\right\|_{L^{2}(0,1)}^{2}\right)dt. \end{aligned}$$

Now by integrating over $(0,T)$, we get

$$\begin{aligned} & \int_{\Omega^{T}}\left(Z^{(n)}\right)^{2}dx+T\int_{\Omega^{T}}\left(\frac{\partial Z^{(n)}}{\partial x}\right)^{2}dxdt\\ \leq\ & 2T\delta^{2}\exp(T)\int_{0}^{T}\left(\left\|Z^{(n-1)}(\bullet,t)\right\|_{L^{2}(0,1)}^{2}+\left\|\frac{\partial Z^{(n-1)}(\bullet,t)}{\partial x}\right\|_{L^{2}(0,1)}^{2}\right)dt. \end{aligned}$$

So, we obtain

$$\begin{aligned} & \int_{0}^{T}\left(\left\|Z^{(n)}(\bullet,t)\right\|_{L^{2}(0,1)}^{2}+\left\|\frac{\partial Z^{(n)}(\bullet,t)}{\partial x}\right\|_{L^{2}(0,1)}^{2}\right)dt\\ \leq\ & \frac{2T\delta^{2}\exp(T)}{\min(1,T)}\int_{0}^{T}\left(\left\|Z^{(n-1)}(\bullet,t)\right\|_{L^{2}(0,1)}^{2}+\left\|\frac{\partial Z^{(n-1)}(\bullet,t)}{\partial x}\right\|_{L^{2}(0,1)}^{2}\right)dt \end{aligned}$$

Finally, we find

$$\left\|Z^{(n)}\right\|_{L^{2}(0,T;\,H^{1}(0,1))}^{2}\leq M\left\|Z^{(n-1)}\right\|_{L^{2}(0,T;\,H^{1}(0,1))}^{2}, \qquad (4.28)$$

where

$$M=\frac{2T\delta^{2}\exp(T)}{\min(1,T)}.$$

From the criteria of convergence of series, we see that the series $\sum_{n=1}^{\infty}Z^{(n)}$ converges if $M<1$, that is if

$$\delta<\sqrt{\frac{\min(1,T)}{2T\exp(T)}}.$$

Since $Z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t)$, then it follows that the sequence $(y^{(n)})_{n\in N}$ defined by

$$y^{(n)}(x,t) = \sum_{i=0}^{n-1} Z^{(i)} + y^{(0)}(x,t),$$

converges to an element $y \in L^2\left(0,T;\ H^1(0,1)\right)$. □

Remains to precise the concept of the solution we are considering. Let $v = v(x,t)$ be any function from $C^1(\Omega)$.

We shall compute the integral $\int_\Omega Gvdxdt$, for this we assume $v_x(0,t) = v_x(1,t) = 0$. By using conditions on y, we have

$$-\int_\Omega \frac{\partial^2 y}{\partial x} vdxdt = \int_\Omega \frac{\partial v}{\partial x}\frac{\partial y}{\partial x} dxdt.$$

Then we put

$$A(y,v) = \int_\Omega \frac{\partial y}{\partial t} vdxdt + \int_\Omega \frac{\partial v}{\partial x}\frac{\partial y}{\partial x} dxdt = \int_\Omega vGdxdt, \tag{4.29}$$

Definition 4.1. *For every $v \in C^1(\Omega)$, a function $y \in L^2(0,T;\,H^1(0,1))$ is called a weak solution of problem* $(4.9) - (4.12)$ *if* (4.30) *holds under the conditions of y.*

Now, we must show that the limit function y is a solution of the problem under study. To do this, we will show that y verifies (4.30) as mentioned in definition 1. So, we consider the weak formulation of problem $(4.9) - (4.12)$:

$$A(y,v) = \int_\Omega vGdxdt. \tag{4.30}$$

From (4.30), we have

$$\begin{aligned} & A\left(y^{(n)},v\right) \\ = \; & A\left(y^{(n)} - y, v\right) + A(y,v) \\ = \; & \int_\Omega v\left[G\left(x,t,y^{(n-1)}\right) - G(x,t,y,)\right] dxdt \\ & \int_\Omega vG(x,t,y)\,dxdt. \end{aligned} \tag{4.31}$$

However, we apply Holder inequality, we get

$$\begin{aligned} & A\left(y^{(n)} - y, v\right) \\ = \; & \int_\Omega v\left[G\left(x,t,y^{(n-1)}\right) - G(x,t,y)\right] dxdt \\ \le \; & \frac{\delta}{2}\|v\|_{L^2(\Omega)}\left\|y^{(n)} - y\right\|_{L^2(\Omega)}. \end{aligned} \tag{4.32}$$

so by passing to the limit in (4.33) as $n \to \infty$, (4.31) become

$$A\left(y^{(n)},v\right) = \int_\Omega vG(x,t,y)\,dxdt. \tag{4.33}$$

Again passing to the limit in (4.31) as $n \to \infty$, we obtain

$$A(y,v) = \int_{\Omega} vG(x,t,y)\,dxdt.$$

Therefore, we have established the following result:

Theorem 4.1. *Assume that condition* (H_2) *holds and*

$$\delta < \sqrt{\frac{\min\left(\frac{1-\varepsilon T}{2}, T\right)}{\frac{T}{\varepsilon}}}$$

then the problem $(4.9)-(4.12)$ *admits a weak solution in* $L^2\left(0,T;\ H^1(0,1)\right).$

It remains to prove that problem $(4.9)-(4.12)$ admits a unique solution.

Theorem 4.2. *Under the condition* $(\mathbf{C}^{**})$, *the solution of the problem* $(4.9)-(4.12)$ *is unique.*

Proof. Suppose that y_1 and y_2 in $L^2\left(0,T;\ H^1(0,1)\right)$ are two solution of $(4.9)-(4.12)$, then $h = y_1 - y_2$ satisfies $h \in L^2\left(0,T;\ H^1(0,1)\right)$ and

$$\frac{\partial h}{\partial t} - \frac{\partial^2 h}{\partial x^2} = \psi(x,t) \qquad (x,t) \in \overline{\Omega}, \tag{4.34}$$

$$h(x,0) = 0, \tag{4.35}$$

$$h_x(0,t) = 0, \tag{4.36}$$

$$h_x(1,t) = 0, \tag{4.37}$$

$$\psi(x,t) = G(x,t,y_1) - G(x,t,y_2).$$

Following the same procedure done in establishing the proof of Lemma 1, then for the problem $(4.35)-(4.38)$, we get

$$\|h\|_{L^2(0,T;\ H^1(0,1))} \leq M\,\|h\|_{L^2(0,T;\ H^1(0,1))}. \tag{4.38}$$

Since $M < 1$, then from (4.39) that

$$(1-M)\,\|h\|_{L^2(0,T;\ H^1(0,1)} \leq 0,$$

from which we conclude that $y_1 = y_2$ in $L^2\left(0,T;\ H^1(0,1)\right).$ $\square$

5. Construction of approximate solutions

In order to solve the problem (2.1) – (2.4) , first we divide the time interval $[0, T]$ into $N \in \mathbb{N}$ equidistant subintervals (t_{j-1}, t_j) for $t_j = j\tau$, where $\tau = \frac{T}{N}$We introduce the following notation

$$u_j = u_j(x) = u(x, t_j),$$

after replacing the derivative $\dfrac{\partial u}{\partial t}$ by backward finite difference approximations $\dfrac{u_j - u_{j-1}}{\tau}$ and the integral by rectangular rule. Then problem (2.1) – (2.4) reduced to the solutions of recurrent system of ODE problems at each successive time point t_j for $j = 1, ..., N$ find, successively for $j = 1, ..., N$; functions $u_j : (0, 1) \to \mathbb{R}$ such that:

$$\frac{u_j - u_{j-1}}{\tau} - \frac{d^2 u_j}{d^2 x} = \tau \sum_{k=0}^{j-1} a(t_j - t_k) g(t_k, u_k) + f(x, t_j) \quad x \in (0, 1) \tag{5.1}$$

$$\frac{du_j}{dx}(0) = \int_0^1 K_0(x, t_j) u(x, t_j) dx \tag{5.2}$$

$$\frac{du_j}{dx}(1) = \int_0^1 K_1(x, t_j) u(x, t_j) dx \tag{5.3}$$

$$u_0(x) = \varphi(x) \quad x \in (0, 1) \tag{5.4}$$

The main numerical difficulty become visible after the full discretization of these nonlocal problem , the presence of an integral BC in the problem gives rise to rows, which are full(see Algorithm 1).

