Int. J. Anal. Appl. (2024), 22:39

International Journal of Analysis and Applications

Some Results of Malcev-Neumann Rings

Kholood Alnefaie[1], **Eltiyeb Ali**[2,3,*]

[1] *Department of Mathematics, College of Science, Taibah University, Madinah 42353, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Education, University of Khartoum, Sudan*
[3] *Department of Mathematics, College of Science and Arts, Najran University, Saudi Arabia*

[*] *Corresponding author:* eltiyeb76@gmail.com

Abstract. Let us consider the function σ, which maps elements from the group G to the group of automorphisms of the ring R. In this paper, we are studying new conditions under which the Malcev-Neumann ring $R*((G))$ is a *PS*, *APP*, *PF*, *PP*, and a Zip rings, respectively. It has been demonstrated that if R is a reduced ring and σ is weakly rigid, then the Malcev-Neumann ring $R*((G))$ over a *PS*-ring is a *PS*. Furthermore, if σ is weakly rigid and the ring R satisfies the descending chain condition on left annihilators, then the Malcev-Neumann ring $R*((G))$ is a right *APP*-ring if and only if, for any G-indexed generated right ideal A of R, $r_R(A)$ is left s-unital. Additionally, we have proven that if R is a commutative ring and σ is weakly rigid, then the Malcev-Neumann ring $R*((G))$ is a *PF* ring if and only if, for any two G-indexed subsets A and B of R such that $B \subseteq \mathrm{ann}_R(A)$, there exists $c \in \mathrm{ann}_R(A)$ such that $bc = b$ for all $b \in B$. These results extend the corresponding findings for polynomial rings and Laurent power series rings.

1. Introduction and Preliminaries

The Malcev-Neumann construction appeared for the first time in the latter part of the 1940's (the Laurent series ring, a particular case of Malcev-Neumann ring, was used before by Hilbert). Using them, Malcev and Neumann independently showed (in 1948 and 1949 resp.) that the group ring of an ordered group over a division ring can be embedded in a division ring. Since then, the construction has appeared in many papers, mainly in the study of various properties of division rings and related topics. For instance, Makar-Limanov in [1] used a particular skew-Laurent series division ring to prove that the skew field of fractions of the first Weyl algebra contains a free noncommutative subalgebra. The study of Malcev-Neumann group ring over arbitrary rings was initiated in [2] by Lorenz while investigating properties of group algebras of nilpotent groups. Other results on Malcev-Neumann rings can be found in Musson and Stafford [3] and Sonin [4].

Received: Dec. 24, 2023.

2020 *Mathematics Subject Classification.* 16W60, 16S50, 16S99.

Key words and phrases. Malcev-Neumann ring; *PS*-ring; *APP*-ring; *PF*-ring; Zip-ring.

https://doi.org/10.28924/2291-8639-22-2024-39

ISSN: 2291-8639

We construct the Malcev-Neumann (group) ring in the following. Let R be a ring, G an ordered group, and suppose that σ is a map from G into the group of automorphisms of R, $x \to \sigma_x$, t is a map from $G \times G$ to $U(R)$, the group of invertible elements of R. Then we can form a Malcev-Neumann ring $R * ((G))$: an element of $R * ((G)$ is a infinite series $f = \sum_{x \in G} r_x x$ with $r_x \in R$ such that the set $supp(f) = \{x \in G \mid r_x \neq 0\}$, called the support of f, is a well ordered subset of G, and the ring structure is given by componentwise addition defined as usual by

$$\sum_{x \in G} a_x x + \sum_{y \in G} b_y y = \sum_{z \in G} (a_z + b_z) z$$

and multiplication is defined by

$$\Big(\sum_{x \in G} a_x x\Big)\Big(\sum_{y \in G} b_y y\Big) = \sum_{z \in G} \Big(\sum_{\{(x,y) \mid xy = z\}} a_x \sigma_x(b_y) t(x,y) \Big) z.$$

In order to insure associativity, it is necessary to impose two additional conditions on σ and t, namely that for all $x, y, z \in G$,

(1) $t(xy,z)\sigma_z(t(x,y)) = t(x,yz)t(y,z)$,
(2) $\sigma_y \sigma_z = \sigma_{yz} \delta(y,z)$,

where $\delta(y,z)$ denotes the automorphism of R induced by the unit $t(y,z)$ by Lemma 1.1 [5]. It is now routine to check that $R * ((G))$ is a ring which we call the Malcev-Neumann ring. This construction has appeared in many papers, mainly in the study various properties of division rings and related topic. For a more comprehensive understanding of this construction and the results associated with it, it is recommended to refer to several scholarly papers on the topic [3], [6], [9], [10], [15] and [16]. The subring of $R * ((G))$ consisting of all finite sums $f = \sum_{x \in G} r_x x$ (i.e., sums of finite support) is just the twisted group ring $R * (G)$. If $G = \mathbb{Z}$, $\sigma_x = id$ for all $x \in G$, $t(x,y) = 1$ for all $x, y \in G$, then $R * ((G))$, is the Laurent series ring. If σ happens to be the trivial homomorphism and $t(x,y) = 1$ for all $x, y \in G$, the resulting untwisted ring will denoted by $R((G))$. As usual, we shall identify R with the subring $R.1 \subseteq R * ((G))$ and identity G with the subgroup $1.G$ of invertible elements in $R * ((G))$.

In this paper, we are studying new conditions under which the Malcev-Neumann ring $R * ((G))$ is a *PS*, *APP*, *PF*, *PP* and a Zip rings, respectively. We prove that, if the ring R satisfies the descending chain condition on left annihilators, then the Malcev-Neumann ring $R * ((G))$ is a right *APP*-ring if and only if, for any G-indexed generated right ideal A of R, $r_R(A)$ is left s-unital. Furthermore, we have proven that if R is a commutative ring and σ is weakly rigid, then the Malcev-Neumann ring $R * ((G))$ is a *PF* ring if and only if, for any two G-indexed subsets A and B of R such that $B \subseteq \mathrm{ann}_R(A)$, there exists $c \in \mathrm{ann}_R(A)$ such that $bc = b$ for all $b \in B$. Additionally, we prove that if R is a Noetherian ring, then $R * ((G))$ is a *PP* ring if and only if R is a *PP* ring, and the Malcev-Neumann ring $R * ((G))$ is a right zip ring if and only if R is a right zip ring. These results extend the corresponding findings for polynomial rings and Laurent power series rings.

Throughout the paper all rings are associative with unity. For a nonempty subset X of a ring R, $r_R(X)$ and $l_R(X)$ denote the right and left annihilators of X in R, respectively. We will denote by

$End(R)$ the monoid of ring endomorphisms of R, and by $Aut(R)$ the group of ring automorphisms of R.

