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SOME INTEGRAL INEQUALITIES FOR BETA-PREINVEX FUNCTIONS

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Abstract. The main objective of this paper is to introduce and study a new class of preinvex functions, which is called beta-preinvex functions. Some Hermite-Hadamard type inequalities for beta-preinvex functions are established. Our results can be viewed as significant and important generalizations of several previously known results. We also establish some integral inequalities involving Euler beta functions for the class of functions whose certain powers of the absolute value are beta-preinvex function. Results proved in this paper may stimulate further research in different areas of pure and applied sciences.

1. Introduction

In recent years, the convex functions and convex sets have been generalized in several directions using novel and innovative techniques to study a wide class of unrelated problem in a unified and general framework. Hanson [2] introduced the concept of invex functions involving the bifunction in study of nonlinear programming. This concept stimulated much interest in applications of these invex function in different branches of pure and applied sciences. Wier and Mond [24] introduced and investigated another class of convex functions, which is called the preinvex functions. They proved that the differentiable preinvex functions are invex functions, but the converse may not true. It is known that the invex functions and preinvex functions are equivalent under some conditions, see [10]. Noor[5] proved that the minimum of a differentiable preinvex functions on the invex sets can be characterized by a class of variational-like inequalities. Noor [7] also proved that a function f is a preinvex function, which satisfies the Hermite-Hadmard type integral inequalities. This result can be viewed as an analogous to the convex functions. These results strongly influenced the recent recent research trends. For the applications, properties and other aspects of the preinvex functions, see [1, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

Various refinements of the Hermite-Hadamard inequalities for the convex functions and their variant forms are being obtained in the literature by many researchers. Tunc et al.[22] introduced the concept of beta-convex functions, which include classical convex functions, s-convex functions, s-Godunova-Levin functions, tgs-convex functions and P-functions as special cases. Noor et al. [16, 16, 17] introduced the concepts of (p, q) and tgs-preinvex f unctions and obtained several integral inequalities via these preinvex functions.

Motivated and inspired by the ideas and techniques of Tunc et. el.[22] and Noor et. al. [16, 17, 18], we introduce some new classes of of beta-preinvex functions and derive some estimates involving the Euler Beta function of the integral $\int_a^{a+\eta(b,a)}(x-a)^p(a+\eta(b,a)-x)^q f(x)\mathrm{d}x$ for the class of functions whose certain powers of the absolute value are beta-preinvex function. This is the main motivation of this paper. Some special cases are discussed. Our results include the previously known results for preinvex functions and their varinat forms as special cases. It is expected that the ideas and techniques considered in this paper be staring for the future research. The interested readers are encouraged to find the novel and innovative applications of these results in other areas.

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2. PRELIMINARIES

Let K_η be a nonempty closed set in $\mathbb{R}$. Let $f : K_\eta \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function and $\eta(\cdot,\cdot) : K_\eta \times K_\eta \to \mathbb{R}$ be a continuous bifunction. In this section, we recall the following new and known concepts.

Definition 2.1. [24]. *A set $K_\eta \subseteq \mathbb{R}$ is said to be an invex set with respect to the bifunction $\eta(\cdot,\cdot)$, if*

$$x + t\eta(y,x) \in K_\eta, \quad \forall x,y \in K_\eta, t \in [0,1].$$

If $\eta(y,x) = y - x$, then invex set K_η reduces to classical convex set. Clearly, every convex set is an invex set but the converse is not true.
We new define some new concepts of beta-preinvex functions and its variant forms.

Definition 2.2. *A function $f : K_\eta \subseteq \mathbb{R} \to \mathbb{R}$ is said to be beta-preinvex function, where $p,q > -1$, if*

$$f\big(x + t\eta(y,x)\big) \le (1-t)^p t^q f(x) + t^p (1-t)^q f(y), \quad \forall x,y \in K_\eta, t \in (0,1). \tag{2.1}$$

If $t = \frac{1}{2}$, then

$$f\left(\frac{2x + \eta(y,x)}{2}\right) \le \frac{f(x) + f(y)}{2^{p+q}}, \quad \forall x,y \in K_\eta, \tag{2.2}$$

which is called the Jensen beta-preinvex function.

We now discuss some special cases of beta-preinvex function, which appears to be new ones.

I). If $p = 1$ and $q = 0$, then Definition 2.2, reduces to:

Definition 2.3. [24]. *A function $f : K_\eta \subseteq \mathbb{R} \to \mathbb{R}$ is said to be preinvex function with respect to the bifunction $\eta(\cdot,\cdot)$, if*

$$f\big(x + t\eta(y,x)\big) \le (1-t)f(x) + tf(y), \qquad \forall x,y \in K_\eta, t \in [0,1].$$

If $t = \frac{1}{2}$, then

$$f\big(\frac{2x + \eta(y,x)}{2}\big) \le \frac{f(x) + f(y)}{2}, \quad \forall \in K_\eta,$$

which is called the Jensen preinvex function.
It has been shown that a function f is a preinvex function, if and only if, it satisfies the inequality of the type

$$f(\frac{2a + \eta(b,a)}{2}) \le \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \le \frac{f(a) + f(b)}{2},$$

which is known as the Hermite-Hadamard-Noor integral inequality. See [5, 6, 7] for the recent results in this direction. It is worth mentioning that the minimum of of a differentiable preinvex functions on the invex sets in a normed space can be characterized by a class of variational inequalities, which is known as the variational-like inequalities. For the formulation, applications and numerical methods for solving the variational-like inequalities and related problems, see [1, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] and the references therein.

