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SOLVABILITY OF EXTENDED GENERAL STRONGLY MIXED VARIATIONAL INEQUALITIES

BALWANT SINGH THAKUR

Abstract. In this paper, a new class of extended general strongly mixed variational inequalities is introduced and studied in Hilbert spaces. An existence theorem of solution is established and using resolvent operator technique, a new iterative algorithm for solving the extended general strongly mixed variational inequality is suggested. A convergence result for the iterative sequence generated by the new algorithm is also established.

1. Introduction and Preliminaries

Variational inequality theory, which was introduced by Stampacchia [24] in 1964, has had a great impact and influence in the development of several branches on pure and applied sciences. A useful and important generalization of variational inequality is the general mixed variational inequality containing a nonlinear term φ. Finding fixed points of a nonlinear mapping is an equally important problem in the functional analysis. Equivalent fixed point formulation of a variational inequality problem, has given a new dimension to the study of solution of variational inequality problems.

In many problems of analysis, one encounters operators who may be split in the form $S = A \pm T$, where A and T satisfies some conditions, and S itself has neither of these properties. An early theorem of this type was given by Krasnoselskii [12], where a complicated operator is split into the sum of two simpler operators. There is another setting arises from perturbation theory. Here the operator equation $Tx \pm Ax = x$ is considered as a perturbation of $Tx = x$ (or $Ax = x$), and one would like to assert that the original unperturbed equation has a solution. In such a situation, there is, in general, no continuous dependence of solutions on the perturbations. For various results in this direction, please see [4, 7, 8, 11, 22, 26]. Another argument is concerned with the approximate solution of the problem: For $f \in H$, find $x \in H$ such that $Tx \pm Ax = f$. Here $T, A : H \to H$ are given operators. Many boundary value problems for quasi linear partial differential equations arising in physics, fluid mechanics and other areas of applications can be formulated as the equation $Tx \pm Ax = f$, see, e.g. Zeidler [28]. Combettes and Hirstoaga [5] showed that the finding of zeros of sum of two operators can be solved via the variational inequality involving sum of two operators. Several authors study this

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type of situations, see, e.g. [6, 21] and references therein. Motivated by these facts, in this paper we study a variational inequality problem involving operator of the form $T - A$.

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function. Let $T : H \to H$ be a nonlinear operator and $g, h : H \to H$ are any mappings. We consider the problem of finding $x^* \in H$ such that

$$(1) \qquad \langle T(x^*) - A(x^*), h(y^*) - g(x^*) \rangle + \varphi(h(y^*)) - \varphi(g(x^*)) \geq 0\,, \quad \forall\, y^* \in H\,,$$

where A is a nonlinear continuous mapping on H and $\partial\varphi$ denotes the subdifferential of φ. We call inequality (1) as *extended general strongly mixed variational inequality.* We now consider some special cases of the problem (1) :

(1) If $A \equiv 0$, then the problem (1) reduces to the extended general mixed variational inequality problem considered in [20]
(2) If h is an identity mapping on H, then the problem (1) reduces to the problem studied by [10].
(3) If $A \equiv 0$ and $h \equiv g$, then the problem (1) reduces to the general mixed variational inequality problem considered in [2, 17, 18, 19].
(4) If h, g be identity mappings on H, then the problem (1) reduces to a class of variational inequality studied by [25].
(5) If $A \equiv 0$ and h, g be identity mappings on H, then the problem (1) reduces to the mixed variational inequality or variational inequality of second kind see [1, 9, 15, 16].

For a multivalued operator $T : H \to H$, we denote by

$$D(T) = \{u \in H : T(u) \neq \emptyset\}\ ,$$

the domain of T,

$$R(T) = \bigcup_{u \in H} T(u)\,,$$

the range of T,

$$\mathrm{Graph}(T) = \{(u, u^*) \in H \times H : u \in D(T) \text{ and } u^* \in T(u)\}\ ,$$

the graph of T.

Definition 1.1. T is called monotone if and only if for each $u \in D(T)$, $v \in D(T)$ and $u^* \in T(u)$, $v^* \in T(v)$, we have

$$\langle v^* - u^*, v - u \rangle \geq 0\,.$$

T is maximal monotone if it is monotone and its graph is not properly contained in the graph of any other monotone operator.

