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A Hybrid Laplace Transform-Optimal Homotopy Asymptotic Method (LT-OHAM) for Solving Integro-Differential Equations of the Second Kind

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ABSTRACT. The new proposed hybrid method between optimal homotopy asymptotic method and Laplace transform namely LT-OHAM is formulated for the first time in our paper. This hybrid method presents significant features of LT-OHAM and its capability of handling IDEs. This formulation is developed to find the solution of IDEs. By using the new presented hybrid method, some applications of IDEs are solved. This hybrid method seems very efficient and easy to solve these types of equations.

1. INTRODUCTION

IDEs model many situations from sciences such as mathematics, natural sciences, chemistry, biology, physics, mechanics, engineering etc. These equations have found applications in many branches include biomechanics, circuit analysis, aerodynamics, hydrology, epidemiology, population dynamics, mathematical modeling of epidemics, diffusion problems, fracture mechanics, control theory, queuing theory, theory, electrostatics, and many others.

Factually, several various problems in different fields are modeled by IDEs. Therefore, the solutions of these equations play an important role in different sciences. In recent decades,

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much work has been carried out by researchers in mathematics and other fields on studding novel methods for solutions of IDEs. Among these are Wavelet-Galerkin method [1], Differential transformation method [2], Homotopy analysis method [3], Variational iteration methods [4], Non-polynomial spline functions [5], Mahgoub transform [6], Laplace Adomian and Laplace modified Adomian decomposition methods [7] and An efficient technique based on least-squares method [8].

Most recently, OHAM was introduced by Marinca et al. [9] and it is used to solve the linear and nonlinear problems. This method is modified to improve the accuracy of the results. (See [10], [11], [12], [13], [14], [15]).

Generally, the Integral transforms play a significant role in applied mathematics. There are several types of integral transforms but in fact the most frequently used is the Laplace transform. Thus, the present paper is dedicated to introduce a new method namely LT-OHAM to find the solution of IDEs.

This paper presents five sections. Section 2 introduces some basic relevant concepts. In section 3, general idea of LT-OHAM is applied to solution of IDEs. In section 4, some applications of IDEs will be presented. Section 5 presents some conclusions.

2. PRELIMINARIES

To introduce our study we present some basic relevant concepts.

Definition 2.1. [16] The general form of *nth* order IDEs of second kind can be expressed in the form of

$$y^{(n)}(x) = f(x) + \int_0^x k(x,t)y(t)dt, \quad 0 \le x \le 1 \tag{1}$$

with IC: $y(a) = \alpha_0, y^{(1)}(a) = \alpha_1, \ldots, y^{(n-1)} = \alpha_{n-1}$.

Definition 2.2. [17] The LT of $f(t)$ is denoted $\mathcal{L}\{f(t)\} = F(s)$ and defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt, \tag{2}$$

where f(t) be a continuous function.

Property 2.1. [17] Let $f(t)$ and $g(t)$ are two continuous functions then

$$\mathcal{L}\{af(t) + bg(t)\} = a\,F(s) + b\,G(s) \tag{3}$$

where a and b are constants.

Proof. Suppose that $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$.

$\mathcal{L}\{af(t) + bg(t)\} = \int_0^\infty ae^{-st}\big(f(t) + bg(t)\big)dt = a\int_0^\infty e^{-st}f(t)dt + b\int_0^\infty e^{-st}g(t)dt$

$\Rightarrow a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\} = a\,F(s) + b\,G(s)$ ■

Property 2.2. [17] Let $f(t), f'(t), \ldots, f^{(n-1)}(t), f^{(n)}(t)$ are continuous functions then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \tag{4}$$

Proof. We can proof this property by definition of LT as

$$\mathcal{L}\{f^{(n)}(t)\} = \int_0^\infty e^{-st} f^{(n)}(t)dt = \lim_{n\to\infty} \int_0^n e^{-st} f^{(n)}(t)dt \ \blacksquare$$

Property 2.3. [17] Let F(s) and G(s) are two LTs then

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\,\mathcal{L}^{-1}\{G(s)\} \quad (5)$$

where a and b are constants.

Proof. From property 2.1, we have $\mathcal{L}\{af(t) + bg(t)\} = a\,F(s) + b\,G(s)$

Take $\mathcal{L}^{-1}\{.\}$ to both sides, then $af(t) + bg(t) = \mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\,\mathcal{L}^{-1}\{G(s)\}$ $\blacksquare$

Property 2.4. [17] Let $f(t) = \int_0^t f(u)du$ then

$$\mathcal{L}\{f(t)\} = \frac{1}{s}F(s) \quad (6)$$

Proof. Suppose that $g(t) = \int_0^t f(u)du$. Then $g'(t) = f(t)$ and $g(0) = 0$.