II). If $p = 0$ and $q = 0$, then

Definition 2.4. [10, 15]. *A function $f : K_\eta \subseteq \mathbb{R} \to \mathbb{R}$ is said to be P-preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\big(x + t\eta(y,x)\big) \le f(x) + f(y), \qquad \forall x,y \in K_\eta.$$

III). If $p = -1$ and $q = 0$, then Definition 2.2 reduces to:

Definition 2.5. [15]. *A function $f : K_\eta \subseteq \mathbb{R} \to \mathbb{R}$ is said to be Godunova-Levin preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\big(x + t\eta(y,x)\big) \le \frac{f(x)}{1-t} + \frac{f(y)}{t}, \qquad \forall x,y \in K_\eta, t \in (0,1).$$

IV). If $p = 1$ and $q = 1$, then Definition 2.2 reduces to:

Definition 2.6. [16]. *A function $f : K_\eta \subseteq \mathbb{R} \to \mathbb{R}$ is said to be tgs-preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\big(x + t\eta(y,x)\big) \leq t(1-t)[f(x) + f(y)], \qquad \forall x, y \in K_\eta, t \in [0,1].$$

V). If $p = s$ and $q = 0$, then Definition 2.2 reduces to:

Definition 2.7. [17]. *A function $f : K_\eta \subseteq \mathbb{R} \to \mathbb{R}$ is said to be s-preinvex function with respect to $\eta(\cdot,\cdot)$, where $s \in [-1,1]$, if*

$$f\big(x + t\eta(y,x)\big) \leq (1-t)^s f(x) + t^s f(y), \qquad \forall x, y \in K_\eta, t \in (0,1).$$

VI). If $p = -s$ and $q = 0$, then Definition 2.2 reduces to:

Definition 2.8. *A function $f : K_\eta \subseteq \mathbb{R} \to \mathbb{R}$ is said to be Godunova-Levin s-preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\big(x + t\eta(y,x)\big) \leq (1-t)^{-s} f(x) + t^{-s} f(y), \qquad \forall x, y \in K_\eta, t \in (0,1).$$

VII). If $p = \frac{1}{2}$ and $q = -\frac{1}{2}$, then Definition 2.2, reduces to:

Definition 2.9. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be generalized MT-preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\big(x + t\eta(y,x)\big) \leq \frac{\sqrt{1-t}}{\sqrt{t}} f(x) + \frac{\sqrt{t}}{\sqrt{1-t}} f(y), \qquad \forall x, y \in K_\eta, t \in (0,1).$$

Definition 2.10. *A function $f : K_\eta \subseteq \mathbb{R} \to \mathbb{R}$ is said to be* log*-beta-preinvex function with respect to $\eta(\cdot,\cdot)$, where $p, q > -1$, if*

$$f\big(x + t\eta(y,x)\big) \leq [f(x)]^{(1-t)^p t^q} [f(y)]^{t^p (1-t)^q}, \qquad \forall x, y \in K_\eta, t \in (0,1).$$

It follows that

$$\log f\big(x + t\eta(y,x)\big) \leq (1-t)^p t^q \log f(x) + t^p (1-t)^q \log f(y).$$

From definition 2.10, we have

$$\begin{aligned} f\big(x + t\eta(y,x)\big) &\leq [f(x)]^{(1-t)^p t^q} [f(y)]^{t^p (1-t)^q} \\ &\leq (1-t)^p t^q f(x) + t^p (1-t)^q f(y). \end{aligned}$$

This shows that, log-beta-preinvex function implies beta-preinvex function, but the converse is not true.
For appropriate and suitable choices of $p, q > -1$, and the invex set, one can obtain several new and known classes of preinvex functions and its variant form from Definition 2.2 and Definition 2.10. This shows that the concept of beta-preinvex functions are quite general and unifying ones.

We recall the following special function which is known as Beta function.

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0,$$

where $\Gamma(.)$ is the Gamma function.

