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# IDENTITIES ON GENOCCHI POLYNOMIALS AND GENOCCHI NUMBERS CONCERNING BINOMIAL COEFFICIENTS

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Abstract. In this paper, the author gives some new identities on Genocchi polynomials and Genocchi numbers.

## 1. Introduction

The researches on Genocchi numbers and Genocchi polynomials have a long history. It can be traced back to Angelo Genocchi (1817–1889). Nowadays, Genocchi numbers and kinds of Genocchi polynomials have become a popular research topic. During these very recent years, some researchers such as Araci [1–7] did many researches on this interesting topic. They studied Genocchi numbers and Genocchi polynomials extensively in many branches of Mathematics, such as elementary number theory, analytic number theory, theory of modular forms, $p$-adic analytic number theory and etc.. Now, let us show the definitions of Genocchi numbers and Genocchi polynomials.

The Genocchi numbers are a sequence of integers that satisfy the following exponential generating function

$$\frac{2t}{e^t+1}=\sum_{n=0}^{\infty}G_n\frac{t^n}{n!},\quad |t|<\pi,$$

with the convention that replacing $G^n$ by $G_n$. The first few Genocchi numbers are

$$G_0=0,\ G_1=1,\ G_2=-1,\ G_3=0,\ G_4=1,\ G_5=0,\ G_6=-3,\ G_7=0,\ G_8=17.$$

The classic Genocchi polynomials are usually defined by mean of the following exponential generating function

$$\frac{2t}{e^t+1}\cdot e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\quad |t|<\pi,$$

with the convention that replacing $G^n(x)$ by $G_n(x)$. It is clear that $G_n(0)=G_n$.

According to the classic Genocchi polynomials, some mathematicians introduced several new polynomials that extended the classic Genocchi polynomials.

Araci [6] and Kim et al. [8] did some researches on the so-called Genocchi polynomials of order $k$, which were defined by

$$\left(\frac{2t}{e^t+1}\right)^k\cdot e^{xt}=\sum_{n=0}^{\infty}G_n^{(k)}(x)\frac{t^n}{n!}.$$

Araci [6] and He [9, 10] introduced the Apostol–Genocchi polynomials defined by

$$\frac{2t}{\lambda e^t+1}\cdot e^{xt}=\sum_{n=0}^{\infty}G_n(x,\lambda)\frac{t^n}{n!}.$$

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Based on which, Araci [7] introduced the high order Apostol–Genocchi polynomials which can be called the generalized Apostol–Genocchi polynomials of order $k \in \mathbb{C}$,

$$\left(\frac{2t}{\lambda e^t + 1}\right)^k \cdot e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x, \lambda)\frac{t^n}{n!}.$$

In [11], Lim defined the degenerated Genocchi polynomials $\mathcal{G}_n^{(k)}(x, \lambda)$ of order $k$ to be

$$\left(\frac{2t}{(1+\lambda t)^{1/\lambda} + 1}\right)^k (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(k)}(x, \lambda)\frac{t^n}{n!}.$$

Besides these generalizations, Araci [1], Duran et al. [12] and Agyuz et al. [13] also introduced the $q$-alanogue of the Genocchi polynomials as follows,

$$\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!} = t\int_{\mathbb{Z}_p} q^{-y} e^{t[y+x]_q} d\mu_{-q}(y),$$

where

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}.$$

This definition used $p$-adic fermionic $q$-integral on $\mathbb{Z}_p$ with respect to $\mu_{-q}$. It can also be defined by

$$\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!} = [2]_q t \sum_{m=0}^{\infty} (-1)^m e^{t[m+x]_q}.$$

In which when we take $x = 0$, it becomes $G_{n,q}(0) := G_{n,q}$, which we call it the $n$-th $q$-Genocchi number.

When it comes to Genocchi numbers, the most common thing comes to our mind is to research the relations between Genocchi numbers, Bernoulli numbers [14–16] and Euler numbers [14, 17]. Indeed, most researches on Genocchi numbers concern the relations between these three kinds of numbers (see for example [2–4, 18, 19]). In other words, there are many literatures that provide identities on these three kinds of numbers. Similarly, when it comes to Genocchi polynomials, the most common thing is to research on the relations between Genocchi polynomials, Bernoulli polynomials and Euler polynomials (see for example [2–4, 9, 18–21]). Even though when it comes to the generalized Genocchi numbers and generalized Genocchi polynomials, it is unavoidable to research the relations as above.