5.1. **Algorithm 1 :(A1).** For The space discretization we use the finite differences scheme . we divide the space interval $[0, 1]$ into $M \in \mathbb{N}$ equidistant subintervals of equal lengths $h = \frac{1}{M}$ second-order difference is used to approximate the second order spatial derivative :

$$\frac{\partial^2 u_{i,j}}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2),$$

where $u_{i,j} = u(x_i, t_j)$, and employing central-differences to approximat the first order spatial derivative in the boundary condition :

$$\frac{\partial u_{i,j}}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2),$$

we construct a difference scheme for the problem (5.1)-(5.4):

$$u_{i,j} - \tau \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} = u_{i,j-1} + \tau^2 \sum_{k=0}^{j-1} a(t_j - t_k) g(t_k, u_{i,k}) + \tau f_{i,j} \quad , i = 0, ..., M \tag{5.5}$$

$$\begin{aligned} \frac{u_{1,j}-u_{-1,j}}{2h} &= \int_0^1 K_0(x,t_j)u(x,t_j)dx && (5.6)\\ \frac{u_{M+1,j}-u_{M-1,j}}{2h} &= \int_0^1 K_1(x,t_j)u(x,t_j)dx && (5.7)\\ u_{i,0} &= \varphi_i \qquad\qquad i=0,...,M \end{aligned}$$

after some rearrangement, the Equation (5.5) becomes :

$$-ru_{i-1,j}+(1+2r)u_{i,j}-ru_{i+1,j} = u_{i,j-1}+\tau^2\sum_{k=0}^{j-1}a(t_j-t_k)g(t_k,u_{i,k}) \tag{5.8}$$
$$+\tau f_{i,j}\ ,\ i=0,...,M$$

where $r=\frac{\tau}{h^2}$. we approximate the integral in (5.6)-(5.7) numerically by the trapezoidal numerical integration rule:

$$\frac{u_{1,j}-u_{-1,j}}{2h} = \int_0^1 K_0(x,t_j)u(x,t_j)dx \tag{5.9}$$
$$= \frac{h}{2}(K_0(x_0,t_j)u_{0,j}+2\sum_{k=1}^{M-1}K_0(x_k,t_j)u_{k,j}+K_0(x_M,t_j)u_{M,j})$$

$$\frac{u_{M+1,j}-u_{M-1,j}}{2h} = \int_0^1 K_1(x,t_j)u(x,t_j)dx \tag{5.10}$$
$$= \frac{h}{2}(K_1(x_0,t_j)u_{0,j}+2\sum_{k=1}^{M-1}K_1(x_k,t_j)u_{k,j}+K_1(x_M,t_j)u_{M,j})$$

which is the same second-order of accuracy in space as the methods used for spatial derivative . Equation (5.9) presents $M+1$ linear equations in $M+3$ unknowns $u_{-1},u_0,...,u_{M+1}$. Eliminating of the "fictitious" value $u_{-1,j}$ beteween $(5.8)_{i=0}$ and (5.9) gives :

$$\begin{aligned} &(1+2r+\tau K_0(x_0,t_j))u_{0,j}+(-2r+2\tau K_0(x_1,t_j))u_{1,j}\\ &+2\tau K_0(x_2,t_j)u_{2,j}+...+2\tau K_0(x_{M-1},t_j)u_{M-1,j}+\tau K_0(x_M,t_j)u_{M,j}\\ &=u_{0,j-1}+\tau^2\sum_{k=0}^{j-1}a(t_j-t_k)g(t_k,u_{0,k})+\tau f_{0,j} \end{aligned} \tag{5.11}$$

Similarly, eliminating $u_{M+1,j}$ beteween $(5.8)_{i=M}$ and (5.10) gives :

$$\begin{aligned} &-\tau K_1(x_0,t_j)u_{0,j}-2\tau K_1(x_1,t_j))u_{1,j}-...-2\tau K_1(x_{M-2},t_j)u_{M-2,j}\\ &+(-2r-2\tau K_1(x_{M-1},t_j))u_{M-1,j}+(1+2r-\tau K_1(x_M,t_j))u_{M,j}\\ &=u_{M,j-1}+\tau^2\sum_{k=0}^{j-1}a(t_j-t_k)g(t_k,u_{M,k})+\tau f_{M,j} \end{aligned} \tag{5.12}$$

Combining (5.11), (5.9), with (5.12) yields an $(M+1)\times(M+1)$ linear system of equations whose coefficient matrix A^j has the form:

$$A^j = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \ldots\ldots & a_{0M} \\ -r & 1+2r & -r & \ldots\ldots & 0 \\ . & . & . & \ldots\ldots & . \\ 0 & \ldots\ldots & -2r & 1+2r & -2r \\ a_{M0} & a_{M1} & a_{M2} & \ldots\ldots & a_{MM} \\ \end{pmatrix}$$

where $a_{00}, a_{01}, ..., a_{0M}$ and $a_{M0}, a_{M1}, ..., a_{MM}$ are the coefficients in (5.11) and (5.12), respectively. We will denote the right-side of the system by $b^j = (b_0, b_1, ..b_M)^T$,
with $b_i = u_{i,j-1} + \tau^2 \sum_{k=0}^{j-1} a(t_j - t_k) g(t_k, u_{i,k}) + \tau f_{i,j}, i = 0, ..., M$. We write the system in the matrix form
:

$$A^j U^j = b^j \tag{5.13}$$

which have to be solved successively with increasing time step $j = 1, .., N$. The main numerical problem is the special character of the algebraic matrix obtained, tridiagonal except that their first and last rows are full,this needs a special solver to get a result. But there exist a simple way how to avoid this complication, we explain it in algorithm 2.