We also recall the well-known assumption about the bifunction $\eta(.,.)$, which plays an important role in the studies of the variational-like inequalities and integral inequalities.
Condition C[4]: Let $I \subseteq \mathbb{R}$ be an invex set with respect to bifunction $\eta(\cdot,\cdot) : I \times I \to \mathbb{R}$. For any $x, y \in I$ and any $t \in [0,1]$, we have

$$\begin{aligned} \eta(y, y + t\eta(x,y)) &= -t\eta(x,y) \\ \eta(x, y + t\eta(x,y)) &= (1-t)\eta(x,y). \end{aligned}$$

3. Hermite-Hadamard Inequalities

In this section, we derive Hermite-Hadamard inequalities for beta-preinvex function. Without loss of generality, we denote by $I = [a, a + \eta(b, a)]$ unless otherwise specified.

Theorem 3.1. *Let $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \to \mathbb{R}$ be beta-preinvex function with $a < a + \eta(b, a)$. If $f \in L[a, a + \eta(b, a)]$ and Condition C holds, then*

$$\begin{aligned} 2^{p+q-1} f\left(\frac{2a + \eta(b,a)}{2}\right) &\leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \\ &\leq \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}[f(a) + f(b)] \end{aligned}$$

Proof. Let f be beta-preinvex function. Then taking $x = a + t\eta(b, a)$ and $y = a + (1 - t)\eta(b, a)$ in (2.2), and using condition C, we have

$$\begin{aligned} f\left(\frac{2a + \eta(b,a)}{2}\right) &\leq \frac{1}{2^{p+q}}\left[f\big(a + t\eta(b,a)\big) + f\big(a + (1-t)\eta(b,a)\big)\right] \\ &= \frac{1}{2^{p+q}}\left[\int_0^1 f\big(a + t\eta(b,a)\big)\mathrm{d}t + \int_0^1 f\big(a + (1-t)\eta(b,a)\big)\,\mathrm{d}t\right] \end{aligned}$$

This implies

$$2^{p+q-1} f\left(\frac{2a + \eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x$$

Now consider

$$\begin{aligned} \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x &= \int_0^1 f\big(a + t\eta(b,a)\big)\mathrm{d}t \\ &\leq f(a)\int_0^1 (1-t)^p t^q \mathrm{d}t + f(b)\int_0^1 t^p(1-t)^q \mathrm{d}t \\ &= [f(a) + f(b)]\beta(p+1, q+1), \end{aligned}$$

which is the required result. □

Theorem 3.2. *Let $f, g : I \subset \mathbb{R} \to \mathbb{R}$ be beta-preinvex functions. If $f, g \in L[a, a + \eta(b, a)]$, then*

$$\begin{aligned} &\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(2a + \eta(b,a) - x)\mathrm{d}x \\ \leq\ &\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)} M(a,b) + \frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)} N(a,b), \end{aligned}$$

where

$$M(a, b) = f(a)g(a) + f(b)g(b) \tag{3.1}$$

$$N(a, b) = f(a)g(b) + f(b)g(a) \tag{3.2}$$

Proof. Let f, g be beta-preinvex functions. Then

$$f\big(a + t\eta(b,a)\big) \leq (1-t)^p t^q f(a) + t^p(1-t)^q f(b) \tag{3.3}$$

$$g\big(a + (1-t)\eta(b,a)\big) \leq t^p(1-t)^q g(a) + (1-t)^p t^q g(b). \tag{3.4}$$

From (3.3) and (3.4), we have

$$\begin{aligned} &f\big(a + t\eta(b,a)\big)g\big(a + (1-t)\eta(b,a)\big) \\ \leq\ &[(1-t)^p t^q f(a) + t^p(1-t)^q f(b)][t^p(1-t)^q g(a) + (1-t)^p t^q g(b)] \end{aligned} \tag{3.5}$$

Integrating both sides of (3.5), we obtain

$$\begin{aligned}
&\int_0^1 f\big(a+t\eta(b,a)\big)g\big(a+(1-t)\eta(b,a)\big)\mathrm{d}t \\
\leq\ & \int_0^1 [(1-t)^p t^q f(a) + t^p(1-t)^q f(b)][t^p(1-t)^q g(a) + (1-t)^p t^q g(b)]\mathrm{d}t \\
=\ & [f(a)g(a)+f(b)g(b)]\int_0^1 t^{p+q}(1-t)^{p+q}\mathrm{d}t + [f(a)g(b)+f(b)g(a)]\int_0^1 t^{2p}(1-t)^{2q}\mathrm{d}t \\
=\ & M(a,b)\beta(p+q+1,p+q+1) + N(a,b)\beta(2p+1,2q+1)
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(2a+\eta(b,a)-x)\mathrm{d}x \\
\leq\ & M(a,b)\beta(p+q+1,p+q+1) + N(a,b)\beta(2p+1,2q+1) \\
=\ & \frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}M(a,b) + \frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}N(a,b),
\end{aligned}$$

which is the required result. □

If $g(2a+\eta(b,a)-x) = g(x)$ in Theorem 3.2, then it reduces to the following result.

