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A NEW STABILITY OF THE S-ESSENTIAL SPECTRUM OF MULTIVALUED LINEAR OPERATORS

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Abstract. We unfold in this paper two main results. In the first, we give the necessary assumptions for three linear relations A, B and S such that $\sigma_{eap,S}(A+B) = \sigma_{eap,S}(A)$ and $\sigma_{e\delta,S}(A+B) = \sigma_{e\delta,S}(A)$ is true. In the second, considering the fact that the linear relations A, B and S are not precompact or relatively precompact, we can show that $\sigma_{eap,S}(A+B) = \sigma_{eap,S}(A)$ is true.

1. Introduction

Assume that A and S are two bounded operators. Accordingly the map $p(\lambda) := \lambda S - A$ is a linear bundle. In fact many problems of mathematical physics (for example quantum theory, transport theory,...) are meant to shed light on the essential spectra of $\lambda S - A$. The spectral theory of Fredholm linear relations is one case worth mentioning given that this type of operators is unstable under the operation closure inverse and conjugate. But this does not hold for the case of multivalued linear operators. On this account, the investigation of the S-essential spectra of multivalued linear operators seems interesting. Historically, in [11] A. Jeribi, N. Moalla, and S. Yengui gave a characterization of the essential spectrum of the operator pencil in order to extend many known results in the literature. In [1] F. Abdmouleh, A. Ammar, and A. Jeribi pursued the study of the S-essential spectra and investigated the S-Browder, the S-upper semi-Browder, and the S-lower semi-Browder essential spectra of bounded linear operators on a Banach space X and they introduced the S-Riesz projection. Moreover, they extended the results of F. Abdmouleh and A. Jeribi [3] to various types of S-essential spectra. In fact, they gave the characterization of the S-essential spectra of the sum of two bounded linear operators. (See for example [10]). In [4] Tereza Alvarez, A. Ammar, and A. Jeribi pursued the study of the S-essential spectra and characterized some S-essential spectra of a closed linear relation in terms of certain linear relations type semi Fredholm. In [6] A. Ammar characterized some essential spectra of a closed linear relation in terms of certain linear relations type $\alpha-$ and $\beta-$ Atkinson.

Throughout this work, let X, Y and Z be three complex normed linear spaces, over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A multivalued linear operator (or a linear relation) A from X to Y is a mapping from a subspace of X,

$$\mathcal{D}(A) := \{x \in X : Ax \neq \emptyset\}$$

called the domain of A, into $\mathcal{P}(Y)\backslash\{\emptyset\}$ (collection of non-empty subsets of Y) such that $A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)$ for all non-zero scalars $\alpha, \beta \in \mathbb{C}$ and $x, y \in \mathcal{D}(A)$. If A maps the points of its domain to singletons, then A is said to be a single valued linear operator (or simply an operator).

A linear relation is uniquely determined by its graph, $G(A)$, which is defined by

$$G(A) := \{(x, y) \in X \times Y : x \in \mathcal{D}(A) \text{ } and \text{ } y \in Ax\}.$$

In this notation, $\mathcal{LR}(X, Y)$ denotes the class of all linear relations on X into Y. If $X = Y$, we would simply note $\mathcal{LR}(X, X) := \mathcal{LR}(X)$.

The inverse of A is the linear relation A^{-1} defined by

$$G(A^{-1}) := \{(y, x) \in Y \times X : (x, y) \in G(A)\}.$$

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The subspace $\mathcal{N}(A) := A^{-1}(0)$ is called the null space of A, and A is called injective if $\mathcal{N}(A) = \{0\}$; i.e., if A^{-1} is a single valued linear operator. The range of A is the subspace $\mathcal{R}(A) := A(\mathcal{D}(A))$, and A is called surjective if $\mathcal{R}(A) = Y$. When A is injective and surjective, we say that A is bijective. The quantities

$$\alpha(A) := dim(\mathcal{N}(A)) \text{ and } \beta(A) := codim(\mathcal{R}(A)) = dim(Y/_{\mathcal{R}(A)})$$

are called the nullity (or the kernel index) and the deficiency of A, respectively. We also write $\overline{\beta}(A) := codim(\overline{\mathcal{R}(A)})$. The index of A is defined by $i(A) := \alpha(A) - \beta(A)$ provided that both $\alpha(A)$ and $\beta(A)$ are not infinite. If $\alpha(A)$ and $\beta(A)$ are infinite, then A is said to have no index. The set of upper semi-Fredholm linear relations from X into Y is defined by:

$$\Phi_+(X,Y) := \{T \in \mathcal{CR}(X,Y) : \ R(T) \ is \ closed, \ and \ \alpha(T) < \infty\},$$

the set of lower semi-Fredholm linear relations from X into Y is defined by:

$$\Phi_-(X,Y) := \{T \in \mathcal{CR}(X,Y) : \ R(T) \ is \ closed, \ and \ \beta(T) < \infty\}.$$