2. Main Results

A ring R is called a left PS-ring if $Soc(_RR)$ is projective. In [12] it was proved that if R is a left PS-ring then so is $R[[x]]$. Xue in [13] showed that for any ring R, $R[[x]]$ is a left PS-ring. If R is a commutative ring and $(S,\leq)$ is a strictly totally ordered monoid which satisfies the condition that $0 \leq s$ for every $s \in S$, then in [14], it was proved that, if R is PS-ring, then the ring $[[R^{S,\leq}]]$ of generalized power series over R is a PS-ring. Firstly, we will consider the PS property of Malcev-Neumann rings.

Let σ be a map from G into the group of automorphisms of R, $x \mapsto \sigma_x$. Then, following Definition 2.1 [8], σ is called weakly rigid if $ab = 0$ implies $a\sigma_x(b) = \sigma_x(a)b = 0$ for any $a, b \in R$ and any $x \in G$. Clearly, if for any $x \in G, \sigma_x = id$, the identity map of R, then σ is weakly rigid. Let α be an endomorphism of R. According to [19], α is called a rigid endomorphism if $r\alpha(r) = 0$ implies $r = 0$ for $r \in R$. A ring R is called α-rigid if there exists a rigid endomorphism α of R. Clearly any rigid endomorphism is a monomorphism and any α-rigid is reduced. Let α be a rigid automorphism of R. It was shown in [19] that if $ab = 0$ then $a\alpha^n(b) = \alpha^n(a)b = 0$ for any positive integer n. Thus the map $\sigma : \mathbb{Z} \to Aut(R) : \sigma(x) = \alpha^x$ is weakly rigid. For more details and examples see [7], [8] and [18].

The following results appeared in [8] and [12] respectively.

Lemma 2.1. *Let R be reduced and σ is weakly rigid. If $f = \sum_{x\in G} a_x x, g = \sum_{y\in G} b_y y \in R*((G))$ are such that $fg = 0$, then $a_x b_y = 0$ for any $x, y \in G$.*

Lemma 2.2. *The following conditions are equivalent for a ring R :*

(1) *R is a right PS-ring.*
(2) *For any maximal right ideal L of R then either $l_R(L) = 0$ or $L = eR$ where $e^2 = e \in R$.*

Theorem 2.3. *Let R be a reduced ring, G an ordered group and σ is weakly rigid. If R is a right PS-ring, then so is $R*((G))$.*

Proof. Let L be a maximal right ideal of $R*((G))$. By Lemma 2.2, it is enough to show that either $l_{R*((G))}(L) = 0$ or $L = \alpha R*((G))$ for some $\alpha^2 = \alpha \in R*((G))$. Let I be the set of all coefficients of 1 of elements of L. Let J be the right ideal of R generated by I. If $J = R$, then there exist $a_1^1, a_1^2, \dots, a_1^n \in I, f_1, f_2, \dots, f_n \in L$ and $r_1, r_2, \dots, r_n \in R$ such that $1 = a_1^1 r_1 + a_1^2 r_2 + \cdots + a_1^n r_n$ with $f_i = \sum_{x\in G} a_x^i x, i = 1, 2, \dots, n$. Suppose that $g = \sum_{y\in G} b_y y \in l_{R*((G))}(L)$. Then $gf_i = 0$. Thus $b_y a_x^i = 0$ by Lemma 2.1. Particularly, $b_y a_1^i = 0$ for any $y \in G$ and any $i = 1, 2, \dots n$. Thus $b_y = b_y(a_1^1 r_1 + a_1^2 r_2 + \cdots + a_1^n r_n) = 0$, and so $g = 0$. Thus $l_{R*((G))}(L) = 0$.

Now suppose that $J \neq R$. We show that J is a maximal right ideal of R. Let $r \in R - J$. Then $r \in R*((G))$. If $r \in L$, then $r \in J$, a contradiction. Thus $r \notin L$. So $R*((G)) = L + rR*((G))$. It follows that there exist $f \in L$ and $h \in R*((G))$ such that $1 = f + rh$. Suppose that $f = \sum_{x\in G} a_x x$ and

$h = \sum_{y \in G} c_y y$. Then $1 = a_1 + r\sigma_1(c_1)t(1,1) \in J + rR$. Thus $R = J + rR$. Hence J is a maximal right ideal of R. Since R is a right PS-ring, it follows that either $l_R(J) = 0$ or there exists an $e^2 = e \in R$ such that $J = eR$.

Case (i). Suppose that $l_R(J) = 0$. We will show that $l_{R*((G))}(L) = 0$. Let $g = \sum_{y \in G} b_y y \in l_{R*((G))}(L), r \in J$. Then there exist $a_1^1, a_1^2, \ldots, a_1^n \in I$, $f_1, f_2, \ldots, f_n \in L$ and $r_1, r_2, \ldots, r_n \in R$ such that $r = a_1^1 r_1 + a_1^2 r_2 + \cdots + a_1^n r_n$, where a_1^i is the constant coefficient of f_i. Since $g \in l_{R*((G))}(L), g f_i = 0$ for every $i = 1, 2, \ldots, n$. By Lemma 2.1, we have $b_y a_1^i = 0$ for any $y \in G$ and any $i = 1, 2, \ldots, n$. Thus $b_y r = b_y(a_1^1 r_1 + a_1^2 r_2 + \cdots + a_1^n r_n) = 0$ for any $y \in G$. This means that $b_y \in l_R(J) = 0$ for any $y \in G$. Thus $g = 0$, and so $l_{R*((G))}(L) = 0$.

Case (ii). Suppose that $J = eR$ where $e^2 = e \in R$. We will show that $L = e(R * ((G)))$. If $e \notin L$, then $R * ((G)) = L + e(R * ((G)))$. Thus $1 = f + eh$, where $f = \sum_{x \in G} a_x x \in L$ and $h = \sum_{y \in G} c_y y \in R * ((G))$, and so $1 = a_1 + e\sigma_1(c_1)t(1,1) \in J + eR = J$, a contradiction. Therefore $e \in L$, and so $e(R * ((G))) \subseteq L$. Conversely, suppose that $f = \sum_{x \in G} a_x x \in L$. For any $x \in G$, there exists $x^{-1} \in G$ such that $xx^{-1} = 1$ since G is a group, and $fx^{-1} \in L$ since L is a right ideal of $R * ((G))$. Thus $a_x \sigma_x(1) t(x, x^{-1}) \in J = eR$ for any $x \in G$. Thus $a_x \in J = eR$ since $t(x, x^{-1})$ is invertible and J is a right ideal of R, and so $a_x = e a_x$. Thus $f = e \sum_{x \in G} \sigma_1^{-1}(a_x t(1, x)^{-1}) x \in e(R * ((G)))$. Thus $L \subseteq eR * ((G))$. Hence $L = e(R * ((G)))$ and the result follows. □