5.2. **Algorithm 2:(A2).** To get rid of the nonlocal BC,we make use of a slightly modified idea of [37], for any given j we introduce three auxiliary problems. The first one with an unknown function v_i is given as:

$$\begin{cases} v_j - \tau \dfrac{d^2 v_j}{d^2 x} = & u_{j-1} + \tau^2 \sum_{k=0}^{j-1} a(t_j - t_k) g(t_k, u_k) \\ \qquad\qquad + & \tau f(x, t_j) \quad x \in (0,1) \\ \dfrac{dv_j}{dx}(0) = 0 & \\ \dfrac{dv_j}{dx}(1) = 0 & \end{cases} \tag{5.14}$$

and the initial condition $\qquad v_0(x) = \varphi(x) \qquad , x \in [0,1]$.
The second one with the unknown z reads as:

$$\begin{cases} z - \tau \dfrac{d^2 z}{d^2 x} = 0 \quad x \in (0,1) \\ \dfrac{dz}{dx}(0) = 1 \\ \dfrac{dz}{dx}(1) = 0 \end{cases} \tag{5.15}$$

The third one with the unknown w reads as

$$\begin{cases} w - \tau \dfrac{d^2 w}{d^2 x} = 0 \quad x \in (0,1) \\ \dfrac{dw}{dx}(0) = 0 \\ \dfrac{dw}{dx}(1) = 1 \end{cases} \tag{5.16}$$

Let us note that the temporary problems are standard problems.

Let α_j and β_j be any real number, the principle of linear superposition gives that $\omega_j := v_j + \alpha_j z + \beta_j w$ is the solution to the following BVP

$$\begin{cases} \omega_j - \tau \dfrac{\partial^2 \omega_j}{\partial^2 x} = & u_{j-1} + \tau^2 \sum_{k=0}^{j-1} a(t_j - t_k) g(t_k, u_k) \\ & +\tau f(x, t_j) \quad x \in (0,1) \\ \dfrac{d\omega_j}{dx}(0) = \alpha_j \\ \dfrac{d\omega_j}{dx}(1) = \beta_j \end{cases} \tag{5.17}$$

and the initial condition $\omega_0(x) = \varphi(x) \qquad , x \in [0,1]$.

We have to pick up the appropriate value of the free parameter α_j and β_j for which the function ω_j be a solution to problem (5.1)-(5.4). We are looking for an α_j and β_j such that

$$\begin{aligned} \alpha_j &= \int_0^1 K_0(x,t_j) u(x,t_j) dx = \int_0^1 K_0(x,t_j) \left(v_j + \alpha_j z(x) + \beta_j w(x) \right) dx \\ \beta_j &= \int_0^1 K_1(x,t_j) u(x,t_j) dx = \int_0^1 K_1(x,t_j) \left(v_j + \alpha_j z(x) + \beta_j w(x) \right) dx \end{aligned}$$

then ω_j will be a solution to problem $(5.1)-(5.3)$ if and only if the pair (α_j, β_j) is a solution of the following system of equations

$$\begin{cases} \alpha_j (1 - \int_0^1 K_0(x,t_j) z dx) - \beta_j \int_0^1 K_0(x,t_j) w dx = \int_0^1 K_0(x,t_j) v_j dx \\ -\alpha_j \int_0^1 K_1(x,t_j) z dx + \beta_j (1 - \int_0^1 K_1(x,t_j) w dx) = \int_0^1 K_1(x,t_j) v_j dx \end{cases} \tag{5.18}$$

we have to check if the determinant

$$D = (1 - \int_0^1 K_0(x,t_j) z dx)(1 - \int_0^1 K_1(x,t_j) w dx) - \int_0^1 K_0(x,t_j) w dx \int_0^1 K_1(x,t_j) z dx \tag{5.19}$$

of system (5.18) is different from zero.

if $D \neq 0$ then we easily deduce :

$$\begin{cases} \alpha_j = \dfrac{(1 - \int_0^1 K_1(x,t_j) w dx) \int_0^1 K_0(x,t_j) v_j dx + \int_0^1 K_1(x,t_j) v_j dx \int_0^1 K_0(x,t_j) w dx}{D} \\ \beta_j = \dfrac{\alpha_j (1 - \int_0^1 K_0(x,t_j) z dx) - \int_0^1 K_0(x,t_j) v_j dx}{\int_0^1 K_0(x,t_j) w dx} \end{cases} \tag{5.20}$$