If $X = Y$, we would simply note $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ by respectively $\Phi_+(X)$ and $\Phi_-(X)$.
Let M be a subspace of X such that $M \cap \mathcal{D}(A) \neq \emptyset$ and let $A \in \mathcal{LR}(X,Y)$. Then, the restriction $A_{|M}$ is the linear relation given by:

$$G(A_{|M}) := \{(m,y) \in G(A) : m \in M\} = G(A) \cap (M \times Y).$$

For A, $B \in \mathcal{LR}(X,Y)$ and $S \in \mathcal{LR}(Y,Z)$, the sum $A+B$ and the product SA are the linear relations defined by

$$G(A+B) := \{(x, y+z) \in X \times Y : (x,y) \in G(A) \ and \ (x,z) \in G(B)\}, \text{ and}$$
$$G(SA) : \{(x,z) \in X \times Z : (x,y) \in G(A), (y,z) \in G(S) \ for \ some \ y \in Y\}$$

respectively. If $\lambda \in \mathbb{K}$, then λA is defined by:

$$G(\lambda A) := \{(x, \lambda y) : (x,y) \in G(A)\}.$$

If $A \in \mathcal{LR}(X)$ and $\lambda \in \mathbb{K}$, then the linear relation $\lambda - A$ is given by:

$$G(\lambda - A) := \{(x, y - \lambda x) : (x,y) \in G(A)\}.$$

We note that $\|Ax\|$ and $\|A\|$ are not real norms. In fact, a nonzero relation can have a zero norm. A is said to be closed if its graph $G(A)$ is a closed subspace of $X \times Y$. The closure of A, denoted by $\overline{A}$, is defined in terms of its graph $G(\overline{A}) := \overline{G(A)}$. We denote by $\mathcal{CR}(X,Y)$ the class of all closed linear relations on X into Y. If $X = Y$, we would simply note $\mathcal{CR}(X,X) := \mathcal{CR}(X)$. If $\overline{A}$ is an extension of A (that is, $\overline{A}_{|\mathcal{D}(A)}$), we say that A is closable.
Let $A \in \mathcal{LR}(X,Y)$. We say that A is continuous if for each neighbourhood V in $\mathcal{R}(A)$, the inverse image $A^{-1}(V)$ is a neighbourhood in $\mathcal{D}(A)$ equivalently $\|A\| < \infty$; open if A^{-1} is continuous; bounded if $\mathcal{D}(A) = X$ and A is continuous; bounded below if it is injective and open; and compact if $\overline{Q_A A(B_{\mathcal{D}(A)})}$ is compact in Y ($B_{\mathcal{D}(A)} := \{x \in \mathcal{D}(A) : \|x\| \leq 1\}$). We denote by $\mathcal{KR}(X,Y)$ the class of all compact linear relations on X into Y. If $X = Y$, we would simply note $\mathcal{KR}(X,X) := \mathcal{KR}(X)$. We say that A is precompact if $Q_T T B_{\mathcal{D}(T)}$ is totally bounded in Y, and strictly singular if there is no infinite dimensional subspace M of $\mathcal{D}(A)$ for which $A_{|M}$ is injective, and open. If X is a normed linear space, then $X^{'}$ will denote the dual norm of X, i.e., the space of all continuous linear functionals x' are defined on X with the norm

$$\|x'\| = \inf\{\lambda : |x'x| \leq \lambda \|x\| \ for \ all \ x \in X\}.$$

If $K \subset X$ and $L \subset X^{'}$, we shall adopt the following notations:

$$\begin{aligned} K^{\perp} &:= \{x' \in X^{'} : x' = 0 \ for \ all \ x \in K\}, \\ L^{\top} &:= \{x \in X : x' = 0 \ for \ all \ x' \in L\}. \end{aligned}$$

Clearly, $K^{\perp}$ and $L^{\top}$ are closed linear subspaces of $X^{'}$ and X respectively.
The adjoint of T, T', is defined by

$$G(A') = G(-A^{-1})^{\perp} \subset Y^{'} \times X^{'}$$

where $\langle (y,x), (y',x') \rangle := \langle x, x' \rangle + \langle y, y' \rangle$. This means that

$$(y', x') \in G(A') \text{ if, and only if, } y'y - x'x = 0 \text{ for all } (x, y) \in G(T).$$

Similarly, we have $y'y = x'x$ for all $y \in Ax$, $x \in \mathcal{D}(A)$. Hence $x' \in A'y$ if, and only if, $y'Ax = x'x$ for all $x \in \mathcal{D}(A)$.