Corollary 2.4. *Let R be a reduced ring and G an ordered group. If R is a right PS-ring, then $R((G))$ is a right PS-ring.*

Corollary 2.5. *Let R be a reduced ring and α is weakly rigid automorphism of R. If R is a PS-ring, then $R[[x, x^{-1}; \alpha]]$ is a PS-ring.*

Proof. Take $G = \mathbb{Z}$ and $t(x, y) = 1$ for any $x, y \in \mathbb{Z}$. For any $x \in \mathbb{Z}$, let $\sigma_x = \alpha^x$. Then σ is weakly rigid. Now the result follows from Theorem 2.3. □

Recall that a ring R is called (resp., quasi-) Baer if the right annihilator of every (resp., right ideal) nonempty subset of R is generated, as a right ideal, by an idempotent of R. In [30] Kaplansky introduced Baer rings to abstract various properties of AW^*-algebras, von Neumann algebras and complete $*$-regular rings. In [33] Clark defined quasi-Baer rings and used them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. A ring R is called a right (resp., left) PP-ring if every principal right (resp., left) ideal is projective (equivalently, if the right (resp., left) annihilator of an element of R is generated (as a right (resp., left) ideal) by an idempotent of R). R is called a PP-ring (also called a Rickart ring [22, p. 18]) if it is both right and left PP. A ring R is called left (resp., right) principally quasi-Baer (or simply left (resp., right) $p.q.$-Baer) if the left (resp., right) annihilator of a principal left (resp., right) ideal is generated as a left (resp., right) ideal by an idempotent. Equivalently, R is right $p.q.$-Baer if R modulo the right annihilator of any principal right ideal is projective. A ring is called $p.q.$-Baer if it is both right and left $p.q.$-Baer. The concept of principally quasi-Baer rings initiated by Birkenmeier, Kim and Park [25]. Following Tominaga [28], an ideal I of R is said to be right s-unital if, for each $a \in I$ there exists an element $x \in I$ such that $ax = a$. A submodule N of a left R-module M is called a pure submodule if $L \bigotimes_R N \to L \bigotimes_R M$ is a monomorphism for every right R-module L. By [21, Proposition 11.3.13], an ideal I is right s-unital if and only if I is pure as a left ideal of R if and only if R/I is flat as a left R-module. According to Liu and Zhao [35], a ring R is called left APP if R has the property that "the left annihilator of a principal ideal is pure as a left ideal". Equivalently, R is a left APP-ring if R modulo the left annihilator of any principal left ideal is flat. Right APP-ring is also defined analogously. A ring is called APP if it is right APP and left APP. By Proposition 2.3 [35], the class of right APP-rings includes both left PP-rings and right $p.q.$-Baer rings (and hence it includes all biregular rings and all quasi-Baer rings), for some details to use this conditions see [11] and [17]. In [20] the authors showed that left $p.q.$-Baer rings are also right APP and provided various examples of commutative APP-rings which are neither $p.q.$-Baer nor PP.

Liu and Zhao Proposition 3.14 [35] proved that, when R is a ring satisfying descending chain condition on left and right annihilators and R is left APP, then $R[[x]]$ is left APP. Zhao Theorem 3 [32] showed that, if $(S, \leq)$ is a strictly totally ordered commutative monoid, $\omega : S \to Aut(R)$ a monoid homomorphism and R satisfying descending chain condition on right annihilators, then the skew generalized power series ring $R[[S^{\leq}, \omega]]$ is left APP if and only if for any S-indexed subset A of R, the left annihilator of the left ideal generated by the set $\{\omega_s(a) \mid a \in A \text{ and } s \in S\}$ is right s-unital. Now we consider the APP-property of Malcev-Neumann rings.

The following result follows from Tominaga Theorem 1 [28].

Lemma 2.6. *An ideal J of a ring R is left(resp., right) s-unital if and only if for any finitly many elements $a_1, a_2, \dots, a_n \in J$, there is an element $e \in J$ such that $a_i = ea_i$(resp., $a_i = a_ie$), for each i.*

Lemma 2.7. *Let R be a ring, G an ordered group and σ is weakly rigid. If R is a right APP-ring, then for any $f = \sum_{x\in G} a_x x$, $g = \sum_{y\in G} b_y y \in R*((G))$, $fR*((G))g = 0$ implies $a_xRb_y = 0$ for any $x, y \in G$.*

Proof. Let $0 \neq f \in R*((G))$ and $0 \neq g \in R*((G))$ be such that $fR*((G))g = 0$. Then for any $r \in R$, from

$$0 = frg = \sum_{z\in G} \sum_{\{(x,y)|xy=z\}} a_x\sigma_x(r\sigma_1(b_y)t(1,y))t(x,y)z$$

it follows that

$$\sum_{\{(x,y)|xy=z\}} a_x\sigma_x(r\sigma_1(b_y)t(1,y))t(x,y) = 0, \forall z \in G.$$

Let x_0 and y_0 denote the minimal elements of $supp(f)$ and $supp(g)$ in the $\leq$. order, respectively. If $x \in supp(f)$ and $y \in supp(g)$ are such that $xy = x_0y_0$, then $x_0 \leq x$ and $y_0 \leq y$. If $x_0 < x$, then $x_0y_0 < xy_0 \leq xy = x_0y_0$, a contradiction. Thus $x = x_0$. Similarly, $y = y_0$. Hence

$$\sum_{\{(x,y)|xy=x_0y_0\}} a_x\sigma_x(r\sigma_1(b_y)t(1,y))t(x,y) = a_{x_0}\sigma_{x_0}(r\sigma_1(b_{y_0})t(1,y_0))t(x_0,y_0) = 0.$$

Thus $a_{x_0}\sigma_{x_0}(r\sigma_1(b_{y_0})t(1,y_0)) = 0$ since $t(x_0,y_0)$ is invertible. Hence, by weakly rigidness of σ we have $\sigma_{x_0}(a_{x_0}r\sigma_1(b_{y_0})t(1,y_0)) = 0$, so $a_{x_0}r\sigma_1(b_{y_0})t(1,y_0) = 0$ since $\sigma_{x_0} \in Aut(R)$. By the way as above, we can get $a_{x_0}rb_{y_0} = 0$, which means that $a_{x_0}Rb_{y_0} = 0$.