Lemma 5.1. *Let* $D = (1 - \int_0^1 K_0(x,t_j)zdx)(1 - \int_0^1 K_1(x,t_j)w_idx) - \int_0^1 K_0(x,t_j)wdx \int_0^1 K_1(x,t_j)zdx$ *there exists* $\tau_0 > 0$ *such that* $D \geqslant \frac{1}{2}$

Proof. One can see that the solution of the second auxiliary problem is :

$$z(x) = \frac{-\sqrt{\tau}ch(\frac{x-1}{\sqrt{\tau}})}{sh(\frac{1}{\sqrt{\tau}})}$$

we have

$$\lim_{\tau\to 0} z(x) = \lim_{\tau\to 0} \frac{-\sqrt{\tau}ch(\frac{x-1}{\sqrt{\tau}})}{sh(\frac{1}{\sqrt{\tau}})} = \lim_{\tau\to 0} \frac{-\sqrt{\tau}(e^{\frac{x-2}{\sqrt{\tau}}} + e^{\frac{-x}{\sqrt{\tau}}})}{e^{\frac{2}{\sqrt{\tau}}} - 1} = 0$$

and the solution of the third auxiliary problem is :

$$w(x) = \frac{\sqrt{\tau}ch(\frac{x}{\sqrt{\tau}})}{sh(\frac{1}{\sqrt{\tau}})}$$

and also

$$\lim_{\tau\to 0} w(x) = \lim_{\tau\to 0} \frac{\sqrt{\tau}ch(\frac{x}{\sqrt{\tau}})}{sh(\frac{1}{\sqrt{\tau}})} = \lim_{\tau\to 0} \frac{\sqrt{\tau}(e^{\frac{x-1}{\sqrt{\tau}}} + e^{\frac{-x-1}{\sqrt{\tau}}})}{e^{\frac{2}{\sqrt{\tau}}} - 1} = 0$$

The variational formulations of temporary problems are:

$$(z, \Phi) + \tau(\frac{dz}{dx}, \frac{d\Phi}{dx}) = -\tau\Phi(0), \quad \textit{for any } \Phi \in H^1(0,1) \tag{5.21}$$

we set $\Phi = z$ into (5.21) and we get

$$\|z\|^2 + \tau\|\frac{dz}{dx}\|^2 = -\tau\Phi(0) = \tau\sqrt{\tau}coth(\frac{1}{\sqrt{\tau}}) \leq C$$

analogously for w

$$(w, \Phi) + \tau(\frac{dw}{dx}, \frac{d\Phi}{dx}) = \tau\Phi(1), \textit{for any } \Phi \in H^1(0,1) \tag{5.22}$$

we set $\Phi = w$ into (5.22) and we get

$$\|w\|^2 + \tau\|\frac{dw}{dx}\|^2 = \tau\Phi(1) = \tau\sqrt{\tau}coth(\frac{1}{\sqrt{\tau}}) \leq C$$

Lebegue domineted theorem says :

$$\lim_{\tau\to 0}\|z\|^2 = \lim_{\tau\to 0}\int_0^1 z^2 = \int_0^1 \lim_{\tau\to 0} z^2 = 0$$

analogously for w

$$\lim_{\tau\to 0}\|w\|^2 = 0$$

cauchy inequality says

$$|\int_0^1 K_0 z| \leq \|K_0\|\|z\| \to 0 \text{ when } \tau \to 0$$

$$|\int_0^1 K_1 w| \leq \|K_1\|\|w\| \to 0 \text{ when } \tau \to 0$$

therefor

$$\lim_{\tau\to 0} D = 1$$

from the definition of a limit we easily arrive at for $\varepsilon = 1/2$ there exists a τ_0 such that: for any $0 < \tau < \tau_0$ we have$D \geqslant 1/2$. □

For the space discretization we use the same scheme in algorithm 1 for a better comparison. We construct a difference scheme for the first auxiliary problem (5.14) :

$$\begin{aligned} v_{i,j} - \tau \frac{v_{i-1,j} - 2v_{i,j} + v_{i+1,j}}{h^2} =& u_{i,j-1} + \tau^2 \sum_{k=0}^{j-1} a(t_j - t_k) g(t_k, u_{i,k}) \\ &+ \tau f_{i,j} \quad ,i \ = 0, ..., M \end{aligned} \tag{5.23}$$