Let X be a complex Banach space and let $A \in \mathcal{CR}(X, Y)$. Suppose that $S \in \mathcal{LR}(X)$ is A-bounded with A-bounded $\delta < 1$ such that $S(0) \subset A(0)$ and $\mathcal{D}(A) \subset \mathcal{D}(S)$. We define the S resolvent set of A by

$$\rho_S(A) := \{\lambda \in \mathbb{C} : \ \lambda S - A \text{ is bijective}\}.$$

In this work, we are concerned with the following S-essential approximate point spectrum of A defined by:

$$\sigma_{eap,S}(A) := \bigcap_{K \in \mathcal{K}_A(X)} \sigma_{ap,S}(A + K).$$

Similarly we are concerned with the following S-essential defect spectrum of A defined by:

$$\sigma_{e\delta,S}(A) := \bigcap_{K \in \mathcal{K}_A(X)} \sigma_{\delta,S}(A + K),$$

where $\mathcal{K}_A(X) := \{K \in \mathcal{KR}(X) : \mathcal{D}(A) \subset \mathcal{D}(K) \textit{ and } K(0) \subset A(0)\}$,

$$\sigma_{ap,S}(A) := \{\lambda \in \mathbb{C} : \lambda S - A \text{ is not bounded below}\},$$

and

$$\sigma_{\delta,S}(A) := \{\lambda \in \mathbb{C} : \lambda S - A \text{ is not surjective}\}.$$

Note that if $S = I$, (the identity operator on X), we recover the usual definition of the essential spectra of a bounded linear operator A.

The purpose of this paper is to extend the results in [8] mentioned above to the general case of S-essentiel stability in the first place. In the second place, in other hypotheses, we show the stability of S-essential approximate point spectrum.

We organize the paper in the following way. Section 2 consists in establishing some preliminary results which will be needed in the sequel. The main results of Section 3 are Lemma 3.1 and Lemma 3.2, which give information concerning the equivalence of norm. In section 4, we investigate the stability of the S-essential approximate point spectrum and the S-essential defect spectrum of closed and closable linear relations under relatively compact and precompact perturbations on a Banach space (see Theorem 4.1), and under different hypotheses we find the stability of the S-essential approximate point spectrum (see Theorem 4.2).

2. Preliminaries

The goal of this section consists in establishing some preliminary results which will be needed in the sequel.

Definition 2.1. [9, Definition, IV.3.1] *Let $A \in \mathcal{LR}(X, Y)$, and let X_A denote the vector space $\mathcal{D}(A)$ normed by*

$$\|x\|_A := \|x\| + \|Ax\|, \textit{ for all } x \in \mathcal{D}(A).$$

Let $G_A \in \mathcal{LR}(X_A, X)$ be the identity injection of $X_A = (\mathcal{D}(A), \|.\|_A)$ into X, i.e.,

$$\mathcal{D}(G_A) = X_A, \ G_A(x) = x, \ \textit{for all } x \in X_A.$$

Definition 2.2. [9, Definition, VII.2.1] *Let A, $B \in \mathcal{LR}(X, Y)$. B is said to be A-bounded (or bounded relative to A) if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and there exist non-negative constants a, and b, such that*

$$\|Bx\| \leq a\|x\| + b\|Ax\| \textit{ for all } x \in \mathcal{D}(A). \tag{2.1}$$

In that case the infimum of all the constant b which satisfies (2.1) is called the A-bound of B.
We note that B is A-bounded if, and only if, $\mathcal{D}(A) \subset \mathcal{D}(B)$, and BG_A is bounded.

Definition 2.3. [9, Definition VII.2.1] *Let $A \in \mathcal{LR}(X, Y)$. A relation $B \in \mathcal{LR}(X, Y)$ is said to be A-compact (or compact relative to A) if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and BG_A is compact.*
B is called A-precompact (or precompact relative to A) if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and BG_A is precompact.

Lemma 2.1. [7, Lemma, 3.1] *Let S, $T \in \mathcal{LR}(X, Y)$ satisfies $S(0) \subset T(0)$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. If S is T-compact, then S is T-bounded.*

Lemma 2.2. [7, Lemma, 3.6] *Let A, B and $S \in \mathcal{LR}(X,Y)$ satisfy $B(0) \cup S(0) \subset A(0)$. Suppose that B is A-bounded with A-bound δ_1, S is A-bounded with A-bound δ_2, and Y is complete.*

(i) *If $\delta_1 + \delta_2 < 1$, and A is closed, then $A + B + S$ is closed.*

(ii) *If $\delta_1 + \delta_2 < \frac{1}{2}$, and $A + B + S$ is closed, then A is closed.*

Lemma 2.3. [7, Lemma, 4.1] *Let $S \in \mathcal{LR}(X,Y)$ and $A \in \mathcal{F}_+(X,Y)$ with $\dim \mathcal{D}(A) = \infty$. If S is precompact, then S is strictly singular.*
If additionally $S(0) \subset \overline{A(0)}$, then $A + S \in \mathcal{F}_+(X,Y)$.