In this paper, we do not want to find relations between the three kinds of numbers or the three kinds of polynomials. We will focus only on Genocchi numbers themselves and Genocchi polynomials themselves. In other words, in this paper, we will give some identities only concern Genocchi numbers and Genocchi polynomials. Actually, by these identities combining with the identities between Genocchi numbers (polynomials), Bernoulli numbers (polynomials) and Euler numbers (polynomials), one can obtain some other identities. While we do not want to show them here since the process of combining two identities is not very novel.

## 2. Identities on Genocchi numbers and Genocchi polynomials

Let us start this section with some straightforward derived identities on Genocchi numbers and Genocchi polynomials.

Differentiating both sides of the exponential generating function for $G_n$ with respect to $x$ yields

$$\frac{d}{dx}G_n(x) = nG_{n-1}(x), \quad \deg G_{n+1}(x) = n.$$

By which we can get

$$\int_a^b G_n(x)dx = \frac{G_{n+1}(b) - G_{n+1}(a)}{n+1}.$$

Thanks to [3, 18], we have

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}.$$

Combining the above two identities and the relation (2.7) below shows

$$\int_0^1 G_n(x)dx = \begin{cases} 0 & n = 0 \\ -2\frac{G_{n+1}}{n+1} & n \geq 1 \end{cases}.$$

Besides these classical identities, one can also find more identities concerning Genocchi numbers and Genocchi polynomials in [4]. Next, we show some new identities on Genocchi numbers and Genocchi polynomials.

**Theorem 2.1.** *For $n \geq 2$, we have*

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_k(x)\cdot G_{n-k+1}(1)}{n-k+1} = x\cdot G_n(x) - \frac{n}{n+1}G_{n+1}(x). \tag{2.1}$$

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{k+1}(x)\cdot G_{n-k}(1)}{k+1} = x\cdot G_n(x) - \frac{n}{n+1}G_{n+1}(x). \tag{2.2}$$

*Proof.* Let us recall the generating function of Genocchi polynomials first,

$$\frac{2t}{e^t+1}\cdot e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}.$$

Taking the partial derivative with respect to $t$ on the right hand side, we deduce that

$$\begin{aligned}
&\frac{\partial}{\partial t}\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \\
=&\frac{\partial}{\partial t}\left(G_0(x) + G_1(x)t + G_2(x)\frac{t^2}{2!} + G_3(x)\frac{t^3}{3!} + \cdots\right) \\
=&G_1(x) + G_2(x)t + G_3(x)\frac{t^2}{2!} + \cdots = \sum_{n=0}^{\infty} G_{n+1}(x)\frac{t^n}{n!}.
\end{aligned} \tag{2.3}$$

Now, let us look at the left hand side.

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{2t}{e^t+1}\cdot e^{xt}\right) = \frac{(2e^{xt} + xe^{xt}\cdot 2t)(e^t+1) - (2te^{xt}\cdot e^t)}{(e^t+1)^2} \\
=&\frac{1}{t}\frac{2t\cdot e^{xt}}{e^t+1} + x\frac{2t\cdot e^{xt}}{e^t+1} - \frac{1}{2t}\frac{2t\cdot e^{xt}}{e^t+1}\frac{2t\cdot e^t}{e^t+1} \\
=&\frac{1}{t}\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} + x\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} - \frac{1}{2t}\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}\sum_{n=0}^{\infty} G_n(1)\frac{t^n}{n!} \\
=&\sum_{n=0}^{\infty}\frac{G_{n+1}(x)}{n+1}\frac{t^n}{n!} + \sum_{n=0}^{\infty} xG_n(x)\frac{t^n}{n!} - \frac{1}{2}\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{G_{n+1}(1)}{n+1}\frac{t^n}{n!} \\
=&\sum_{n=0}^{\infty}\left[\frac{G_{n+1}(x)}{n+1} + x\cdot G_n(x) - \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}G_k(x)\frac{G_{n-k+1}(1)}{n-k+1}\right]\frac{t^n}{n!}.
\end{aligned} \tag{2.4}$$

In the second to last step, we used the fact that $G_0(x) = 0$.

Comparing the coefficients of $\frac{t^n}{n!}$ in (2.3) and (2.4) yields

$$\frac{G_{n+1}(x)}{n+1} + x\cdot G_n(x) - \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}G_k(x)\frac{G_{n-k+1}(1)}{n-k+1} = G_{n+1}(x).$$

Then (2.1) follows from rearranging the terms in this identity.