$$\frac{v_{1,j} - v_{-1,j}}{2h} = 0 \tag{5.24}$$

$$\frac{v_{M+1,j} - v_{M-1,j}}{2h} = 0 \tag{5.25}$$

$$v_{i,0} = \varphi_i \qquad i = 0, ..., M$$

after some rearrangement, the Equation (5.23) becomes :

$$-rv_{i-1,j} + (1 + 2r)v_{i,j} - rv_{i+1,j} = u_{i,j-1} + \tau^2 \sum_{k=0}^{j-1} a(t_j - t_k) g(t_k, u_{i,k}) + \tau f_{i,j} \ , \ i = 0, M \tag{5.26}$$

where $r = \frac{\tau}{h^2}$.There are $M + 1$ linear equations in $M + 3$ unknowns $v_{-1,j}, v_{0,j}, ..., v_{M+1;j}$ Eliminating of the "fictitious" value $v_{-1,j}$ beteween $(5.23)_{i=0}$ and (5.24) gives :

$$(1 + 2r)v_{0,j} - 2rv_{1,j} = u_{0,j-1} + \tau^2 \sum_{k=0}^{j-1} a(t_j - t_k) g(t_k, u_{0,k}) + \tau f_{0,j}, \tag{5.27}$$

Eliminating $v_{M+1,j}$ beteween $(5.23)_{i=M}$ and (5.25) gives :

$$-2rv_{M-1,j} - (1 + 2r)v_{M,j} = u_{M,j-1} + \tau^2 \sum_{k=0}^{j-1} a(t_j - t_k) g(t_k, u_{M,k}) + \tau f_{M,j}, \tag{5.28}$$

Combining (5.27), (5.26), with (5.27) yields an $(M + 1) \times (M + 1)$ linear system of equations, we write the system in the matrix form :

$$A^j V^j = B^j \quad j = 1, n \tag{5.29}$$

where

$$A^j = \begin{pmatrix} 1 + 2r & -2r & 0 & ...... & 0 \\ -r & 1 + 2r & -r & ...... & 0 \\ . & . & . & ...... & . \\ 0 & 0 & ..... & -2r & 1 + 2r \end{pmatrix}$$

$$B^j = \begin{pmatrix} u_{0,j-1} + \tau^2 \sum_{k=0}^{j-1} a(t_j - t_k) g(t_k, u_{0,k}) + \tau f_{0,j} \\ u_{1,j-1} + \tau^2 \sum_{k=0}^{j-1} a(t_j - t_k) g(t_k, u_{1,k}) + \tau f_{1,j} \\ . \\ u_{M,j-1} + \tau^2 \sum_{k=0}^{j-1} a(t_j - t_k) g(t_k, u_{M,k}) + \tau f_{M,j} \end{pmatrix}$$

and

$$V^j = \begin{pmatrix} v_{0,j} \\ v_{1,j} \\ . \\ v_{M,j} \end{pmatrix}$$

Then at each time level, the difference scheme can be written as systems of $M+1$ tridiagonal linear algebraic equations, which is solved by Thomas' algorithm. After that computing the value of α_j and β_j from equation (5.20) . The integrals are approximated by the composite trapezoidal rule:

$$\int_{x_0}^{x_M} f(x)dx = \frac{h}{2}[f(x_0) + f(x_M) + 2\sum_{i=1}^{M-1} f(x_i)] + O(h^2)$$

then the approximative solution of (2.1)-(2.4) is obtained by:

$$u_{i,j} = v_{i,j} + \alpha_j z_i + \beta_j w_i, i = 0..., M, j = 1, ..., N.$$

6. Numerical experiment

To test the above algorithms , we use two examples as follows:

Example 1. Consider (2.1) – (2.4) in $\Omega = (0,1) \times (0,1)$, with

$$\begin{aligned} a(t-s) &= (t-s)^2 \\ g(t, u(x,t)) &= 2u(x,t) \\ f(x,t) &= -(x(x-1)-2)(-3e^{-t} - 4t + 2t^2 + 4) - 2e^{-t} \\ K_0(x,t) &= \frac{6}{13} \\ K_1(x,t) &= -\frac{6}{13} \\ \varphi(x) &= x(x-1) - 2 \end{aligned}$$