Proposition 2.1. [5, Theorem 2.17] *Let $B \in \mathcal{LR}(X,Y)$, $A \in \mathcal{F}_+(X,Y)$ with $G(B) \subset G(A)$, and $\dim D(B) = \infty$, then $B \in \mathcal{F}_+(X,Y)$.*

Lemma 2.4. [2, Lemma 2.3] *Let X be complete, $T \in \mathcal{CR}(X)$, and $K \in \mathcal{K}_T(X)$.*

(i) *If $T \in \Phi_+(X)$, then $T + K \in \Phi_+(X)$ with $i(T + K) = i(T)$.*

(ii) *If $T \in \Phi_-(X)$, then $T + K \in \Phi_-(X)$ with $i(T + K) = i(T)$.*

Proposition 2.2. [4, Theorem 3.1] *Let X be complete, $A \in \mathcal{CR}(X)$ and $\lambda \in \mathbb{C}$. If $S \in \mathcal{LR}(X)$ is A-bounded with A-bounded $\delta < 1$ such that $S(0) \subset A(0)$ and $\mathcal{D}(A) \subset \mathcal{D}(S)$, then*

(i) *$\lambda \notin \sigma_{eap,S}(A)$ if, and only if, $\lambda S - A \in \Phi_+(X)$ and $i(\lambda S - A) \leq 0$.*

(ii) *$\lambda \notin \sigma_{e\delta,S}(A)$ if, and only if, $\lambda S - A \in \Phi_-(X)$ and $i(\lambda S - A) \geq 0$.*

To end this section, we present the following Proposition suggested by Cross in [9].

Proposition 2.3. *Let A, $B \in \mathcal{LR}(X,Y)$*

(i) [9, Corollary V.2.5] *$A \in \mathcal{F}_+(X,Y)$ if, and only if, $AG_A \in \mathcal{F}_+(X_A,Y)$.*

(ii) [9, Corollary V.2.3] *If A is precompact, then A is continuous.*

(iii) [9, Proposition III.1.5] *Let $\mathcal{D}(A) \subset \mathcal{D}(B)$. If B is continuous, then $(A + B)' = A' + B'$.*

(iv) [9, Proposition V.5.15] *Let $A \in \mathcal{CR}(X,Y)$. $A \in \mathcal{KR}(X,Y)$ if, and only if, $A' \in \mathcal{KR}(Y',X')$.*

(v) [9, Proposition V.7.5] *$A \in \mathcal{F}_+(X,Y)$ if, and only if, $A' \in \mathcal{F}_-(Y^{'},X^{'})$ and $A^{'} \in \mathcal{F}_+(Y^{'},X^{'})$ if, and only if, $A \in \mathcal{F}_-(X,Y)$.*

(vi) [9, Proposition V.7.8] *If $\dim B(0) < \infty$, then $A + B - B \in \mathcal{F}_+(X,Y)$ if, and only if, $A \in \mathcal{F}_+(X,Y)$.*

(vii) [9, Proposition V.5.27] *If A is closable. Then $A \in \mathcal{F}_-(X,Y)$ if, and only if, $AG_A \in \mathcal{F}_-(X_A,Y)$.*

$(viii)$ [9, Proposition V.5.12] *Let $\mathcal{D}(A) \subset \mathcal{D}(B)$, and let $A \in \mathcal{F}_-(X,Y)$. If B is precompact, then $A + B \in \mathcal{F}_-(X,Y)$.*

3. Main results

In [9], book1 claims that $\|A\| - \|B\| \leq \|A - B\|$ is not in general true. He gives an example (see [9, Exercise, II.1.12]).
In the first Lemma in this section, we give a necessary and sufficient condition for two linear relations A and B so that the equality of $\|A\| - \|B\| \leq \|A - B\|$ become justified.

Lemma 3.1. *Let A, $B \in \mathcal{LR}(X,Y)$. If $B(0) \subset A(0)$ and $\mathcal{D}(A) \subset \mathcal{D}(B)$, then*

(i) *$\|Ax\| - \|Bx\| \leq \|Ax - Bx\|$, for $x \in \mathcal{D}(A)$.*

(ii) *$\|Ax\| - \|Bx\| \leq \|Ax + Bx\|$, for $x \in \mathcal{D}(A)$.*

Proof We have for $x \in \mathcal{D}(A)$, by Lemma [4, Lemma 2.2 (iii)], we get $(A - B + B)x = Ax$, then

$$\|(A - B + B)x\| = \|Ax\|, \tag{3.1}$$

(i) Using [9, Proposition, II.1.5] and from Eqs (3.1), we obtain $\|Ax\| \leq \|(A - B)x\| + \|Bx\|$. So $\|Ax\| - \|Bx\| \leq \|Ax - Bx\|$.

(ii) Using [9, Proposition, II.1.5] and from Eqs (3.1), we obtain $\|Ax\|\| \leq \|(A+B)x\| + \|Bx\|$. So $\|Ax\| - \|Bx\| \leq \|Ax + Bx\|$.