T^{-1} is the operator defined by

$$v \in T^{-1}(u) \Leftrightarrow u \in T(v)\,.$$

Definition 1.2 (See [3])**.** For a maximal monotone operator T, the resolvent operator associated with T, for any $\sigma > 0$, is defined as

$$J_T(u) = (I + \sigma T)^{-1}(u)\,, \ \ \forall\, u \in H\,.$$

It is known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. Furthermore, the resolvent operator is single-valued and nonexpansive i.e. $\|J_T(x) - J_T(y)\| \leq \|x - y\|$, $\forall x, y \in H$. In particular, it is well known that the subdifferential $\partial\varphi$ of φ is a maximal monotone operator; see [13].

Lemma 1.3. [3] *For a given* $z \in H$, $u \in H$ *satisfies the inequality*

$$\langle u - z, x - u\rangle + \lambda\varphi(x) - \lambda\varphi(u) \geq 0\,, \quad \forall x \in H$$

if and only if $u = J_\varphi(z)$, *where* $J_\varphi = (I + \lambda\,\partial\varphi)^{-1}$ *is the resolvent operator and* $\lambda > 0$ *is a constant.*

Inequality (1), can be written in an equivalent form as follows:

Find $x^* \in H$ such that

(2)
$$\langle \rho(T(x^*) - A(x^*)) + g(x^*) - h(x^*), h(y^*) - g(x^*)\rangle + \rho\varphi(h(y^*)) - \rho\varphi(g(x^*)) \geq 0\,,$$

for all $y^* \in H$.

This equivalent formulation plays an important role in the development of iterative methods for solving the mixed variational inequality problem (1).

Using Lemma 1.3, we will establish following important relation:

Lemma 1.4. $x^* \in H$ *is a solution of* (2) *if and only if* x^* *satisfies the following relation*

$$g(x^*) = J_\varphi\left(h(x^*) - \rho(T(x^*) - A(x^*))\right)\,, \tag{3}$$

where $\rho > 0$ *is a constant and* $J_\varphi = (I + \rho\partial\varphi)^{-1}$ *is the proximal mapping,* I *stands for the indentity operator on* H.

Proof. Let $x^* \in H$ be a solution of problem (2), then

(4)
$$\langle g(x^*) - (h(x^*) - \rho\,(T(x^*) - A(x^*)))\,, h(y^*) - g(x^*)\rangle + \rho\varphi(h(y^*)) - \rho\varphi(g(x^*)) \geq 0\,,$$

for all $y^* \in H$. Applying Lemma 1.3 for $\lambda = \rho$, inequality (4) is equivalent to

$$g(x^*) = J_\varphi\left(h(x^*) - \rho\,(T(x^*) - A(x^*))\right)\,,$$

the required result. □

Lemma 1.4 implies that the problem (2) is equivalent to the fixed point problem (3). This alternative equivalent formulation provides a natural connection between variatonal inequality problem (2) and the fixed point theory which will be used to prove existence result. The following lemma is in this sense :

Lemma 1.5. $x^* \in H$ *is a solution of* (2) *if and only if* x^* *is a fixed point of the mapping* F *given by*

$$F(u) = u - g(u) + J_\varphi\left(h(u) - \rho(T(u) - A(u))\right)\,, \quad u \in H\,. \tag{5}$$

Proof. Let $x^* \in H$ be a fixed point of the mapping F. Then

$$g(x^*) = J_\varphi\left(h(x^*) - \rho(T(x^*) - A(x^*))\right)\,.$$

From Lemma 1.4, x^* is a solution of (2). □

We now recall some some definitions:

Definition 1.6. An operator $T : H \to H$ is said to be :

(i) strongly monotone, if for each $x \in H$, there exists a constant $\nu > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq \nu \|x - y\|^2$$

holds, for all $y \in H$;

(ii) $\phi-$cocoercive, if for each $x \in H$, there exists a constant $\phi > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq -\phi \|T(x) - T(y)\|^2$$

holds, for all $y \in H$;