It is easy to check that the exact solution of this test problem is

$$u^*(x,t) = (x(x-1)-2)e^{-t}$$

Algorithm	A1	A2	A1	A2	A1	A2
M \ (x,t)	(0.2,0.5)	(0.2,0.5)	(0.6,0.5)	(0.6,0.5)	(1,0.5)	(1,0.5)
20	-1.3206835	-1.3206835	-1.3695993	-1.3695993	-1.2228648	-1.2228648
40	-1.3153454	-1.3153454	-1.3640651	-1.3640651	-1.2179125	-1.2179125
80	-1.3127135	-1.3127135	-1.3613348	-1.3613348	-1.2154743	-1.2154743
160	-1.3114068	-1.3114068	-1.3599787	-1.3599787	-1.2142647	-1.2142647
320	-1.3107557	-1.3107557	-1.3593029	-1.3593029	-1.2136622	-1.2136622
640	-1.3104308	-1.3104308	-1.3589656	-1.3589656	-1.2133615	-1.2133615
1280	-1.3102684	-1.3102684	-1.3587971	-1.3587971	-1.2132114	-1.2132114
$u^*(x,0.05)$	-1.3101060	-1.3101060	-1.3586290	-1.3586290	-1.2130610	-1.2130610

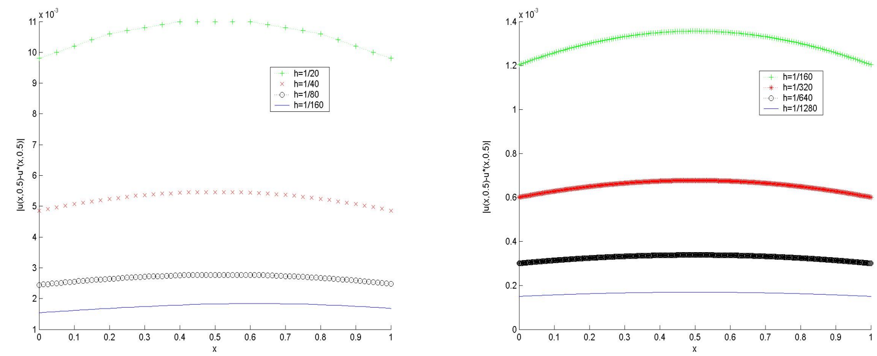
TABLE 1. Some numerical results at $t = 0.5$ with $\tau = \dfrac{h}{2}$ for Example 1.

Algorithm	A1 and A2	A1 and A2	A1 and A2	A1	A2
M \ (x,t)	(0.2,0.5)	(0.6,0.5)	(1.0,0.5)	CPU time (s)	CPU time (s)
20	0.0105773	0.0109706	0.0098035	0.265	0.218
40	0.0052392	0.0054364	0.0048512	0.733	0.53
80	0.0026073	0.0027061	0.0024130	1.982	1.7
160	0.0013006	0.00135003	0.0012034	6.896	6.631
320	0.0006495	0.0006742	0.0006009	30.152	25.787
640	0.0003245	0.0003369	0.0003002	183.41	114.48
1280	0.0001500	0.0001684	0.0001622	1309.21	438.62

TABLE 2. The absolute errors of some numerical solutions at $t = 0.5$ with $\tau = \dfrac{h}{2}$ and CPU-times for Example 1.

M	10	20	40	80	160
N	40	160	640	2560	10240
$\|u - u_{h\tau}\|_\infty$	2.7724e-2	6.8285e-3	1.7008e-3	4.2479e-04	1.0617e-04

TABLE 3. The maximum errors of the numerical solutions for Example 1.



(a) (b)

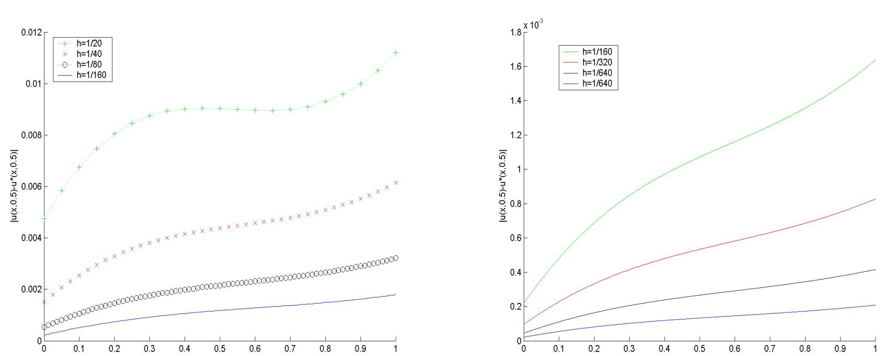
FIGURE 1. The errors of the numerical solutions at t=0.5 for example1.

Example 2. Now, consider problem (2.1) − (2.4) in$\Omega = (0, 1) \times (0, 1)$ with

$$\begin{aligned}
a(t-s) &= e^{(t-s)} \\
g(t, u(x,t)) &= u(x,t) + 3 \\
f(x,t) &= e^t(\pi^2 cos(\pi x) + (cos(\pi x) + x)(1-t) - 3) + 3 \\
K_0(x,t) &= \frac{1+\pi^2}{\pi^2 - e} e^x \\
K_1(x,t) &= \frac{1-\pi^2}{-sin(1)\pi^2 + (1-\pi^2)(cos(1)-1)} cos(x) \\
\varphi(x) &= cos(\pi x) + x
\end{aligned}$$

It is easy to check that the exact solution of this test problem is

$$u^*(x,t) = (cos(\pi x) + x)e^t$$



(a) (b)

FIGURE 2. The errors of the numerical solutions at t=0.5 for example2.