Lemma 3.2. *Let A, B, and $S \in \mathcal{LR}(X,Y)$ verifying $B(0) \subset A(0)$ and $\lambda \in \mathbb{C}$.*
If S is A-bounded with A-bound δ_1 and B is A-bounded with A-bound δ_2 such that $\delta_2 + |\lambda|\delta_1 < 1$, then $\|.\|_A$ and $\|.\|_{\lambda S-(A+B)}$ are equivalent.
In particular, $\|.\|_A$ and $\|.\|_{\lambda S-A}$ are equivalent.

Proof Since S is A-bounded with bound δ_1 and B is A-bounded with bound δ_2, there exist non-negative constants a, b, a_1 and b_1 such that, for $x \in \mathcal{D}(A)$, $\|Sx\| \leq a\|x\| + b\|Ax\|$ and $\|Bx\| \leq a_1\|x\| + b_1\|Ax\|$. So we have $-\|Bx\| \geq -a_1\|x\| - b_1\|Ax\|$, thus $\|Ax\| - \|Bx\| \geq -a_1\|x\| + (1-b_1)\|Ax\|$. Using Lemma 3.1 (ii), we get $\|Ax + Bx\| \geq -a_1\|x\| + (1-b_1)\|Ax\|$. On the other hand,

$$\begin{aligned}
\|x\|_{\lambda S-(A+B)} &= \|x\| + \|(\lambda S - (A+B))x\|,\\
&\geq \|x\| + \|(A+B))x\| - |\lambda|\|Sx\|,\\
&\geq \|x\| - a_1\|x\| + (1-b_1)\|Ax\| - |\lambda|\|Sx\|,\\
&\geq \|x\| - a_1\|x\| + (1-b_1)\|Ax\| - |\lambda|a\|x\| - b|\lambda|\|Ax\|,\\
&\geq (1 - a_1 - |\lambda|a)\|x\| + (1 - b_1 - |\lambda|b)|\|Ax\|,\\
&\geq \min(1 - a_1 - |\lambda|a, 1 - b_1 - |\lambda|b)|(\|x\| + \|Ax\|).
\end{aligned}$$

Therefore, $\|x\|_{\lambda S-(A+B)} \geq K|\|x\|_A$, with $K = \min(1 - a_1 - |\lambda|a, 1 - b_1 - |\lambda|b)$.
On the other hand, we obtain

$$\begin{aligned}
\|x\|_{\lambda S-(A+B)} &= \|x\| + \|(\lambda S - (A+B))x\|,\\
&\leq \|x\| + \|Ax\| + \|Bx\| + |\lambda|\|Sx\|,\\
&\leq \|x\| + a_1\|x\| + b_1\|Ax\| + \|Ax\| + |\lambda|a\|x\| + b|\lambda|\|Ax\|,\\
&\leq (1 + a_1 + |\lambda|a)\|x\| + (1 + b_1 + |\lambda|b)|\|Ax\|,\\
&\leq \max(1 + a_1 + |\lambda|a, 1 + b_1 + |\lambda|b)|(\|x\| + \|Ax\|).
\end{aligned}$$

Therefore, $\|x\|_{\lambda S-(A+B)} \leq H|\|x\|_A$, with $H = \max(1 + a_1 + |\lambda|a, 1 + b_1 + |\lambda|b)$.
We deduce that $\|.\|_A$ and $\|.\|_{\lambda S-(A+B)}$ are equivalent.

Lemma 3.3. *Let A, B, and $S \in \mathcal{LR}(X)$ and let $\lambda \in \mathbb{C}$.*

(i) $\mathcal{R}((\lambda S - A)G_B) = \mathcal{R}(\lambda S - A)$.

(ii) $\mathcal{N}((\lambda S - A)G_B) = \mathcal{N}(\lambda S - A)$.

Proof (i) Using the fact that $G_B x = (G_B)^{-1}x = x$, $\mathcal{R}(A) = A\mathcal{D}(A)$ and $\mathcal{D}(AB) = B^{-1}\mathcal{D}(A)$.

$$\begin{aligned}
\mathcal{R}((\lambda S - A)G_B) &= (\lambda S - A)G_B\mathcal{D}((\lambda S - A)G_B),\\
&= (\lambda S - A)\mathcal{D}((\lambda S - A)G_B),\\
&= (\lambda S - A)G_B\mathcal{D}(\lambda S - A),\\
&= (\lambda S - A)\mathcal{D}(\lambda S - A),\\
&= \mathcal{R}(\lambda S - A).
\end{aligned}$$

$$(ii)\mathcal{N}((\lambda S - A)G_B) = \{x \in \mathcal{D}((\lambda S - A)G_B), (\lambda S - A)G_B(x) = (\lambda S - A)G_B(0)\},$$

$$= \{x \in \mathcal{D}(\lambda S - A), (\lambda S - A)(x) = (\lambda S - A)(0)\},$$

$= \mathcal{N}(\lambda S - A)$.