(iii) relaxed $(\phi, \gamma)-$cocoercive or relaxed cocoercive with respect to constant (ϕ, γ), if for each $x \in H$, there exists constants $\gamma > 0$ and $\phi > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq -\phi \|T(x) - T(y)\|^2 + \gamma \|x - y\|^2$$

holds, for all $y \in H$;

(iv) $\mu-$Lipschitz continuous or Lipschitz with respect to constant μ, if for each $x, y \in H$, there exists a constant $\mu > 0$ such that

$$\|T(x) - T(y)\| \leq \mu \|x - y\| \ .$$

2. Main results

Lemma 1.5, is the main motivation for our next result:

Theorem 2.1. *Let H be a real Hilbert space and $T, A, g, h : H \to H$ are operators. Suppose that the following assumptions are satisfied :*

(i) *T, g, h are relaxed cocoercive with constants (ϕ_T, γ_T), (ϕ_g, γ_g), (ϕ_h, γ_h) respectively,*

(ii) *T, A, g, h are Lipschitz mappings with constants $\mu_T, \mu_A, \mu_g, \mu_h$ respectively.*

If

$$1 + \mu_g^2(1 + 2\phi_g) > 2\gamma_g \ , \quad 1 + \mu_h^2(1 + 2\phi_h) > 2\gamma_h \ ,$$

and

$$\rho \in \left(\frac{(\gamma_T - \phi_T\mu_T^2) - \sqrt{d}}{\mu_T^2 + \mu_A^2}, \frac{(\gamma_T - \phi_T\mu_T^2) + \sqrt{d}}{\mu_T^2 + \mu_A^2} \right) , \tag{6}$$

where

$$d := (\phi_T\mu_T^2 - \gamma_T)^2 - \frac{1}{2}(\mu_T^2 + \mu_A^2)(1 + \kappa(2 - \kappa)) > 0$$

$$\kappa = \sqrt{1 - 2\gamma_g + \mu_g^2(1 + 2\phi_g)} + \sqrt{1 - 2\gamma_h + \mu_h^2(1 + 2\phi_h)} \, ,$$

then the problem (2) *has a unique solution.*

Proof. It is enough to show that the mapping F defined by (5) has a fixed point. For $u \in H$, set $p(u) = T(u) - A(u)$.

For all $x \neq y \in H$, we have

$$\begin{aligned}\|F(x)-F(y)\| &\leq \|x-y-(g(x)-g(y))\| \\ &\quad + \|J_\varphi(h(x)-\rho(p(x))) - J_\varphi(h(y)-\rho(p(y)))\| \\ &\leq \|x-y-(g(x)-g(y))\| + \|h(x)-h(y)-\rho(p(x)-p(y))\| \\ &\leq \|x-y-(g(x)-g(y))\| + \|x-y-(h(x)-h(y))\| \\ &\quad + \|x-y-\rho(p(x)-p(y))\| \ . \end{aligned} \tag{7}$$

Since g is relaxed $(\phi_g, \gamma_g)-$cocoercive and μ_g-Lipschitz mapping, we can compute the following:

$$\begin{aligned}\|x-y-(g(x)-g(y))\|^2 &= \|x-y\|^2 - 2\langle g(x)-g(y), x-y\rangle + \|g(x)-g(y)\|^2 \\ &\leq (1+\mu_g^2)\|x-y\|^2 + 2\phi_g\|g(x)-g(y)\|^2 - 2\gamma_g\|x-y\|^2 \\ &\leq \left(1-2\gamma_g+\mu_g^2(1+2\phi_g)\right)\|x-y\|^2 \ . \end{aligned} \tag{8}$$

Similarly,

$$\|x-y-(h(x)-h(y))\|^2 \leq \left(1-2\gamma_h+\mu_h^2(1+2\phi_h)\right)\|x-y\|^2 \ . \tag{9}$$

Also,

$$\begin{aligned}\|x-y-\rho(p(x)-p(y))\|^2 &= \|x-y-\rho(T(x)-T(y))+\rho(A(x)-A(y))\|^2 \\ &\leq 2\|x-y-\rho(T(x)-T(y))\|^2 + 2\rho^2\|A(x)-A(y)\|^2 \\ &\leq 2\|x-y-\rho(T(x)-T(y))\|^2 + 2\rho^2\mu_A^2\|x-y\|^2 \ . \end{aligned} \tag{10}$$