Note that the second to last step can also be written as

$$\frac{\partial}{\partial t}\left(\frac{2t}{e^t+1}\cdot e^{xt}\right) = \sum_{n=0}^{\infty}\frac{G_{n+1}(x)}{n+1}\frac{t^n}{n!} + \sum_{n=0}^{\infty} xG_n(x)\frac{t^n}{n!} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{G_{n+1}(x)}{n+1}\frac{t^n}{n!}\sum_{n=0}^{\infty} G_n(1)\frac{t^n}{n!},$$

which gives us

$$\frac{G_{n+1}(x)}{n+1} + x \cdot G_n(x) - \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{k+1}(x)}{k+1}G_{n-k}(1) = G_{n+1}(x).$$

Rearranging the terms above yeilds (2.2).

This completes the proof. □

**Remark 2.1.** *According to the process of the proof above, one can also obtain that*

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{n-k}(x)\cdot G_{k+1}(1)}{k+1} = x\cdot G_n(x) - \frac{n}{n+1}G_{n+1}(x), \tag{2.5}$$

*and*

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{n-k+1}(x)\cdot G_k(1)}{n-k+1} = x\cdot G_n(x) - \frac{n}{n+1}G_{n+1}(x). \tag{2.6}$$

*But we should notice that (2.5) and (2.6) are equivalent to (2.1) and (2.2), respectively. This is because when $k$ goes from $0$ to $n$, $n-k$ also goes from $0$ to $k$. Hence if we replace $k$ by $n-k$ in (2.1) and (2.2), we can then get (2.5) and (2.6) respectively. From this point of view, we do not regard (2.5) and (2.6) as new identities.*

**Corollary 2.1.** *For $n \geq 2$, we have*

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_k\cdot G_{n-k+1}(1)}{n-k+1} = -\frac{n}{n+1}G_{n+1}.$$

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{G_{k+1}\cdot G_{n-k}(1)}{k+1} = -\frac{n}{n+1}G_{n+1}.$$

*Proof.* This lemma follows from taking $x = 0$ in Theorem 2.1. □

Having developed to this point, it is necessary to say something about $G_n(1)$. Since

$$\frac{2t}{e^t+1}\cdot e^t = \sum_{n=0}^{\infty}G_n(1)\frac{t^n}{n!}.$$

Then

$$\sum_{n=0}^{\infty}\left(G_n + G_n(1)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}G_n\frac{t^n}{n!} + \sum_{n=0}^{\infty}G_n(1)\frac{t^n}{n!} = 2t.$$

Thus, $G_1 + G_1(1) = 2$ and for $n \geq 2$, $G_n + G_n(1) = 0$, which means

$$G_n(1) = \begin{cases} 1, & n = 1, \\ -G_n, & n \geq 2. \end{cases} \tag{2.7}$$

So, in this sense, we can call the integer sequence $G_n(1)$ the negative Genocchi numbers.

With this fact, we can obtain

**Corollary 2.2.** *For $n \geq 2$, we have*

$$\frac{1}{2}\sum_{k=0}^{n-1}\binom{n}{k}\frac{G_k(x)\cdot G_{n-k+1}}{n-k+1} = (\frac{1}{2} - x)\cdot G_n(x) + \frac{n}{n+1}G_{n+1}(x). \tag{2.8}$$

$$\frac{1}{2}\sum_{k=0}^{n-2}\binom{n}{k}\frac{G_{k+1}(x)\cdot G_{n-k}}{k+1} = (\frac{1}{2} - x)\cdot G_n(x) + \frac{n}{n+1}G_{n+1}(x). \tag{2.9}$$

*Proof.* Since $G_n(1) = -G_n$ except for $n = 1$. Then we can replace $G_{n-k+1}(1)$ by $-G_{n-k+1}$ except for $k = n$. This gives us

$$\frac{1}{2}G_n(x) - \frac{1}{2}\sum_{k=0}^{n-1}\binom{n}{k}\frac{G_k(x)\cdot G_{n-k+1}}{n-k+1} = x\cdot G_n(x) - \frac{n}{n+1}G_{n+1}(x),$$

which means (2.8) holds true.

Similarly, we can show (2.9) through (2.2). □

If we take $x = 0$ in Corollary 2.2, we can arrive at the following conclusion.