Now suppose that $w \in G$ is such that for any $x \in supp(f)$ and $y \in supp(g)$ with $xy < w$, $a_xRb_y = 0$. We will show that $a_xRb_y = 0$ for any $x \in supp(f)$ and $y \in supp(g)$ with $xy = w$. If there are not $x \in supp(f)$ and $y \in supp(g)$ such that $xy = w$, then clearly the conclusion holds. Now suppose that $x \in supp(f)$ and $y \in supp(g)$ are such that $xy = w$. For convenience we write $\{(x,y) \mid xy = w\}$ as $\{(x_i,y_i) \mid i = 1,2,\dots,n\}$ with $x_1 < x_2 < \cdots < x_n$. Then for any $r \in R$, from

$$\sum_{\{(x,y)|xy=w\}} a_x\sigma_x(r\sigma_1(b_y)t(1,y))t(x,y) = 0$$

it follows that

$$\sum_{i=1}^{n} a_{x_i}\sigma_{x_i}(r\sigma_1(b_{y_i})t(1,y_i))t(x_i,y_i) = 0. \tag{2.1}$$

Note that $x_1y_i < x_iy_i = \omega$ for any $i = 2,3,\dots,n$. By the hypothesis, we have $a_{x_1}Rb_{y_i} = 0$ for $i = 2,3,\dots,n$. Since R is right APP, by Lemma 2.6, there exists $e_{x_1} \in r_R(a_{x_1}R)$ such that $b_{y_i} = e_{x_1}b_{y_i}$ for any $i = 2,3,\dots,n$. Let $r' \in R$ and take $r = r'e_{x_1}$ in Eq. (2.1). Thus from $a_{x_1}r'e_{x_1} = 0$ it follows that $a_{x_1}\sigma_{x_1}(r'e_{x_1}\sigma_1(b_{y_1})t(1,y_1)) = 0$ since σ is weakly rigid. Hence

$$\sum_{i=2}^{n} a_{x_i}\sigma_{x_i}(r'e_{x_1}\sigma_1(b_{y_i})t(1,y_i))t(x_i,y_i) = 0. \tag{2.2}$$

On the other hand, since $b_{y_i} = e_{x_1}b_{y_i}$ for any $i = 2,3,\dots,n$, and σ is weakly rigid, one gets $a_{x_i}\sigma_{x_i}(r'(1-e_{x_1})\sigma_1(b_{y_i})t(1,y_i)) = 0$ and so

$$a_{x_i}\sigma_{x_i}(r'e_{x_1}\sigma_1(b_{y_i})t(1,y_i)) = a_{x_i}\sigma_{x_i}(r'\sigma_1(b_{y_i})t(1,y_i))$$

for all $i = 2,3,\dots,n$. Therefore Eq. (2.2) becomes

$$\sum_{i=2}^{n} a_{x_i}\sigma_{x_i}(r'\sigma_1(b_{y_i})t(1,y_i))t(x_i,y_i) = 0. \tag{2.3}$$

Since $x_2y_i < x_iy_i = \omega$ for $i = 3,4,\dots,n$, by the hypothesis, there exists $e_{x_2} \in r_R(a_{x_2}R)$ such that $b_{y_i} = e_{x_2}b_{y_i}$ for each $i \geq 3$. So if we take $r' = pe_{x_2}$ in Eq. (2.3), we have

$$a_{x_2}\sigma_{x_2}(pe_{x_2}\sigma_1(b_{y_2})t(1,y_2)) = 0,$$

and

$$\sum_{i=3}^{n} a_{x_i}\sigma_{x_i}(pe_{x_2}\sigma_1(b_{y_i})t(1,y_i))t(x_i,y_i) = \sum_{i=3}^{n} a_{x_i}\sigma_{x_i}(p\sigma_1(b_{y_i})t(1,y_i))t(x_i,y_i) = 0.$$

Continuing in this manner yields that $a_{x_n}\sigma_{x_n}(q\sigma_1(b_{y_n})t(1,y_n))t(x_n,y_n) = 0$, where q is an arbitrary element of R. Consequently, $a_{x_n}qb_{y_n} = 0$. Hence $a_{x_{n-1}}qb_{y_{n-1}} = 0, \dots, a_{x_1}qb_{y_1} = 0$. Therefore, by transfinite induction, we have shown that $a_xRb_y = 0$ for any $x, y \in G$. □

Lemma 2.8. *Let R be a ring, G an ordered group and σ is weakly rigid. Then the following conditions are equivalent:*

(1) *For any* $f = \sum_{x\in G} a_x x, g = \sum_{y\in G} b_y y \in R*((G))$, $fR*((G))g = 0$ *implies* $a_x R b_y = 0$ *for any* $x, y \in G$.
(2) *For any* $f = \sum_{x\in G} a_x x \in R*((G))$, $r_{R*((G))}(fR*((G))) = r_R(I)*((G))$, *where* I *is the right ideal of* R *generated by* $\{a_x \mid x \in G\}$.

Proof. (1)⇒(2) Assume that $g = \sum_{y\in G} b_y y \in r_{R*((G))}(fR*((G)))$ with $f \in R*((G))$. By (1), $a_x R b_y = 0$ for all x and y. Thus $b_y \in r_R(I)$, and so $g \in r_R(I)*((G))$. Hence $r_{R*((G))}(fR*((G))) \subseteq r_R(I)*((G))$. Conversely, suppose that $g = \sum_{y\in G} b_y y \in r_R(I)*((G))$. Then $b_y \in r_R(I)$ for all $y \in G$. Thus $a_x R b_y = 0$ for all x and y. Since R is σ-weakly rigid, then for any $h = \sum_{z\in G} c_z z \in R*((G))$, we have $a_x\sigma_x(c_z)\sigma_x\sigma_z(b_y) = 0$ for any $x, y, z \in G$. Thus, $a_x\sigma_x(c_z)\sigma_x\sigma_z(b_y)\sigma_x(t(z,y))t(x,p) = 0$ for any $x, y, z, p \in G$. Hence

$$fhg = \Big(\sum_{x\in G} a_x x\Big)\Big(\sum_{p\in G}\sum_{\{(z,y)|zy=p\}} c_z\sigma_z(b_y)t(z,y)p\Big)$$
$$= \sum_{q\in G}\sum_{\{(x,p)|xp=q\}}\sum_{\{(z,y)|zy=p\}} a_x\sigma_x(c_z)\sigma_x\sigma_z(b_y)\sigma_x(t(z,y))t(x,p)q = 0.$$

This means that $g \in r_{R*((G))}(fR*((G)))$. So $r_{R*((G))}(fR*((G))) = r_R(I)*((G))$.

(2)⇒(1) Suppose that $f = \sum_{x\in G} a_x x, g = \sum_{y\in G} b_y y$ in $R*((G))$ are such that $fR*((G))g = 0$. Thus $g \in r_{R*((G))}(fR*((G)))$. By (2) $g \in r_R(I)*((G))$, where I be the right ideal of R generated by $\{a_x \mid x \in G\}$. Hence $b_y \in r_R(I)$. So $a_x R b_y = 0$ for all $x, y \in G$. □

Lemma 2.9. *Let* R *be a ring,* G *an ordered group and* σ *is weakly rigid. Then for any* $a \in R$, $r_R(aR)*((G)) = r_{R*((G))}(aR*((G)))$.