Now, we estimate

$$\begin{aligned}\|x-y-\rho(T(x)-T(y))\|^2 &\leq \|x-y\|^2 - 2\rho\langle T(x)-T(y), x-y\rangle \\ &\quad + \rho^2\|T(x)-T(y)\|^2 \\ &\leq \left(1-2\rho\gamma_T+2\rho\mu_T^2\phi_T+\rho^2\mu_T^2\right)\|x-y\|^2 \ . \end{aligned} \tag{11}$$

Substituting (11) into (10), gives

$$\|x-y-\rho(p(x)-p(y))\| \leq \sqrt{2\left(1-2\rho\gamma_T+2\rho\mu_T^2\phi_T+\rho^2(\mu_T^2+\mu_A^2)\right)}\,\|x-y\| \ . \tag{12}$$

Substituting (8), (9), (12) into (7), we have

$$\|F(x)-F(y)\| \leq (\kappa + f(\rho))\|x-y\| \ ,$$

where

$$\kappa = \sqrt{1-2\gamma_g+\mu_g^2(1+2\phi_g} + \sqrt{1-2\gamma_h+\mu_h^2(1+2\phi_h)}\,,$$

and

$$f(\rho) = \sqrt{2\left(1-2\rho\gamma_T+2\rho\mu_T^2\phi_T+\rho^2(\mu_T^2+\mu_A^2)\right)}\,.$$

From (6), we get that $(\kappa + f(\rho)) < 1$, thus F is a contraction mapping and therefore has a unique fixed point in H, which is a solution of variational inequality (2). □

Remark 2.2. Theorem 2.1, extend and improve Theorem 3.1 of [20].

If K is closed convex set in H and $\varphi(x) = \delta_K(x)$, for all $x \in K$, where δ_K is the indicator function of K defined by

$$\delta_K(x) = \begin{cases} 0, & \text{if } x \in K\,; \\ +\infty, & \text{otherwise}\,, \end{cases}$$

then the problem (2) reduces to the following variational inequality problem: Consider the problem of finding $x^* \in K$

$$\langle \rho(T(x^*) - A(x^*)) + g(x^*) - h(x^*),\, h(y^*) - g(x^*)\rangle \geq 0\,, \quad \forall\, y^* \in K\,. \tag{13}$$

We immediately obtain following result from Theorem 2.1 :

Corollary 2.3. *Let H be a real Hilbert space, K be a nonempty closed convex subset of H and $T, A : H \to H$ and $g, h : K \to K$ are operators. Suppose that following assumptions are satisfied :*

(i) *T, g, h are relaxed cocoercive with constants (ϕ_T, γ_T), (ϕ_g, γ_g), (ϕ_h, γ_h) respectively,*

(ii) *T, A, g, h are Lipschitz mappings with constants $\mu_T, \mu_A, \mu_g, \mu_h$ respectively.*

If (6) *holds, then the problem* (13) *has a unique solution.*

If we take h as identity mapping in (13), we get an inequality, equivalent to the general strongly nonlinear variational inequality studied by Siddiqi and Ansari [23]. Corollary 2.3 partially extends and improves the result of [14, 23].

3. Iterative algorithm and convergence

We rewrite the relation (3) in the following form

$$x^* = x^* - g(x^*) + J_\varphi\left(h(x^*) - \rho(T(x^*) - A(x^*))\right)\,. \tag{14}$$

Using the fixed point formulation (14), we now suggest and analyze the following iterative methods for solving the variational inequality problem (2).

Algorithm 1. For a given $x_0 \in H$, find the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - g(x_n) + J_\varphi\left(h(x_n) - \rho\left(T(x_n) - A(x_n)\right)\right)\,, \quad n = 0, 1, 2, \ldots$$

which is called explicit iterative method.