Proposition 3.1. *Let X be complete, let A, B, $S \in \mathcal{LR}(X)$ satisfy $B(0) \subset A(0)$ and let $\lambda \in \mathbb{C}$. If B is A-precompact, then $i(\lambda S - A) = i(\lambda S - (A+B))$.*

Proof Since B is A-precompact, then BG_A is precompact, and X is complete. By Remark [9, Note V.1 p 134] BG_A is compact.

$$\begin{aligned} i(\lambda S - A) &= i((\lambda S - A)G_A), \text{ by Lemma 3.3}, \\ &= i((\lambda S - A)G_A + BG_A), \text{ by Lemma 2.4 } (BG_A \text{ is compact}), \\ &= i((\lambda S - (A+B))G_A), \\ &= i(\lambda S - (A+B)), \text{ by Lemma 3.3}. \end{aligned}$$

Lemma 3.4. *Let A, $B \in \mathcal{LR}(X,Y)$ such that $G(A) \subsetneq G(B)$. We have*

(i) $\alpha(A) \leq \alpha(B)$.

(ii) $\beta(B) \leq \beta(A)$.

(iii) $i(A) \leq i(B)$.

Proof *(i)* We have $\alpha(A) := \dim(\mathcal{N}(A))$. Then

$$\begin{aligned} \mathcal{N}(A) &:= \{x \in \mathcal{D}(A) : \ (x,0) \ \in G(A)\}, \\ &\subsetneq \{x \in \mathcal{D}(A) : \ (x,0) \ \in G(B)\}, \\ &= \mathcal{N}(B_{|\mathcal{D}(A)}), \\ &\subset \mathcal{N}(B). \end{aligned}$$

So, $\alpha(A) \leq \alpha(B)$.
(ii) We have $\beta(A) := codim(\mathcal{R}(A)) = dim(Y/_{\mathcal{R}(A)})$. Let $y \in \mathcal{R}(A)$. Then, $y \in Ax$ for all $x \in \mathcal{D}(A)$. We get by $G(A) \subsetneq G(B)$, $y \in Bx$ for all $x \in \mathcal{D}(A)$. So, $y \in \mathcal{R}(B_{|\mathcal{D}(A)})$. Thus $y \in \mathcal{R}(B)$. We infer that $Y/_{\mathcal{R}(B)} \subset Y/_{\mathcal{R}(A)}$. Then $\beta(B) \leq \beta(A)$.
(iii) $i(A) := \alpha(A) - \beta(A) \leq \alpha(B) - \beta(B) = i(B)$.

Lemma 3.5. *Let A, $B \in \mathcal{LR}(X,Y)$. If $G(A) \subsetneq G(B)$, then $G(A) \subsetneq G(A+B)$.*

Proof

$$\begin{aligned} G(A) &:= \{(x,y) \in X \times Y : \ x \ \in \mathcal{D}(A) \subsetneq \mathcal{D}(B) \text{ and } y \in Ax \subsetneq Bx\}, \\ &\subsetneq \{(x,y) \in X \times Y : \ x \in \mathcal{D}(A) \cap \mathcal{D}(B) \text{ and } y \in Ax + Bx\}, \\ &:= \{(x,y) \in X \times Y : \ x \in \mathcal{D}(A+B) \text{ and } y \in (A+B)x\}, \\ &:= G(A+B). \end{aligned}$$

4. Stability of $\sigma_{eap,S}(.)$ and $\sigma_{e\delta,S}(.)$

In this section, on one level, we study the stability of the S-essential approximate point spectrum and the S-essential defect spectrum of closed and closable linear relations under relatively precompact perturbations on a Banach space. On another level, we study the stability of the S-essential approximate point spectrum but under assumptions different from those adopted above.

Theorem 4.1. *Let X be complete, $A \in \mathcal{CR}(X)$, B, $S \in \mathcal{LR}(X)$ satisfy $B(0) \subset S(0) \subset A(0)$ and $\dim \mathcal{D}(B) = \infty$, and $\lambda \in \mathbb{C}$.*
If S is A-bounded with A-bound δ_1 and B is A-precompact with A-bound δ_2 such that $\delta_2 + |\lambda|\delta_1 < 1$, then

$$\sigma_{eap,S}(A+B) = \sigma_{eap,S}(A),$$

and

$$\sigma_{e\delta,S}(A+B) = \sigma_{e\delta,S}(A).$$

Proof Let B be A-precompact, then BG_A is precompact, and X and X_A are complete. By Remark [9, Note V.1 p 134], we get BG_A is compact. By Lemma 2.1, we get BG_A is bounded, then B is A-bounded with A-bound δ_2. Using the fact that S is A-bounded with A-bound δ_1 and $\delta_2 + |\lambda|\delta_1 < 1$ and by applying Lemma 2.2 *(i)*, we obtain $\lambda S - (A+B)$ is closed.
Suppose that $\lambda \notin \sigma_{eap,S}(A)$, then by Proposition 2.2, $\lambda S - A \in \Phi_+(X)$. By Proposition 2.3 *(i)*, we get $(\lambda S - A)G_{\lambda S - A} \in \Phi_+(X_A)$, which gives $(\lambda S - A)G_A \in \Phi_+(X_A)$ by referring to Lemma 3.2. Since BG_A is compact and $\dim \mathcal{D}(B) = \dim \mathcal{D}(BG_A) = \infty$, then using Lemma 2.3 it follows that $(\lambda S - (A+B))G_A \in \Phi_+(X_A)$. By Lemma 3.2, we obtain $(\lambda S - (A+B))G_{\lambda S-(A+B)} \in \Phi_+(X_A)$. Using