Proof. Let $g = \sum_{y\in G} b_y y \in r_{R*((G))}(aR*((G)))$. Then for any $r \in R$, $ar\sigma_1(b_y)t(1,y) = 0$. Thus $arb_y = 0$ since $t(1,y)$ is invertible and R is σ-weakly rigid. Hence $b_y \in r_R(aR)$. So $g \in r_R(aR)*((G))$. Conversely, suppose that $g = \sum_{y\in G} b_y y \in r_R(aR)*((G))$. Then $aRb_y = 0$. Hence for any $f = \sum_{x\in G} c_x x \in R*((G))$,

$$a\sigma_1(c_x)t(1,x)\sigma_x(b_y)t(x,y) = 0.$$

Thus

$$afg = \sum_{z\in G}\sum_{\{(x,y)|xy=z\}} a\sigma_1(c_x)t(1,x)\sigma_x(b_y)t(x,y)z = 0.$$

Hence $g \in r_{R*((G))}(aR*((G)))$. So, $r_R(aR)*((G)) = r_{R*((G))}(aR*((G)))$. □

In order to prove the main result, we first give the necessity of the ring $R*((G))$ to be right APP-ring.

Proposition 2.10. *Let* R *be a ring,* G *an ordered group and* σ *is weakly rigid. If* $R*((G))$ *is a right* APP*-ring, then* R *is a right* APP*-ring.*

Proof. Let $a, b \in R$ be such that $a \in r_R(bR)$. Then $a \in r_R((bR)*((G)))$. By Lemma 2.9, $a \in r_{R*((G))}(bR*((G)))$. Since $R*((G))$ is right APP, then there exists an $f = \sum_{x\in G} c_x x \in r_{R*((G))}(bR*((G)))$ such that $a = fa$. Thus $brf = 0$ for any $r \in R$. Hence $br\sigma_1(c_x)t(1,x) = 0$, and so $brc_x = 0$ for any

$x \in G$. In particular, $c_1 \in r_R(bR)$. On the other hand, for $a = fa$ it follows that $(1-f)a = 0$. Thus, $(1-c_1)\sigma_1(a)t(1,1) = 0$, and so $a = c_1 a$. Therefore R is a right APP-ring. □

Let R be a ring and G an ordered group. We say a nonempty subset X of R is G-indexed, if there exists a well-ordered subset I of G such that X is indexed by I. We say an ideal J of R is G-indexed left (resp. right) s-unital if for any G-indexed subset $\{a_s \mid s \in I\}$ of J, there exist a $c \in J$ such that $a_s = ca_s$ (resp., $a_s = a_s c$).

Lemma 2.11. *Let R be a ring and G an ordered group. If R satisfies the descending chain condition on left (resp. right) annihilators, then for any ideal J of R, J is left (resp. right) s-unital if and only if J is G-indexed left (resp. right) s-unital.*

Proof. $\Leftarrow$). Obviously since any singleton is G-indexed.

$\Rightarrow$) Let J be a left s-unital ideal of R and $A = \{a_x \mid x \in I\}$ a G-indexed subset of J. Define a set of left annihilators

$$H = \{l_R(X) \mid X \subseteq A, \mid X \mid < \infty\}.$$

Since R satisfies the descending chain condition on left annihilators, H has a minimal element, say $l_R(X_0)$. Assume that $X_0 = \{a_{x_1}, a_{x_2}, \ldots, a_{x_n}\}$. Since J is left s-unital, by Lemma 2.6, there exists $c \in J$ such that $a_{x_i} = ca_{x_i}$ for all $i = 1, 2, \ldots, n$. So $(1-c) \in l_R(X_0)$. If there exists $a_x \in A \backslash X_0$. Then by the minimality of $l_R(X_0)$, we have $l_R(a_x, a_{x_1}, \ldots, a_{x_n}) = l_R(X_0)$. Thus $a_x = ca_x$. This implies that $a_x = ca_x$ for any $a_x \in A$. Therefore J is a G-indexed left s-unital ideal. □

Theorem 2.12. *Let G be an ordered group and σ is weakly rigid. If R satisfies the descending chain condition on left annihilators, then the following conditions are equivalent:*

(1) *$R * ((G))$ is a right APP.*
(2) *For any G-indexed generated right ideal A of R, $r_R(A)$ is left s-unital.*

Proof. (1)$\Rightarrow$(2) Let $A = \sum_{x \in I} a_x R$, where I is well-ordered subset of G. Define $f = \sum_{x \in G} a_x x \in R * ((G))$, where $a_x = 0$ if $x \in G \backslash I$. Since $R * ((G))$ is right APP, by Proposition 2.10, R is APP. Thus, $r_{R*((G))}(fR * ((G))) = r_R(A) * ((G))$ by Lemma 2.7 and Lemma 2.8. Hence, by (1) $r_R(A) * ((G))$ is left s-unital. Now we prove $r_R(A)$ is left S-unital.

Let $b \in r_R(A)$. Then $b \in r_R(A) * ((G))$. Thus there exists an $h = \sum_{z \in G} c_z z \in r_R(A) * ((G))$ such that $b = hb$. Consequently, $c_1 \in r_R(A)$ and $b = c_1 b$. Hence $r_R(A)$ is left s-unital.

(2)$\Rightarrow$(1) Let $f = \sum_{x \in G} a_x x, g = \sum_{y \in G} b_y y \in R * ((G))$ be such that $g \in r_{R*((G))}(fR * ((G)))$. Then, by (2) and Lemma 2.7, we have $a_x R b_y = 0$ for any $x, y \in G$. Thus $b_y \in r_R(\sum_{x \in supp(f)} a_x R)$ for any $y \in supp(g)$. By (2), $r_R(\sum_{x \in supp(f)} a_x R)$ is left s-unital. So $r_R(\sum_{x \in supp(f)} a_x R)$ is G-indexed left s-unital by Lemma 2.11. Hence There exists $c \in r_R(\sum_{x \in supp(f)} a_x R)$ such that $b_y = cb_y$ for any $y \in supp(g)$.