Algorithm 2. For a given $x_0 \in H$, find the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - g(x_n) + J_\varphi\left(h(x_{n+1}) - \rho\left(T(x_{n+1}) - A(x_{n+1})\right)\right)\,, \quad n = 0, 1, 2, \ldots$$

which is an implicit iterative method.

Now, we use Algorithm 1 as predictor and Algorithm 2 as a corrector to obtain the following predictor-corrector method for solving variational inequality problem (1).

Algorithm 3. For a given $x_0 \in H$, find the approximate solution x_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= x_n - g(x_n) + J_\varphi\left(h(x_n) - \rho(Tx_n - Ax_n)\right) \\ x_{n+1} &= x_n - g(x_n) + J_\varphi\left(h(y_n) - \rho(Ty_n - Ay_n)\right)\,, \quad n = 0, 1, 2, \ldots\,. \end{aligned}$$

Using Algorithm 3, we can suggest following :

Algorithm 4. For a given $x_0 \in H$, find the approximate solution x_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= x_n - g(x_n) + J_\varphi\left(h(x_n) - \rho(Tx_n - Ax_n)\right) \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n\left(x_n - g(x_n) + J_\varphi\left(h(y_n) - \rho(Ty_n - Ay_n)\right)\right), \end{aligned}$$

where $n = 0, 1, 2, \ldots,$ $\{\alpha_n\}$ is sequences in $[0, 1]$, satisfying certain conditions.

Now, we define a more general predictor-corrector iterative method for approximate solvability of variational inequality problem (1).

Algorithm 5. For a given $x_0 \in H$, find the approximate solution x_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n\left(x_n - g(x_n) + J_\varphi\left(h(x_n) - \rho(Tx_n - Ax_n)\right)\right) \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n\left(x_n - g(x_n) + J_\varphi\left(h(y_n) - \rho(Ty_n - Ay_n)\right)\right), \end{aligned} \tag{15}$$

where $n = 0, 1, 2, \ldots,$ $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$, satisfying certain conditions.

We need following result to prove the next result :

Lemma 3.1. [27] *Let $\{a_n\}$ be a non negative sequence satisfying*

$$a_{n+1} \le (1-c_n)a_n + b_n\,,$$

with $c_n \in [0,1]$, $\sum_{n=0}^\infty c_n = \infty$, $b_n = o(c_n)$. Then $\lim_{n\to\infty} a_n = 0$.

Theorem 3.2. *Let T, A, g, h satisfy all the assumptions of Theorem 2.1, also condition (6) holds and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ for all $n \ge 0$ such that $\sum_{n=0}^\infty \alpha_n = \infty$. Then the approximate sequence $\{x_n\}$ constructed by the Algorithm 5 converges strongly to a solution x^* of (2).*

Proof. For $u \in H$, set $pu = Tu - Au$. Since $x^* \in H$ is a solution of (1), by (14), we have

$$x^* = x^* - g(x^*) + J_\varphi\left(h(x^*) - \rho(T(x^*) - A(x^*))\right).$$

Using (15), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - (g(x_n) - g(x^*))\| \\ &\quad + \alpha_n\|J_\varphi(h(y_n) - \rho p(y_n)) - J_\varphi(h(x^*) - \rho p(x^*))\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\sqrt{1 - 2\gamma_g + \mu_g^2(1+2\phi_g)}\,\|x_n - x^*\| \\ &\quad + \alpha_n\|h(y_n) - h(x^*) - \rho(p(y_n) - p(x^*))\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\sqrt{1 - 2\gamma_g + \mu_g^2(1+2\phi_g)}\,\|x_n - x^*\| \\ &\quad + \alpha_n\|y_n - x^* - (h(y_n) - h(x^*))\| \\ &\quad + \alpha_n\|y_n - x^* - \rho(p(y_n) - p(x^*))\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\sqrt{1 - 2\gamma_g + \mu_g^2(1+2\phi_g)}\,\|x_n - x^*\| \\ &\quad + \alpha_n\sqrt{1 - 2\gamma_h + \mu_h^2(1+2\phi_h)}\,\|y_n - x^*\| \\ &\quad + \alpha_n\sqrt{2\left(1 - 2\rho\gamma_T + 2\rho\mu_T^2\phi_T + \rho^2(\mu_T^2 + \mu_A^2)\right)}\,\|y_n - x^*\| \\ &= (1-\alpha_n)\|x_n - x^*\| + \alpha_n\theta_g\|x_n - x^*\| + \alpha_n(\theta_h + f(\rho))\|y_n - x^*\|, \end{aligned} \tag{16}$$