Algorithm	A1	A2	A1	A2	A1	A2
M \ (x,t)	(0.2,0.5)	(0.2,0.5)	(0.6,0.5)	(0.6,0.5)	(1,0.5)	(1,0.5)
20	1.6716326	1.6716326	0.4887113	0.4887113	0.0112091	0.0112091
40	1.6668822	1.6668822	0.4843230	0.4843230	0.0061441	0.0061441
80	1.6650529	1.6650529	0.4820596	0.4820596	0.0032070	0.0032070
160	1.6642748	1.6642748	0.4809105	0.4809105	0.0016372	0.0016372
320	1.6639199	1.6639199	0.4803316	0.4803316	0.0008270	0.0008270
640	1.6637510	1.6637510	0.4800411	0.4800411	0.0004156	0.0004156
1280	1.6636686	1.6636686	0.4798955	0.4798955	0.0002083	0.0002083
$u^*(x, 0.05)$	1.6635880	1.6635880	0.4797498	0.4797498	0.0000000	0.0000000

TABLE 4. Some numerical results at $t = 0.5$ with $\tau = \frac{h}{2}$ a for Example 2.

Algorithm	A1 and A2	A1 and A2	A1 and A2	A1	A2
M \ (x,t)	(0.2,0.5)	(0.6,0.5)	(1.0,0.5)	CPU time (s)	CPU time (s)
20	0.0080448	0.0089614	0.0112091	0.22	0.19
40	0.0032945	0.0045731	0.0061441	0.54	0.49
80	0.0026073	0.0027061	0.0024130	1.63	1.57
160	0.0013006	0.0013500	0.0012034	6.80	6.15
320	0.0006495	0.0006742	0.0006009	29.53	24.87
640	0.0001632	0.0002912	0.0004156	169.47	109.80
1280	0.0000808	0.0001457	0.0002083	1271.96	464.24

TABLE 5. The absolute errors of some numerical solutions at $t = 0.5$ with $\tau = \frac{h}{2}$ and CPU-times for Example 2.

M	10	20	40	80	160
N	40	160	640	2560	10240
$\|u - u_{h\tau}\|_\infty$	3.8541e-2	9.5205e-3	2.3728e-3	5.9274e-04	1.4815e-04

TABLE 6. The maximum errors of the numerical solutions for Example 2.

Our numerical experiment are performed using Matlab and we used an Intel Core i3 with 2.1 GHz . Table 1 and table 4 gives some numerical results and exact values at some points at the time $t = 0.5$. Table 2 and table 5 gives the absolute errors of the numerical solutions at some points at the time $t = 0.5$, and this is also shown in figure 1 and figure 2. Table 3 and table 6 gives the maximum errors of the numerical solutions. The maximum error is defined as follows:

$$e(h,\tau) = \|u - u_{h\tau}\|_\infty = \underset{0\leq i\leq M}{max}\{\underset{0\leq j\leq N}{max} u(x_i,t_j) - u_{ij}\}$$

The results obtained using algorithm 1 and algorithm 2 have the same accuracy . It is also noted that the algorithm 2 will require less CPU time than algorithm 1 (see table 2 and table 5). From table 3 and table 6,we may see the errors decrease about by a factor of 4 as the spatial mesh size is reduced by a factor of 2 and the time mesh size is reduced by a factor of 4.

Conclusion

It is important to note that, for non-local problems, there is not yet a general theory analogous to that of classical problems. This is due to the relative novelty of this topic on the one hand and to the complexity of the questions it raises on the other hand. Each problem then requires a specific treatment, which highlights the topicality of the subject tackled in this article. Especially, when combined a parabolic integrodifferential equation with the second kind integral condition. So in this paper, we establish sufficient conditions for the existence, uniqueness and numerical solution for a parabolic integrodifferential equation with the second kind integral condition. For the theoretical studies we use the energy inequality and fixed point theorem methods. Also we construct a new numerical scheme to solve parabolic integrodifferential equation with the second kind integral condition, which has the following advantage: The coefficient matrices of the scheme is tridiagonal,to solve the linear system of equations by Thomas algorithm the cost is about $8M - 7$ (M the order of The coefficient matrices) , will save remarkable CPU time.

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