Now for any $h = \sum_{r \in G} r_z z \in R * ((G))$,

$$fhc = \sum_{q \in G} \sum_{\{(x,z) \mid xz = q\}} a_x \sigma_x(r_z) t(x,z) \sigma_z(c) t(z,1)) q = 0,$$

and from $b_y = cb_y$ it follows that $(1-c)\sigma_1(b_y) = 0$ for any $b \in G$ since σ is weakly rigid, so

$$(1-c)g = \sum_{y \in G} (1-c)\sigma_1(b_y)t(1,y)y = 0,$$

which imply that $c \in r_{R*((G))}(fR*((G)))$ and $g = c_g g$. Hence $R*((G))$ is right *APP*. □

Corollary 2.13. *Let R be a ring and G an ordered group. If R satisfies the descending chain condition on left annihilators, then $R((G))$ is right APP if and only if for any G-indexed generated right ideal A of R, $r_R(A)$ is left s-unital.*

Let R be a commutative ring with identity. Then R is called a *PF*-ring (resp., *PP*-ring) if every principal ideal of R is a flat (resp., projective) R-module. It is well-known that if R is Noetherian, then these two notions are equal (see Corollary 4.3 [24]). It is proved in [26] that a ring R is a *PF*-ring if and only if the annihilator of each element $r \in R$, $ann_R(r)$ is a pure ideal, that is, for all $b \in ann_R(r)$ there exists $c \in ann_R(r)$ such that $bc = b$. Also proved that in [27], the power series ring $R[[X]]$ is a *PF*-ring if and only if for any two countable subsets $A = \{a_0, a_1, \ldots\}$ and $B = \{b_0, b_1, \ldots\}$ of R such that $A \subseteq ann_R(B)$, there exists $r \in ann_R(B)$ such that $ar = a$ for all $a \in A$. J. Kim Theorem 3 and Theorem 4 [31] proved that for a Noetherian ring R, $R[[X]]$ is a *PF* (resp., *PP*) ring if and only if R is a *PF* (resp., *PP*) ring. Liu and Ahsan proved in [34] that the ring $[[R^{S,\leq}]]$ of generalized power series is a *PP*-ring if and only if R is a *PP*-ring and every S-indexed subset C of $B(R)$ (the set of all idempotents of R) has a least upper bound in $B(R)$. Also in [29], it was proved that, if R is a commutative ring with identity and $(S, \leq)$ is a strictly totally ordered monoid, then the ring $[[R^{S,\leq}]]$ of generalized power series is a *PF*-ring if and only if for any two S-indexed subsets A and B of R such that $B \subseteq ann_R(A)$, there exists $c \in ann_R(A)$ such that $bc = b$ for all $b \in B$, and that for a Noetherian ring R, $[[R^{S,\leq}]]$ is a *PP* ring if and only if R is a *PP*-ring. Under some conditions, *PF* (resp., *PP*) properties of Malcev-Neumann rings we have the following.

Lemma 2.14. *[27, Lemma 1]. Any PF-ring is reduced.*

Theorem 2.15. *Let R be a commutative ring and G an ordered group. Then $R*((G))$ is a PF-ring if and only if for any two G-indexed subsets A and B of R such that $B \subseteq ann_R(A)$, there exists $c \in ann_R(A)$ such that $bc = b$ for all $b \in B$.*

Proof. ⇐) Let $f = \sum_{x \in G} a_x x$, $g = \sum_{y \in G} b_y y \in R*((G))$ and let $g \in ann_{R*((G))}(f)$. Then

$$0 = gf = \sum_{z \in G} \sum_{\{(y,x) | yx = z\}} b_y \sigma_y(a_x) t(y,x) z.$$

Note that, in particular, R is a *PF*-ring, so by Lemma 2.14, R is reduced. Thus by Lemma 2.1, $b_y a_x = 0$ for all $x, y \in G$. Let $A = \{a_x \mid x \in supp(f)\}$ and $B = \{b_y \mid y \in supp(g)\}$. Then A and B are G-indexed and $B \subseteq ann_R(A)$. So by hypothesis, there exists $c \in ann_R(A)$ such that $b_y c = b_y$ for all $y \in G$. So $c\sigma_x(a_x)t(1,x) = 0$ for any $x \in G$ and $b_y \sigma_y(1-c)t(y,1) = 0$ for any $y \in G$. Thus

$$cf = \sum_{x \in G} c\sigma_x(a_x)t(1,x)x = 0$$

and

$$g(1-c)=\sum_{y\in G} b_y\sigma_y(1-c)t(y,1)y=0,$$

which implies that $c\in ann_{R*((G))}(f)$ and $gc=g$. Therefore $R*((G))$ is a PF-ring.

⇒) Assume that $R*((G))$ is a PF-ring. Let $A=\{a_x \mid x\in I\}, B=\{b_y \mid y\in J\}$ be two G-indexed subsets of R such that $B\subseteq ann_R(A)$, where I and J are well-ordered subsets of G. Define $f=\sum_{x\in G} a_x x\in R*((G))$, where $a_x=0$ if $x\in G\backslash I$, and $g=\sum_{y\in G} b_y y\in R*((G))$, where $b_y=0$ if $y\in G\backslash J$. Then

$$gf=\sum_{z\in G}\sum_{\{(y,x)|yx=z\}} b_y\sigma_y(a_x)t(y,x)z=0.$$

Therefore $g\in ann_{R*((G))}(f)$. Thus by the assumption, there exists $h=\sum_{u\in G} d_u u\in ann_{R*((G))}(f)$ such that $gh=g$. Therefore we have $0=hf$ and $0=g(h-1)$. Since, by Lemma 2.14 and Lemma 2.1, R is reduced, $d_u a_x=0$ for any $u,x\in G$, and $b_y(d_1-1)=0$ for any $y\in G$. So $d_1\in ann_R(A)$ and $bd_1=b$ for all $b\in B$. Therefore the result holds. □

Corollary 2.16. *Let R be a commutative ring and G an ordered group. If R is a PF-ring, then $R((G)))$ is a PF.*

Corollary 2.17. *Let R be a commutative ring and α is weakly rigid automorphism of R. Then the following conditions are equivalent:*

(1) *For any countable subset A and B of R such that $B\subseteq ann_R(A)$, there exists $c\in ann_R(A)$ such that $bc=b$ for all $b\in B$.*
(2) *$R[[x,x^{-1};\alpha]]$ is a PF.*

Proof. Take $G=\mathbb{Z}$ and $t(x,y)=1$ for any $x,y\in\mathbb{Z}$. For any $x\in\mathbb{Z}$, let $\sigma_x=\alpha^x$. Then σ is weakly rigid. Now the result follows from Theorem 2.15. □

Lemma 2.18. *[8, Corollary 3.2]. Let R be a reduced ring and σ is weakly rigid. If $\phi\in R*((G))$ is an idempotent, then there exists an idempotent $e\in R$ such that $\phi=e$.*

Theorem 2.19. *Let R be a Noetherian ring, G an ordered group and σ is weakly rigid. Then $R*((G))$ is a PP-ring if and only if R is a PP.*

Proof. Suppose that $R*((G))$ is a PP-ring. Let $a\in R$. Then $ann_{R*((G))}(a)=\phi(R*((G)))$ for some idempotent $\phi=\sum_z d_z z\in R*((G))$. By Lemma 2.18, there exists and idempotent $e\in R$ such that $\phi=e$, we claim that $ann_R(a)=eR$. If $b\in ann_R(a)$, then $b\in ann_{R*((G))}(a)=e(R*((G)))$, and so we have $b=eh$ for some $h=\sum_{y\in G} b_y y$. Thus, $b=e\sigma_1(b_1)t(1,1)\in eR$. Hence $ann_R(a)\subseteq eR$. For the opposite inclusion is clear. So $ann_R(a)=eR$. Therefore R is a PP-ring.