Proposition 2.3 (i), we get $\lambda S-(A+B)\in\Phi_+(X)$ and we have $i(\lambda S-A)=i(\lambda S-(A+B))$ by Proposition 3.1, that is $\lambda\notin\sigma_{eap,S}(A+B)$ by Proposition 2.2. So $\sigma_{eap,S}(A+B)\subseteq\sigma_{eap,S}(A)$. Conversely, let $\lambda\notin\sigma_{eap,S}(A+B)$. Then by proposition 2.2, we have $\lambda S-(A+B)\in\Phi_+(X)$. Using Proposition 2.3 (i), we get $(\lambda S-(A+B))G_{\lambda S-(A+B)}\in\Phi_+(X_A)$, which gives $(\lambda S-(A+B))G_A\in\Phi_+(X_A)$ by referring to Lemma 3.2. Since BG_A is compact, then by Lemma 2.3, it follows that $(\lambda S-A)G_A\in\Phi_+(X_A)$. By Lemma 3.2, we obtain $(\lambda S-A)G_{\lambda S-A}\in\Phi_+(X_A)$. Using Proposition 2.3 (i), we get $\lambda S-A\in\Phi_+(X)$. We have $i(\lambda S-A)=i(\lambda S-(A+B))$ by Proposition 3.1, that is $\lambda\notin\sigma_{eap,S}(A)$ by Proposition 2.2. We infer that

$$\sigma_{eap,S}(A+B)=\sigma_{eap,S}(A).$$

Now suppose that $\lambda\notin\sigma_{e\delta,S}(A)$, then by Proposition 2.2, we have $\lambda S-A\in\Phi_-(X)$. Applying Proposition 2.3 (vii), we obtain $(\lambda S-A)G_{\lambda S-A}\in\Phi_-(X_A)$. Using Lemma 3.2, we get $(\lambda S-A)G_A\in\Phi_-(X_A)$. Since BG_A is precompact, then by Proposition 2.3 $(viii)$, we obtain $(\lambda S-(A+B))G_A\in\Phi_-(X_A)$. Resorting to Lemma 3.2, we get $(\lambda S-(A+B))G_{\lambda S-(A+B)}\in\Phi_-(X_A)$. So applying Proposition 2.3 (vii), we get $(\lambda S-(A+B))\in\Phi_-(X)$. We have $i(\lambda S-A)=i(\lambda S-(A+B))$ by Proposition 3.1, that is $\lambda\notin\sigma_{e\delta,S}(A+B)$ by Proposition 2.2. Then

$$\sigma_{e\delta,S}(A+B)\subset\sigma_{e\delta,S}(A).$$

Conversely, let $\lambda\notin\sigma_{e\delta,S}(A+B)$, then by Proposition 2.2, we obtain $\lambda S-(A+B)\in\Phi_-(X)$. Using Proposition 2.3 (vii), we get $(\lambda S-(A+B))G_{\lambda S-(A+B)}\in\Phi_-(X_A)$. Applying Lemma 3.2, we get $(\lambda S-(A+B))G_A\in\Phi_-(X_A)$.

The latter holds if, and only if, $((\lambda S-(A+B))G_A)^{'}\in\Phi_+(X_A^{'})$ by Proposition 2.3 (v). Subsequently, using Proposition 2.3 (ii) and (iii), we get $((\lambda S-A)G_A)^{'}+(BG_A)^{'}\in\Phi_+(X_A^{'})$. Since BG_A is precompact, then by Proposition 2.3 (iv) we have $(BG_A)'$ is precompact. Applying Lemma 2.4, we have $((\lambda S-A)G_A)^{'}\in\Phi_+(X_A^{'})$. Besides using Proposition 2.3 (v), we get $(\lambda S-A)G_A\in\Phi_-(X_A)$. So by Proposition 2.3 (vii), $((\lambda S-A)\in\Phi_-(X)$. We have $i(\lambda S-A)=i(\lambda S-(A+B))$ by Proposition 3.1. That is $\lambda\notin\sigma_{e\delta,S}(A)$ by Proposition 2.2. We conclude that

$$\sigma_{e\delta,S}(A+B)=\sigma_{e\delta,S}(A).$$

Theorem 4.2. *Let $A\in\mathcal{CR}(X)$, B, $S\in\mathcal{LR}(X)$ and let $\lambda\in\mathbb{C}$. Suppose that S is A-bounded with A-bound δ_1 and B is A-bounded with A-bound δ_2 such that $\delta_2+|\lambda|\delta_1<1$. If $G(B)\subsetneq G(\lambda S)\subsetneq G(A)$ and $\dim\mathcal{D}(B)=\infty$, then*

(i) $\sigma_{eap,S}(A+B)\subset\sigma_{eap,S}(A)$.