where $\theta_g = \sqrt{1 - 2\gamma_g + \mu_g^2(1 + 2\phi_g)}$, $\theta_h = \sqrt{1 - 2\gamma_h + \mu_h^2(1 + 2\phi_h)}$
and $f(\rho) = \sqrt{2\left(1 - 2\rho\gamma_T + 2\rho\mu_T^2\phi_T + \rho^2(\mu_T^2 + \mu_A^2)\right)}$.

Similarly, we have

$$\begin{aligned}
\|y_n - x^*\| &\le (1-\beta_n)\|x_n - x^*\| + \beta_n \|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \beta_n \|J_\varphi(h(x_n) - \rho p(x_n)) - J_\varphi(h(x^*) - \rho p(x^*))\| \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\theta_g\|x_n - x^*\| \\
&\quad + \beta_n \|h(x_n) - h(x^*) - \rho(p(x_n) - p(x^*))\| \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\theta_g\|x_n - x^*\| \\
&\quad + \beta_n \|x_n - x^* - (h(x_n) - h(x^*))\| \\
&\quad + \beta_n \|x_n - x^* - \rho(p(x_n) - p(x^*))\| \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\theta_g\|x_n - x^*\| \\
&\quad + \beta_n\theta_h\|x_n - x^*\| + \beta_n f(\rho)\|x_n - x^*\| \\
&= (1-\beta_n)\|x_n - x^*\| + \beta_n(\kappa + f(\rho))\|x_n - x^*\| \\
&\le (1-\beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^*\| \\
&= \|x_n - x^*\| . \qquad (17)
\end{aligned}$$

Substituting (17) into (16), yields that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n(\theta_g + \theta_h + f(\rho))\|x_n - x^*\| \\
&= (1 - \alpha_n(1 - (\kappa + f(\rho))))\|x_n - x^*\| . \qquad (18)
\end{aligned}$$

By virtue of Lemma 3.1, we get from (18) that, $\lim_{n\to\infty}\|x_{n+1} - x^*\| = 0$, i.e. $x_n \to x^*$, as $n \to \infty$. This completes the proof. □

Remark 3.3. Theorem 3.2, extend and improve Theorem 2.1 of [10] and Theorem 3.2 of [20].

It is well known that, if $\varphi(\cdot)$ is the indicator function of K in H, then $J_\varphi = P_K$, the projection operator of H onto the closed convex set K, and consequently, the following result can be obtain from Theorem 3.2.

Corollary 3.4. *Let T, A, g, h satisfy all the assumptions of Corollary 2.3. Let $x_0 \in K$, construct a sequence $\{x_n\}$ in K by*

$$\begin{aligned}
&y_n = x_n - g(x_n) + P_K(h(x_n) - \rho(Tx_n - Ax_n)) \\
&x_{n+1} = (1-\alpha_n)x_n + \alpha_n(x_n - g(x_n) + P_K(h(y_n) - \rho(Ty_n - Ay_n))) , \quad n = 0, 1, 2, \ldots ,
\end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ for all $n \ge 0$ such that $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to a solution x^* of (13).

References

[1] Baiocchi,C., Capelo,A.: Variational and Quasi Variational Inequalies. J. Wile and Sons, New York (1984).

[2] Bnouhachem,A., Noor,M.A., Al-Shemas,E.H.: On self-adaptive method for general mixed variational inequalities, Math. Prob. Engineer. (2008), doi: 10.1155/2008/280956.

[3] Brezis,H.: Opérateurs maximaux monotone et semi-groupes de contractions dans les espaces de Hilbert. In: North-Holland Mathematics Studies. 5, Notas de matematics, vol. 50, North-Holland, Amsterdam (1973).