Conversely, assume that R is a PP-ring. Let $h=\sum_{y\in G} b_y y\in R*((G))$. We will show that there exists $e^2=e\in R$ such that $ann_{R*((G))}(h)=e(R*((G)))$.

Since R is Noetherian, $c(h)$ is finitely generated, say $c(h)=<b_{y_1},b_{y_2},\dots,b_{y_n}>$, where $y_1,y_2,\dots,y_n\in G$. Let $N=ann_R(b_{y_1},\dots,b_{y_n})=\cap_{i=1}^n ann_R(b_{y_i})$. Since R is PP, there exist idempotent

$e_1, e_2, \ldots, e_n \in R$ such that $ann_R(b_{y_i}) = e_iR$, for $i = 1, 2, \ldots, n$. Take $e = e_1e_2 \cdots e_n$. Then $N = eR$ and $e^2 = e \in R$. Now we show that $ann_{R*((G))}(h) = e(R * ((G)))$. Let $f = \sum_{x\in G} a_x x \in ann_{R*((G))}(h)$. Then $a_x b_y = 0$ for any $x, y \in G$ since R is reduced by Lemma 2.14. Thus $a_x \in N$ for any $x \in G$, so $a_x = ea_x$ for any $x \in G$. Hence $f = e(\sum_{x\in G} \sigma_1^{-1}(a_x t(1, x)^{-1})x) \in e(R * ((G)))$. Therefore $ann_{R*((G))}(h) \subseteq e(R * ((G)))$. From $e \in N = ann_R(b_{y_1}, \ldots, b_{y_n})$ it follows that $e \in ann_{R*((G))}(h)$. Hence $ann_{R*((G))}(h) = e(R * ((G)))$ and so $R * ((G))$ is a PP-ring. □

Theorem 2.20. *Let R be a ring, G an ordered group and σ is weakly rigid. Then $R * ((G))$ is a reduced left PP-ring if and only if R is a reduced left PP-ring and every G-indexed subset C of $B(R)$ has a least upper bound in B.*

Proof. It follows from Theorem 2.3 [34] and Theorem 2.19. □

Corollary 2.21. *Let R be a commutative PP-ring and G an ordered group. If every subset of $B(R)$ has a least upper bound in $B(R)$, then $R * ((G))$ is a PP-ring.*

Following to Faith [23], a ring R is called right *Zip* provided that if the right annihilator $r_R(X)$ of a subset X of R is zero, then there exists a finite subset $Y \subseteq X$ such that $r_R(Y) = 0$; equivalently, for a left ideal J of R with $r_R(J) = 0$, there exists a finitely generated left ideal $J_1 \subseteq J$ such that $r_R(J_1) = 0$. R is *Zip* if it is right and left *Zip*. Faith [23] it was proved that if R is a commutative *Zip* ring and G is a finite abelian group, then the group ring $R[G]$ of G over R is a *Zip*-ring.

Proposition 2.22. *Let R be a reduced ring, G an ordered group and σ is weakly rigid. Then $R * ((G))$ is a right Zip-ring if and only if R is a right Zip.*

Proof. $\Rightarrow$) Suppose that $R * ((G))$ is *Zip* and $X \subseteq R$ with $r_R(X) = 0$. If $f = \sum_{x\in G} a_x x \in r_{R*((G))}(X)$, then $cf = 0$ for all $c \in X$, and so $ca_x = 0$ for all $c \in X$ and all $x \in supp(f)$. Thus for all $x \in supp(f)$, $0 = a_x \in r_R(X)$, and so $f = 0$. Hence $r_{R*((G))}(X) = 0$. Since $R * ((G))$ is *Zip*, there exists a finite subset $X_0 \subseteq X$ such that $r_{R*((G))}(X_0) = 0$. Hence $r_R(X_0) = r_{R*((G))}(X_0) \cap R = 0$. Therefore R is *Zip*

$\Leftarrow$) Assume that R is a *Zip*, and V is a subset of $R * ((G))$ with $r_{R*((G))}(V) = 0$. For any $f = \sum_{x\in G} a_x x \in R * ((G))$, let C_f denote the set $\{a_x \mid x \in supp(f)\}$, and for the subset $V \subseteq R * ((G))$, let C_V denote the set $\cup_{f\in V} C_f$. Now we show that $r_R(C_V) = 0$. If $r \in r_R(C_V)$, then $ar = 0$ for all $a \in C_V$. So for any $f = \sum_{x\in G} a_x x \in V$, we have $a_x r = 0$ for all $x \in supp(f)$, and so $fr = 0$ by Lemma 2.1. Hence $0 = r \in r_{R*((G))}(V)$, and so $r_R(C_V) = 0$ is proved. Since R is *Zip*, there exists a finite subset $U_0 = \{q_1, q_2, \ldots, q_n\} \subseteq C_V$ such that $r_R(U_0) = 0$. Let $f_i (1 \le i \le n)$ be an element of V such that there exists $u_i \in supp(f)$ with q_i is the coefficients of u_i. Let $V_0 = \{f_1, f_2, \ldots, f_n\}$. Then V_0 is a finite subset of V and $C_{V_0} \supseteq U_0$. Then $r_R(C_{V_0}) \subseteq r_R(U_0) = 0$. Now we show that $r_{R*((G))}(V_0) = 0$. Suppose $g = \sum_{y\in G} b_y y \in r_{R*((G))}(V_0)$. Then $f_i g = 0$ for any $f_i = \sum_{x\in G} a_x^i x \in V_0$. By Lemma 2.1, $a_x^i b_y = 0$ for all $x \in supp(f_i)$ and any $y \in supp(g)$. Hence $0 = b_y \in r_R(C_{V_0})$ for all $y \in supp(g)$, and so $g = 0$. Hence $r_{R*((G))}(V_0) = 0$. Therefore $R * ((G))$ is a *Zip*. □

Corollary 2.23. *Let R be a reduced ring and α is weakly rigid automorphism of R. Then the following conditions are equivalent:*

(1) *R is a right Zip.*
(2) *$R[[x, x^{-1}; \alpha]]$ is a right Zip.*

Proof. Take $G = \mathbb{Z}$ and $t(x, y) = 1$, for any $x, y \in \mathbb{Z}$. For any $x \in \mathbb{Z}$, let $\sigma_x = \alpha^x$. Then σ is weakly rigid. Now the result follows from Proposition 2.22. □

Corollary 2.24. *Let G be an ordered group and R a reduced ring. Then R is a right Zip-ring if and only if $R((G))$ is a right Zip.*

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] L. Makar-Limanov, The Skew Field of Fractions of the Weyl Algebra Contains a Free Noncommutative Subalgebra, Commun. Algebra. 11 (1983), 2003–2006. https://doi.org/10.1080/00927878308822945.