(ii) *If $\dim B(0)<\infty$, then $\sigma_{eap,S}(A+B)=\sigma_{eap,S}(A)$.*

Proof Since S is A-bounded with A-bound δ_1 and B is A-bounded with A-bound δ_2 such that $\delta_2+|\lambda|\delta_1<1$, then applying Lemma 2.2, we obtain $\lambda S-(A+B)$ is closed.

(i) Suppose that $\lambda\notin\sigma_{eap,S}(A)$, then by Proposition 2.2, $\lambda S-A\in\Phi_+(X)$ and $i(\lambda S-A)\leq 0$. Since $G(B)\subsetneq G(\lambda S)$ and $G(\lambda S)\subsetneq G(A)$, then applying Lemma 3.5, we get $G(\lambda S)\subsetneq G(\lambda S-A)$, and then $G(B)\subsetneq G(\lambda S-A)$. On the one hand, we have

$$\begin{aligned}G(\lambda S-(A+B)) \ &:=\ \{(x,y)\in X\times X:(x,y_1)\in G(\lambda S-A)\ \mathit{and}\\ &\qquad (x,y_2)\in G(B)\subsetneq G(\lambda S-A),\ \mathit{where}\ y=y_1+y_2\},\\ &\subsetneq\ G(\lambda S-A).\end{aligned}$$

On the other hand, $\dim\mathcal{D}(\lambda S-(A+B))=\dim(\mathcal{D}(\lambda S-A)\cap\mathcal{D}(B))=\dim\mathcal{D}(B)=\infty$. Then by Proposition 2.1, we obtain $\lambda S-(A+B)\in\Phi_+(X)$.

We have $G(\lambda S-(A+B))\subsetneq G(\lambda S-A)$, using Lemma 3.4, we get $i(\lambda S-(A+B))\leq(\lambda S-A)\leq 0$. So by Proposition 2.2, we obtain $\lambda\notin\sigma_{eap,S}(A+B)$. Then

$$\sigma_{eap,S}(A+B)\subset\sigma_{eap,S}(A).$$

(ii) Since $G(B) \subsetneq G(\lambda S)$ and $G(\lambda S) \subsetneq G(A)$, then applying Lemma 3.5, we get $G(\lambda S) \subsetneq G(\lambda S - A)$ and $G(B) \subsetneq G(\lambda S - A)$. By Lemma 3.5, we obtain $G(B) \subsetneq G(\lambda S - (A + B))$. On the one hand,

$$\begin{aligned} G(\lambda S - A - B + B) &:= \{(x, y + z) \in X \times Y : (x, y) \in G(\lambda S - (A + B)) \textit{ and} \\ & \quad\; (x, z) \in G(B)\} \\ &\subsetneq \{(x, y + z) \in X \times Y : (x, y) \in G(\lambda S - (A + B)) \textit{ and} \\ & \quad\; (x, z) \in G(S) \subsetneq G(\lambda S - (A + B))\} \\ &:= G(\lambda S - (A + B)). \end{aligned}$$

On the other hand, $\dim \mathcal{D}(\lambda S - A - B + B) = \dim(\mathcal{D}(\lambda S) \cap \mathcal{D}(A) \cap \mathcal{D}(B)) = \dim \mathcal{D}(B) = \infty$. Let $\lambda \notin \sigma_{eap,S}(A + B)$. Then by proposition 2.2, we have $\lambda S - (A + B) \in \Phi_+(X)$. Since $\lambda S - A$ is closed and $\dim B(0) < 0$, then $\lambda S - A - B + B$ is closed and we have $G(\lambda S - A - B + B) \subsetneq G(\lambda S - (A + B))$, $\dim \mathcal{D}(\lambda S - A - B + B) = \infty$, then by Proposition 2.1, we get $\lambda S - A - B + B \in \Phi_+(X)$. Thus by Proposition 2.3 (vi), we obtain $\lambda S - A \in \Phi_+(X)$. Using Lemma 3.4, we get $i(\lambda S - A) \leq i(\lambda S - A - B + B) \leq (\lambda S - (A + B)) \leq 0$. So by Proposition2.2, we obtain $\lambda \notin \sigma_{eap,S}(A)$. Thus, $\sigma_{eap,S}(A) \subset \sigma_{eap,S}(A + B)$. We infer that

$$\sigma_{eap,S}(A + B) = \sigma_{eap,S}(A).$$

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