Corollary 3.1. *Let $f, g : I \subset \mathbb{R} \to \mathbb{R}$ be beta-preinvex functions. If $f, g \in L[a, a+\eta(b,a)]$, then*

$$\begin{aligned}
&\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}x \\
\leq\ & \frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}M(a,b) + \frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}N(a,b),
\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (3.1) *and* (3.2) *respectively.*

Theorem 3.3. *Let $f, g : I \subset \mathbb{R} \to \mathbb{R}$ be beta-preinvex functions. If $fg \in L[a, a+\eta(b,a)]$ and Condition C holds, then*

$$\begin{aligned}
&2^{2(p+q)-1} f\left(\frac{2a+\eta(b,a)}{2}\right) g\left(\frac{2a+\eta(b,a)}{2}\right) \\
&-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}x \\
\leq\ & \frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}M(a,b) + \frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}N(a,b),
\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (3.1) *and* (3.2) *respectively.*

Proof. Let f be beta-preinvex function. Then taking $x = a+t\eta(b,a)$ and $y = a+(1-t)\eta(b,a)$ in (2.2) and using condition C, we have

$$\begin{aligned}
f\left(\frac{2a+\eta(b,a)}{2}\right) &\leq \frac{1}{2^{p+q}}\big[f\big(a+t\eta(b,a)\big) + f\big(a+(1-t)\eta(b,a)\big)\big], \\
g\left(\frac{2a+\eta(b,a)}{2}\right) &\leq \frac{1}{2^{p+q}}\big[g\big(a+t\eta(b,a)\big) + g\big(a+(1-t)\eta(b,a)\big)\big].
\end{aligned}$$

Consider

$$
\begin{aligned}
& f\left(\frac{2a+\eta(b,a)}{2}\right)g\left(\frac{2a+\eta(b,a)}{2}\right) \\
\leq\ & \frac{1}{2^{2p+2q}}\left[f\big(a+t\eta(b,a)\big)+f\big(a+(1-t)\eta(b,a)\big)\right] \\
& \left[g\big(a+t\eta(b,a)\big)+g\big(a+(1-t)\eta(b,a)\big)\right] \\
\leq\ & \frac{1}{2^{2p+2q}}\left[f\big(a+t\eta(b,a)\big)g\big(a+t\eta(b,a)\big)\right. \\
& +f\big(a+(1-t)\eta(b,a)\big)g\big(a+(1-t)\eta(b,a)\big) \\
& +[(1-t)^p t^q f(a)+t^p(1-t)^q f(b)][t^p(1-t)^q g(a)+(1-t)^p t^q g(b)] \\
& \left.+[t^p(1-t)^q f(a)+(1-t)^p t^q f(b)][(1-t)^p t^q g(a)+t^p(1-t)^q g(b)]\right].
\end{aligned}
$$

Integrating over $[0,1]$, we have

$$
\begin{aligned}
& \int_0^1 f\left(\frac{2a+\eta(b,a)}{2}\right)g\left(\frac{2a+\eta(b,a)}{2}\right)\mathrm{d}t \\
\leq\ & \frac{1}{2^{2p+2q}}\left[\int_0^1 f\big(a+t\eta(b,a)\big)g\big(a+t\eta(b,a)\big)\mathrm{d}t\right. \\
& +\int_0^1 f\big(a+(1-t)\eta(b,a)\big)g\big(a+(1-t)\eta(b,a)\big)\mathrm{d}t \\
& +2[f(a)g(a)+f(b)g(b)]\int_0^1 t^{p+q}(1-t)^{p+q}\mathrm{d}t \\
& \left.+2[f(a)g(b)+f(b)g(a)]\int_0^1 t^{2p}(1-t)^{2q}\mathrm{d}t\right] \\
=\ & \frac{1}{2^{2p+2q}}\left[\int_0^1 f\big(a+t\eta(b,a)\big)g\big(a+t\eta(b,a)\big)\mathrm{d}t\right. \\
& +\int_0^1 f\big(a+(1-t)\eta(b,a)\big)g\big(a+(1-t)\eta(b,a)\big)\mathrm{d}t \\
& \left.+2M(a,b)\beta(p+q+1,p+q+1)+2N(a,b)\beta(2p+1,2q+1)\right] \\
=\ & \frac{1}{2^{2p+2q-1}}\left[\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}t\right. \\
& \left.+M(a,b)\beta(p+q+1,p+q+1)+N(a,b)\beta(2p+1,2q+1)\right].
\end{aligned}
$$