[2] M. Lorenz, Division Algebras Generated by Finitely Generated Nilpotent Groups, J. Algebra. 85 (1983), 368–381. https://doi.org/10.1016/0021-8693(83)90101-1.

[3] I.M. Musson, K. Stafford, Malcev-Neumann Group Rings, Commun. Algebra. 21 (1993), 2065–2075. https://doi.org/10.1080/00927879308824665.

[4] C. Sonin, Krull Dimension of Malcev-Neumann Rings, Commun. Algebra. 26 (1998), 2915–2931. https://doi.org/10.1080/00927879808826317.

[5] D.S. Passman, Infinite Crossed Pure and Applied Mathematics, 135, Academic Press, Boston, MA, 1989.

[6] D.S. Passman, The Algebraic Structure of Group Rings, John Wiley, New York, (1977).

[7] E. Ali, The Reflexive Condition on Skew Monoid Rings, Eur. J. Pure Appl. Math. 16 (2023), 1878–1893.

[8] Z.K. Liu, Principal Quasi-Baerness of Malcev-Neumann Rings, J. Math. (Wuhan, China), 25 (2005), 237–246.

[9] Y.V. Merekin, On a Matrix Representation of Recursive Schemes for Word Generation, Southeast Asian Bull. Math. 31 (2007), 1167–1172.

[10] E. Ali, A. Elshokry, Some Results on a Generalization of Armendariz Rings, Asia Pac. J. Math. 6 (2019), 1. https://doi.org/10.28924/APJM/6-1.

[11] E. Ali, A Note on Skew Generalized Power Serieswise Reversible Property, Int. J. Anal. Appl. 21 (2023), 69. https://doi.org/10.28924/2291-8639-21-2023-69.

[12] W.K. Nicholson, J.F. Watters, Rings With Projective Socle, Proc. Amer. Math. Soc. 102 (1988), 443–450.

[13] X. Weimin, Modules With Projective Socles, Riv. Mat. Univ. Parma, 1(5) (1992), 311–315.

[14] L. Zhongkui, L. Fang, Ps-Rings of Generalized Power Series, Commun. Algebra. 26 (1998), 2283–2291. https://doi.org/10.1080/00927879808826276.

[15] E. Ali, On Crossed Product Rings Over p.q.-Baer and Quasi-Baer Rings, Int. J. Anal. Appl. 21 (2023), 108. https://doi.org/10.28924/2291-8639-21-2023-108.

[16] E. Ali, A. Elshokry, Some Properties of Quasi-Armendariz Rings and Their Generalizations, Asia P. J. Math. 5 (2018), 14–26.

[17] E. Ali, Generalized Reflexive Structures Properties of Crossed Products Type, Eur. J. Pure Appl. Math. 16 (2023), 2156–2168. https://doi.org/10.29020/nybg.ejpam.v16i4.4918.

[18] E. Ali, A. Elshokry, A Note on (S, ω)-Quasi-Armendariz Rings, Palestine J. Math. 12 (2023), 452–464.

[19] C.Y. Hong, N.K. Kim, T.K. Kwak, Ore Extensions of Baer and *p.p.*-Rings, J. Pure Appl. Algebra. 151 (2000), 215–226. https://doi.org/10.1016/s0022-4049(99)00020-1.

[20] A. Majidinya, A. Moussavi, K. Paykan, Generalized APP-Rings, Commun. Algebra. 41 (2013), 4722–4750. https://doi.org/10.1080/00927872.2011.636414.

[21] B. Stenstrom, Rings of Quotients, Springer-Verlag, Berlin, Heidelberg, 1975.

[22] C.E. Rickart, Banach Algebras With an Adjoint Operation, Ann. Math. 47 (1946), 528–550. https://doi.org/10.2307/1969091.

[23] C. Faith, Rings With Zero Intersection Property on Annihilators: Zip Rings, Publ. Mat. 33 (1989), 329–338. https://www.jstor.org/stable/43737136.

[24] J.J. Rotman, An Introduction to Homological Algebra, Academic Press, San Diego, 1993.

[25] G.F. Birkenmeier, J.Y. Kim, J.K. Park, Principally Quasi-Baer Rings, Commun. Algebra. 29 (2001), 639–660. https://doi.org/10.1081/agb-100001530.

[26] H. Al-Ezeh, On Some Properties of Polynomials Rings, Int. J. Math. Math. Sci. 10 (1987), 311–314. https://doi.org/10.1155/s0161171287000371.

[27] H. Al-Ezeh, Two Properties of the Power Series Ring, Int. J. Math. Math. Sci. 11 (1988), 9–13. https://doi.org/10.1155/s0161171288000031.

[28] H. Tominaga, On S-Unital Rings, Math. J. Okayama Univ. 18 (1976), 117–134.

[29] H. Kim and T.I. Kwon. PF-Rings of Generalized Power Series, Kyungpook Math. J. 47 (2007), 127–132.

[30] I. Kaplansky, Rings of Operators, Benjamin, New York, (1965).

[31] J.H. Kim, A Note on the Quotient Ring R((X)) of the Power Series Ring R[[X]], J. Korean Math. Soc. 25 (1998), 265–271.

[32] R. Zhao, Left APP-Rings of Skew Generalized Power Series, J. Algebra Appl. 10 (2011), 891–900. https://doi.org/10.1142/s0219498811005014.

[33] W.E. Clark, Twisted Matrix Units Semigroup Algebras, Duke Math. J. 34 (1967), 417–423. https://doi.org/10.1215/s0012-7094-67-03446-1.

[34] Z.K. Liu, J. Ahsan, PP-Rings of Generalized Power Series, Acta Math. Sinica. 16 (2000), 573–578. https://doi.org/10.1007/s1011400000884.

[35] Z.K. Liu, Z. Renyu, a Generalization OF PP-Rings and p.q.-Baer Rings, Glasgow Math. J. 48 (2006), 217–229. https://doi.org/10.1017/s0017089506003016.