**Corollary 2.3.** *For $n \geq 2$, we have*

$$\frac{1}{2}\sum_{k=0}^{n-1}\binom{n}{k}\frac{G_k \cdot G_{n-k+1}}{n-k+1} = \frac{1}{2}G_n + \frac{n}{n+1}G_{n+1}.$$

$$\frac{1}{2}\sum_{k=0}^{n-2}\binom{n}{k}\frac{G_{k+1} \cdot G_{n-k}}{k+1} = \frac{1}{2}G_n + \frac{n}{n+1}G_{n+1}.$$

Next, let us talk about $G_n(x+y)$ which is given by

$$\frac{2t}{e^t+1} \cdot e^{(x+y)t} = \sum_{n=0}^{\infty} G_n(x+y)\frac{t^n}{n!}.$$

As the basic properties we have mentioned for $G_n(x)$, $G_n(x+y)$ has the same properties, such as

$$\int_c^d \int_a^b G_n(x+y)dxdy = \frac{G_{n+2}(a+c) - G_{n+2}(b+c)}{(n+2)(n+1)} - \frac{G_{n+2}(a+d) - G_{n+2}(b+d)}{(n+2)(n+1)}.$$

Now, we would like to show some identities on $G_n(x+y)$.

**Theorem 2.2.** *For $y \neq 0$,*

$$G_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}G_k(x)y^{n-k}. \tag{2.10}$$

*Conversely, we have*

$$G_n(x) = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}G_k(x+y)y^{n-k}. \tag{2.11}$$

*Symmetrically, when $x \neq 0$, we have*

$$G_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}G_k(y)x^{n-k}, \tag{2.12}$$

*and*

$$G_n(y) = (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}G_k(x+y)x^{n-k}. \tag{2.13}$$

*Proof.* Since

$$\sum_{n=0}^{\infty} G_n(x+y)\frac{t^n}{n!} = \frac{2t}{e^t+1}e^{(x+y)t} = \frac{2t}{e^t+1}e^{xt} \cdot e^{yt}$$
$$= \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}\sum_{n=0}^{\infty} y^n\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}G_{n-k}(x)y^{n-k}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ shows (2.10) holds true.

The binomial inverse formula [22, pp.192, (5.48)] reads as

$$a_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k b_k \Leftrightarrow b_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k.$$

Equation (2.10) can be rewritten as

$$\frac{G_n(x+y)}{y^n} = \sum_{k=0}^{n}\binom{n}{k}\frac{G_k(x)}{y^k}.$$

Taking $a_k = \frac{G_k(x+y)}{y^k}$ and $b_k = (-1)^k \frac{G_k(x)}{y^k}$ gives us

$$(-1)^n \frac{G_n(x)}{y^n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{G_k(x+y)}{y^k},$$

which shows (2.11) holds.

Since $x$ and $y$ are symmetric, then we can obtain (2.12) and (2.13) by changing the position of $x$ and $y$. □

**Remark 2.2.** *If we want (2.11) and (2.13) to be more beautiful, we can replace $k$ by $n-k$. Then we can have*

$$G_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} G_{n-k}(x+y) y^k,$$

*and*

$$G_n(y) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} G_{n-k}(x+y) x^k.$$

**Corollary 2.4.**

$$\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} G_k = (-1)^n G_n + 2n.$$

*Proof.* Thanks to [3, 18], we have

$$\frac{G_n(x+1) + G_n(x)}{n} = 2x^{n-1}. \tag{2.14}$$

Taking $x = -1$ in (2.14) shows

$$G_n + G_n(-1) = 2n \cdot (-1)^{n-1}. \tag{2.15}$$

Let $x = 0$ and $y = -1$ in (2.10), we deduce that

$$G_n(-1) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} G_k. \tag{2.16}$$

Plugging (2.16) in (2.15) shows

$$G_n + \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} G_k = 2n \cdot (-1)^{n-1}.$$

Then multiplying $(-1)^{n-1}$ on the both sides proves this corollary. □

**Corollary 2.5.** *For $n \geq 2$, we have*

$$G_n + \sum_{k=0}^{n} \binom{n}{k} G_k = 0.$$

*Proof.* Let $x = 1$ and $y = 0$ in (2.12), we can get that

$$G_n(1) = \sum_{k=0}^{n} \binom{n}{k} G_k.$$

Since we have mentioned above that $G_n(1) + G_n = 0$ when $n \geq 2$. Then the conclusion follows. □

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