This completes the proof. □

Theorem 3.4. *Let $f,g:I\subset\mathbb{R}\to\mathbb{R}$ be beta-preinvex functions. If $fg\in L[a,a+\eta(b,a)]$, then*

$$
\begin{aligned}
& \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} \mu(x)[f(a)g(x)+f(b)g(x)]\mathrm{d}x \\
& +\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} \mu(x)[g(a)f(x)+g(b)f(x)]\mathrm{d}x \\
\leq\ & \frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}M(a,b)+\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}N(a,b) \\
& +\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}x,
\end{aligned}
$$

where $M(a,b)$ and $N(a,b)$ are given by (3.1) and (3.2) respectively and

$$
\mu(x)=\left(\frac{((a+\eta(b,a))-x)^p(x-a)^q}{\eta(b,a)^{p+q}}\right).
$$

Proof. Let f, g be beta-preinvex functions. Then, we have

$$\begin{aligned} f\big(a+t\eta(b,a)\big) &\leq (1-t)^p t^q f(a) + t^p(1-t)^q f(b), \\ g\big(a+t\eta(b,a)\big) &\leq (1-t)^p t^q g(a) + t^p(1-t)^q g(b). \end{aligned}$$

Now, using $\langle x_1 - x_2, x_3 - x_4 \rangle \geq 0, (x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$\begin{aligned} & f\big(a+t\eta(b,a)\big)[(1-t)^p t^q g(a) + t^p(1-t)^q g(b)] \\ & +g\big(a+t\eta(b,a)\big)[(1-t)^p t^q f(a) + t^p(1-t)^q f(b)] \\ \leq\ & [(1-t)^p t^q f(a) + t^p(1-t)^q f(b)][(1-t)^p t^q g(a) + t^p(1-t)^q g(b)] \\ & +f\big(a+t\eta(b,a)\big)g\big(a+t\eta(b,a)\big) \\ =\ & [f(a)g(a)+f(b)g(b)]t^{2p}(1-t)^{2q} + [f(a)g(b)+f(b)g(a)]t^{p+q}(1-t)^{p+q} \\ & +f\big(a+t\eta(b,a)\big)g\big(a+t\eta(b,a)\big) \end{aligned}$$

Integrating over $[0,1]$, we have

$$\begin{aligned} & g(a)\int_0^1 (1-t)^p t^q f\big(a+t\eta(b,a)\big)\mathrm{d}t \\ & +g(b)\int_0^1 t^p(1-t)^q f\big(a+t\eta(b,a)\big)\mathrm{d}t \\ & +f(a)\int_0^1 (1-t)^p t^q g\big(a+t\eta(b,a)\big)\mathrm{d}t \\ & +f(b)\int_0^1 t^p(1-t)^q g\big(a+t\eta(b,a)\big)\mathrm{d}t \\ \leq\ & [f(a)g(a)+f(b)g(b)]\int_0^1 t^{2p}(1-t)^{2q}\mathrm{d}t \\ & +[f(a)g(b)+f(b)g(a)]\int_0^1 t^{p+q}(1-t)^{p+q}\mathrm{d}t \\ & +\int_0^1 f\big(a+t\eta(b,a)\big)g\big(a+t\eta(b,a)\big)\mathrm{d}t \end{aligned}$$

This implies

$$\begin{aligned} & \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} \mu(x)[f(a)g(x)+f(b)g(x)]\mathrm{d}x \\ & +\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} \mu(x)[g(a)f(x)+g(b)f(x)]\mathrm{d}x \\ \leq\ & M(a,b)\beta(2p+1,2q+1) + N(a,b)\beta(p+q+1,p+q+1) \\ & +\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}x, \end{aligned}$$

which is the required result. $\square$

Theorem 3.5. *Let $f,g : I \subset \mathbb{R} \to \mathbb{R}$ be beta-preinvex functions. If $fg \in L[a, a+\eta(b,a)]$ and condition C holds, then*

$$\begin{aligned}
& f\left(\frac{2a+\eta(b,a)}{2}\right)\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} g(x)\mathrm{d}x \\
& +g\left(\frac{2a+\eta(b,a)}{2}\right)\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \\
\leq\ & \frac{1}{2^{p+q}}\left[\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}x\right. \\
& \left.+\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}M(a,b)+\frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}N(a,b)\right] \\
& +2^{p+q-1}f\left(\frac{2a+\eta(b,a)}{2}\right)g\left(\frac{2a+\eta(b,a)}{2}\right),
\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (3.1) and (3.2) respectively.

Proof. Let f, g be beta-preinvex function. Then taking $x = a + t\eta(b,a)$ and $y = a + (1-t)\eta(b,a)$ in (2.2) and using condition C, we have