[4] Browder,F.E.: Fixed point theorems for nonlinear semicontractive mappings in Banach spaces. Arch. Rat. Mech. Anal. **21**, 259–269 (1966).
[5] Combettes,P.L., Hirstoaga,S.A.: Visco-penalization of the sum of two monotone operators. Nonlinear Anal. **69**, 579–591 (2008).
[6] Dhage,B.C.: Remarks on two fixed-point theorems involving the sum and the product of two operators. Comput. Math. Appl. **46**, 1779–1785 (2003).
[7] Fucik,S.: Fixed point theorems for a sum of nonlinear mapping. Comment. Math. Univ. Carolinae **9**, 133-143 (1968).
[8] Fucik,S.: Solving of nonlinear operator equations in Banach space. Comment. Math. Univ. Carolinae **10**, 177–186 (1969).
[9] Glowinski,R., Lions,J.L., Tremolieres,R.: Numerical Analysis of Variational Inequalities. North-Holland, Amesterdam, Holland (1981).
[10] Hassouni,A., Moudafi,A.: Perturbed algorithm for variational inclusions. J. Math. Anal. Appl. **185**, 706-712 (1994).
[11] Kirk,W.A.: On nonlinear mappings of strongly semicontractive type. J. Math. Anal. Appl. **27**, 409–412 (1969).
[12] Krasnoselskii,M.A.: Two remarks of the method of successive approximations. Uspeki Mat. Nauk **10**, 123–127 (1955).
[13] Minty,H.J.: On the monotonicity of the gradient of a convex function. Pacific J. Math. **14**, 243–247 (1964).
[14] Noor,M.A.: Strongly nonlinear variational inequalities. C.R. Math. Rep. Acad. Sci. Canad. **4**, 213–218 (1982).
[15] Noor,M.A.: On a class of variational inequalities. J. Math. Anal. Appl. **128**, 135–155 (1987).
[16] Noor,M.A.: A class new iterative methods for general mixed variational inequalities. Math. Comput. Modell. **31**, 11–19 (2000).
[17] Noor,M.A.: Modified resolvent splitting algorithms for general mixed variational inequalities. J. Comput. Appl. Math. **135**, 111–124 (2001).
[18] Noor,M.A.: Operator-splitting methods for general mixed variational inequalities. J. Ineq. Pure Appl. Math. **3**(5), Art.67, 9p. (2002) http://eudml.org/doc/123617.
[19] Noor,M.A.: Psueudomontone general mixed variational inequalities. Appl. Math. Comput. **141**, 529–540 (2003).
[20] Noor,M.A., Ullah,S., Noor,K.I., Al-Said,E.: Iterative methods for solving extended general mixed variational inequalities. Comput. Math. Appl. **62**, 804–813 (2011).
[21] O'Regan, D.: Fixed point theory for the sum of two operators. Appl. Math. Lett. **9**, 1–8 (1996).
[22] Petryshyn,W.V.: Remarks on fixed point theorems and their extensions. Trans. Amer. Math. Soc. **126**, 43–53 (1967).
[23] Siddiqi,A.H., Ansari,Q.H.: General strongly nonlinear variational inequalities. J. Math. Anal. Appl. **166**, 386–392 (1992).
[24] Stampacchia,G.: Formes bilineares sur les ensemble convexes. C. R. Acad. Sci. Paris **285**, 4413–4416 (1964).
[25] Verma,R.U.: Generalized auxiliary problem principle and solvability of a class of nonlinear variational inequalities involoving cocoercive and co-Lipschitzian mappings. J. Ineq. Pure Appl. Math. **2**(3), Art.27, 9p. (2001) http://eudml.org/doc/122114.
[26] Webb,J.R.L.: Fixed point theorems for nonlinear semicontractive operators in Banach spaces. J. London Math. Soc. **1**, 683–688 (1969).
[27] Weng,X.L.: Fixed point iteration for local stricly pseudo-contractive mappings. Proc. Amer. Math. Soc. **113**, 727–731 (1991).
[28] Zeidler,E.: Nonlinear functional analysis and its applications, II/B : Nonlinear monotone operators. Springer, New York (1990).

School of Studies in Mathematics, Pt.Ravishankar Shukla University, Raipur, 492010, India