$$\begin{aligned}
f\left(\frac{2a+\eta(b,a)}{2}\right) &\leq \frac{1}{2^{p+q}}\big[f\big(a+t\eta(b,a)\big)+f\big(a+(1-t)\eta(b,a)\big)\big], \\
g\left(\frac{2a+\eta(b,a)}{2}\right) &\leq \frac{1}{2^{p+q}}\big[g\big(a+t\eta(b,a)\big)+g\big(a+(1-t)\eta(b,a)\big)\big].
\end{aligned}$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0, (x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$\begin{aligned}
& \frac{1}{2^{p+q}}f\left(\frac{2a+\eta(b,a)}{2}\right)\big[g\big(a+t\eta(b,a)\big)+g\big(a+(1-t)\eta(b,a)\big)\big] \\
& +\frac{1}{2^{p+q}}g\left(\frac{2a+\eta(b,a)}{2}\right)\big[f\big(a+t\eta(b,a)\big)+f\big(a+(1-t)\eta(b,a)\big)\big] \\
\leq\ & \frac{1}{2^{2p+2q}}\big[f\big(a+t\eta(b,a)\big)+f\big(a+(1-t)\eta(b,a)\big)\big]\big[g\big(a+t\eta(b,a)\big) \\
& +g\big(a+(1-t)\eta(b,a)\big)\big]+f\left(\frac{2a+\eta(b,a)}{2}\right)g\left(\frac{2a+\eta(b,a)}{2}\right) \\
\leq\ & \frac{1}{2^{2p+2q}}\big[f\big(a+t\eta(b,a)\big)g\big(a+t\eta(b,a)\big) \\
& +f\big(a+(1-t)\eta(b,a)\big)g\big(a+(1-t)\eta(b,a)\big) \\
& +2[f(a)g(a)+f(b)g(b)]t^{p+q}(1-t)^{p+q} \\
& +2[f(a)g(b)+f(b)g(a)]t^{2p}(1-t)^{2q}\big] \\
& +f\left(\frac{2a+\eta(b,a)}{2}\right)g\left(\frac{2a+\eta(b,a)}{2}\right)
\end{aligned}$$

Integrating over [0, 1], we have

$$\begin{aligned}
& \frac{1}{2^{p+q}} f\left(\frac{2a+\eta(b,a)}{2}\right) \int_0^1 \left[g\big(a+t\eta(b,a)\big)+g\big(a+(1-t)\eta(b,a)\big)\right]\mathrm{d}t \\
& +\frac{1}{2^{p+q}} g\left(\frac{2a+\eta(b,a)}{2}\right) \int_0^1 \left[f\big(a+t\eta(b,a)\big)+f\big(a+(1-t)\eta(b,a)\big)\right]\mathrm{d}t \\
\leq\ & \frac{1}{2^{2p+2q}}\left[\int_0^1 f\big(a+t\eta(b,a)\big)g\big(a+t\eta(b,a)\big)\mathrm{d}t\right. \\
& +\int_0^1 f\big(a+(1-t)\eta(b,a)\big)g\big(a+(1-t)\eta(b,a)\big)\mathrm{d}t \\
& +2[f(a)g(a)+f(b)g(b)]\int_0^1 t^{p+q}(1-t)^{p+q}\mathrm{d}t \\
& \left.+2[f(a)g(b)+f(b)g(a)]\int_0^1 t^{2p}(1-t)^{2q}\mathrm{d}t\right] \\
& +\int_0^1 f\left(\frac{2a+\eta(b,a)}{2}\right)g\left(\frac{2a+\eta(b,a)}{2}\right)\mathrm{d}t \\
=\ & \frac{1}{2^{2p+2q}}\left[\int_0^1 f\big(a+t\eta(b,a)\big)g\big(a+t\eta(b,a)\big)\mathrm{d}t\right. \\
& +\int_0^1 f\big(a+(1-t)\eta(b,a)\big)g\big(a+(1-t)\eta(b,a)\big)\mathrm{d}t \\
& \left.+2M(a,b)\beta(p+q+1,p+q+1)+2N(a,b)\beta(2p+1,2q+1)\right] \\
& +f\left(\frac{2a+\eta(b,a)}{2}\right)g\left(\frac{2a+\eta(b,a)}{2}\right)
\end{aligned}$$

From the above inequality, it follows that

$$\begin{aligned}
& f\left(\frac{2a+\eta(b,a)}{2}\right)\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} g(x)\mathrm{d}x \\
& +g\left(\frac{2a+\eta(b,a)}{2}\right)\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \\
\leq\ & \frac{1}{2^{p+q}}\left[\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)\mathrm{d}x\right. \\
& \left.+\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}M(a,b)+\frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}N(a,b)\right] \\
& +2^{p+q-1} f\left(\frac{2a+\eta(b,a)}{2}\right)g\left(\frac{2a+\eta(b,a)}{2}\right),
\end{aligned}$$

which is the requires result. □

Remark 3.1. *If we take $\eta(b,a)=b-a$, in above results, we obtain the known integral inequalities for the class of beta-convex functions, see* [22].

4. Integral Inequalities

We need the following Lemma in order to obtain new integral inequalities related to beta-preinvex function, which can be proved using the technique of Liu[3].

Lemma 4.1. *If $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \to \mathbb{R}$ is a function such that $f \in L[a, a + \eta(b, a)]$, then the following equality holds for some fixed $\alpha, \beta > 0$.*

$$\begin{aligned}&\int_a^{a+\eta(b,a)} (x-a)^\alpha (a+\eta(b,a)-x)^\beta f(x)\mathrm{d}x \\ = \;& \eta^{\alpha+\beta+1}(b,a)\int_0^1 t^\alpha (1-t)^\beta f(a+t\eta(b,a))\mathrm{d}t,\end{aligned}$$

Theorem 4.1. *Let $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on the interior I° of I. If $f \in L[a, a + \eta(b, a)]$ and $|f|$ is beta-preinvex function on $[a, a + \eta(b, a)]$ and $\alpha, \beta > 0$, then*

$$\begin{aligned}&\int_a^{a+\eta(b,a)} (x-a)^\alpha (a+\eta(b,a)-x)^\beta f(x)\mathrm{d}x \\ \leq \;& \eta^{\alpha+\beta+1}(b,a)\big(|f(a)|\varphi_1(t;a,b) + |f(b)|\varphi_2(t;a,b)\big),\end{aligned}$$

where

$$\begin{aligned}\varphi_1(t;a,b) &= \int_0^1 t^{\alpha+q}(1-t)^{\beta+p}\mathrm{d}t \\ &= \beta(\alpha+q+1, \beta+p+1) \qquad (4.1)\end{aligned}$$

$$\begin{aligned}\varphi_2(t;a,b) &= \int_0^1 t^{\alpha+p}(1-t)^{\beta+q}\mathrm{d}t \\ &= \beta(\alpha+p+1, \beta+q+1) \qquad (4.2)\end{aligned}$$

Proof. Using Lemma 4.1 and beta-preinvexity of $|f|$, we have

$$\begin{aligned}&\int_a^{a+\eta(b,a)} (x-a)^\alpha (a+\eta(b,a)-x)^\beta f(x)\mathrm{d}x \\ = \;& \eta^{\alpha+\beta+1}(b,a)\int_0^1 t^\alpha (1-t)^\beta \big|f(a+t\eta(b,a))\big|\mathrm{d}t \\ \leq \;& \eta^{\alpha+\beta+1}(b,a)\int_0^1 t^\alpha (1-t)^\beta \big\{(1-t)^p t^q |f(a)| \\ & + t^p(1-t)^q|f(b)|\big\}\mathrm{d}t \\ = \;& \eta^{\alpha+\beta+1}(b,a)\bigg(|f(a)|\int_0^1 t^{\alpha+q}(1-t)^{\beta+p}\mathrm{d}t \\ & + |f(b)|\int_0^1 t^{\alpha+p}(1-t)^{\beta+q}\mathrm{d}t\bigg) \\ = \;& \eta^{\alpha+\beta+1}(b,a)\big(|f(a)|\varphi_1(t;a,b) + |f(b)|\varphi_2(t;a,b)\big).\end{aligned}$$

This completes the proof. □

Theorem 4.2. *Let $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on the interior I° of I. If $f \in L[a, a + \eta(b, a)]$ and $|f|^\lambda$ is beta-preinvex function on $[a, a + \eta(b, a)]$ and $\alpha, \beta > 0$, $\lambda \geq 1$, then*

$$\begin{aligned}&\int_a^{a+\eta(b,a)} (x-a)^\alpha (a+\eta(b,a)-x)^\beta f(x)\mathrm{d}x \\ \leq \;& \eta^{\alpha+\beta+1}(b,a)\big(\varphi_3(t;a,b)\big)^{1-\frac{1}{\lambda}} \\ & \big(|f(a)|^\lambda\varphi_1(t;a,b) + |f(b)|^\lambda\varphi_2(t;a,b)\big)^{\frac{1}{\lambda}},\end{aligned}$$

where $\varphi_1(t; a, b)$, $\varphi_2(t; a, b)$ are given by (4.1) and (4.2) respectively, and

$$\begin{aligned}\varphi_3(t;a,b) &= \int_0^1 t^\alpha(1-t)^\beta\mathrm{d}t \\ &= \beta(\alpha+1, \beta+1).\end{aligned}$$

Proof. Using Lemma 4.1, beta-preinvexity of $|f|^\lambda$ and power mean inequality, we have

$$\begin{aligned}
& \int_a^{a+\eta(b,a)} (x-a)^\alpha (a+\eta(b,a)-x)^\beta f(x)\mathrm{d}x \\
= \;& \eta^{\alpha+\beta+1}(b,a) \int_0^1 t^\alpha (1-t)^\beta \big| f(a+t\eta(b,a)) \big| \mathrm{d}t \\
\leq \;& \eta^{\alpha+\beta+1}(b,a) \bigg(\int_0^1 t^\alpha (1-t)^\beta \mathrm{d}t \bigg)^{1-\frac{1}{\lambda}} \\
& \bigg(\int_0^1 t^\alpha (1-t)^\beta \big| f(a+t\eta(b,a)) \big|^\lambda \mathrm{d}t \bigg)^{\frac{1}{\lambda}} \\
\leq \;& \eta^{\alpha+\beta+1}(b,a) \bigg(\int_0^1 t^\alpha (1-t)^\beta \mathrm{d}t \bigg)^{1-\frac{1}{\lambda}} \\
& \bigg(\int_0^1 t^\alpha (1-t)^\beta \big\{ (1-t)^p t^q |f(a)|^\lambda + t^p (1-t)^q |f(b)|^\lambda \big\} \mathrm{d}t \bigg)^{\frac{1}{\lambda}} \\
= \;& \eta^{\alpha+\beta+1}(b,a) \bigg(\int_0^1 t^\alpha (1-t)^\beta \mathrm{d}t \bigg)^{1-\frac{1}{\lambda}} \\
& \bigg(|f(a)|^\lambda \int_0^1 t^{\alpha+q} (1-t)^{\beta+p} \mathrm{d}t + |f(b)|^\lambda \int_0^1 t^{\alpha+p} (1-t)^{\beta+q} \mathrm{d}t \bigg)^{\frac{1}{\lambda}} \\
= \;& \eta^{\alpha+\beta+1}(b,a) \big(\varphi_3(t;a,b) \big)^{1-\frac{1}{\lambda}} \\
& \big(|f(a)|^\lambda \varphi_1(t;a,b) + |f(b)|^\lambda \varphi_2(t;a,b) \big)^{\frac{1}{\lambda}},
\end{aligned}$$

which the the required result. □

Theorem 4.3. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on the interior I° of I. If $f \in L[a, a+\eta(b,a)]$ and $|f|^\lambda$ is beta-preinvex function on $[a, a+\eta(b,a)]$ and $\alpha, \beta > 0$, then*

$$\begin{aligned}
& \int_a^{a+\eta(b,a)} (x-a)^\alpha (a+\eta(b,a)-x)^\beta f(x)\mathrm{d}x \\
\leq \;& \eta^{\alpha+\beta+1}(b,a) \big(\varphi_4(t;a,b) \big)^{\frac{1}{\mu}} \\
& \times \big(|f(a)|^\lambda + |f(b)|^\lambda \beta(p+1, q+1) \big)^{\frac{1}{\lambda}},
\end{aligned}$$

where $\frac{1}{\lambda} + \frac{1}{\mu} = 1$ and

$$\begin{aligned}
\varphi_4(t;a,b) &= \int_0^1 t^{\alpha\mu} (1-t)^{\beta\mu} \mathrm{d}t \\
&= \beta(\alpha\mu+1, \beta\mu+1).
\end{aligned}$$

Proof. Using Lemma 4.1, beta-preinvexity of $|f|^\lambda$ and the Holder's integral inequality, we have

$$\begin{aligned}
& \int_a^{a+\eta(b,a)} (x-a)^\alpha (a+\eta(b,a)-x)^\beta f(x)\mathrm{d}x \\
= \; & \eta^{\alpha+\beta+1}(b,a)\int_0^1 t^\alpha(1-t)^\beta \big|f(a+t\eta(b,a))\big|\mathrm{d}t \\
\leq \; & \eta^{\alpha+\beta+1}(b,a)\bigg(\int_0^1 t^{\alpha\mu}(1-t)^{\beta\mu}\mathrm{d}t\bigg)^{\frac{1}{\mu}} \\
& \bigg(\int_0^1 \big|f(a+t\eta(b,a))\big|^\lambda \mathrm{d}t\bigg)^{\frac{1}{\lambda}} \\
\leq \; & \eta^{\alpha+\beta+1}(b,a)\bigg(\int_0^1 t^{\alpha\mu}(1-t)^{\beta\mu}\mathrm{d}t\bigg)^{\frac{1}{\mu}} \\
& \bigg(\int_0^1 [(1-t)^p t^q |f(a)|^\lambda + t^p(1-t)^q|f(b)|^\lambda]\mathrm{d}t\bigg)^{\frac{1}{\lambda}} \\
= \; & \eta^{\alpha+\beta+1}(b,a)\big(\varphi_4(t;a,b)\big)^{\frac{1}{\mu}} \\
& \times\big(|f(a)|^\lambda + |f(b)|^\lambda \beta(p+1,q+1)\big)^{\frac{1}{\lambda}}.
\end{aligned}$$

This completes the proof. □

Remarks

From Definitions 2.2 and 2.3, we see that the function f satisfies the relation

$$f(\frac{2x+\eta(y,x)}{2}) = \frac{f(x)+f(y)}{2},$$

which is called the preinvex functional equation.

We remark that, if $\eta(y,x) = y-x$, then the invex set K_η becomes convex set K. In this case, the preinvex functional equations collapses to the following equation

$$f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2},$$

which is called the Jensen-Cauchy functional equation, see[19].

This has motivated us to consider the additive preinvex functional equation of the type

$$f(2x+\eta(y,x)) = f(x)+f(y). \tag{4.3}$$

It is an interesting problem to study the stability criteria of the additive preinvex functional equations (4.3) using the technique of Rassias[20], Ulam[23] and Small[21]. We leave this to the interested readers to explore this aspects of the additive preinvex functional equations.

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