By using property 2.2, we have $\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\}$.

Since $g'(t) = f(t)$ and $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{g'(t)\} = \mathcal{L}\{f(t)\} = F(s) = s\mathcal{L}\{g(t)\}$

i.e., $\frac{F(s)}{s} = \mathcal{L}\{g(t)\} = \mathcal{L}\left\{\int_0^t f(u)du\right\}$ $\blacksquare$

Definition 2.3. [9] Let the differential equation $L(u(t)) + N(u(t)) + g(t) = 0$, $B\left(u, \frac{du}{dt}\right) = 0$, where L is a linear operator, N is a nonlinear operator and B is a boundary operator. OHAM in a general form

$$(1-p)\left[L(u(t,p)) + g(t)\right] = H(p)\left[L(u(t,p)) + g(t) + N(u(t,p))\right], \quad (7)$$

where the parameter $p \in [0,1]$, $H(p)$ denotes a non-zero auxiliary function for $p \neq 0$ and $H(0) = 0$.

3. GENERAL IDEA of LT-OHAM

Here, we have applied the LT-OHAM for the solution of IDEs. Rewrite Eq. (1) as.

$$y^{(n)}(x) - f(x) - \int_0^x k(x,t)y(t)dt = 0, \ \ 0 \le x \le 1 \quad (8)$$

with IC: $y(a) = \alpha_0, y'(a) = \alpha_1, \dots, y^{(n-1)} = \alpha_{n-1}$.

By performing the LT on both sides of Eq. (8), yields

$$\mathcal{L}\{y^{(n)}(x)\} - \mathcal{L}\{f(x)\} - \mathcal{L}\{\int_0^x k(x,t)y(t)dt\} = 0, \quad 0 \le x \le 1 \quad (9)$$

Using properties of LT, we get

$$\sum_{i=1}^n s^{(i-1)}\, y^{(i-1)}(0) - \mathcal{L}\{f(x)\} - \mathcal{L}\{y(x)\}\mathcal{L}\{k(x)\} = 0 \quad (10)$$

By simplifying, we get

$$\sum_{i=1}^n s^{(i-1)}\, y^{(i-1)}(0) - F(s) - Y(s)K(s) = 0 \quad (11)$$

Applying OHAM to Eq. (11)

$L(y(x,p)) = \mathcal{L}\{y^{(n)}(x)\} = \sum_{i=1}^n s^{(i-1)}\, y^{(i-1)}(0)$, $\quad N(y(x,p)) = -\mathcal{L}\{y(x)\}\mathcal{L}\{k(x)\} = Y(s)K(s)$, $g(x) = -F(s)$

which satisfies

$$(1-p)\left[\sum_{i=1}^n s^{(i-1)}\, y^{(i-1)}(0) - F(s)\right] = H(p)\left[\sum_{i=1}^n s^{(i-1)}\, y^{(i-1)}(0) - F(s) - Y(s)K(s)\right] \quad (12)$$

If $p = 0$ and $p = 1$, then

$$\mathcal{L}\{y(x,0)\} = \mathcal{L}\{y_0(x)\} = Y_0(s), \qquad \mathcal{L}\{ y(x,1)\} = \mathcal{L}\{y(x)\} = Y(s)$$

respectively. Define $H(p)$ as

$$H(p) = \sum_{j=1}^{m} c_j\, p^j, \tag{13}$$

where c_j, $i = 1,2, \dots$ are constants.

Now, we can use Taylor's series

$$\mathcal{L}\left(y(x,p,c_j)\right) = \mathcal{L}\big(y_0(x,t)\big) + \sum_{k=1}^{\infty} \mathcal{L}\{y_k(x,c_j)\}\, p^k \tag{14}$$

When $p = 1$, yields

$$\mathcal{L}\left(y(x,1,c_j)\right) = \mathcal{L}\big(y_0(x,t)\big) + \mathcal{L}\big(\sum_{k=1}^{\infty} y_k(x,c_j)\, \big) \tag{15}$$

Substituting Eq.(14) into Eq.(10), it holds that

$P^0 : \mathcal{L}\big(y_0(x)\big) = \mathcal{L}\{f(x)\} = F(s) \Longrightarrow\ y_0(x) = \mathcal{L}^{-1}\{F(s)\}$

$P^1 : \mathcal{L}\big(y_1(x)\big) = -c_1\mathcal{L}\{ y_0(x)\}\mathcal{L}\{k(x)\} \Longrightarrow\ y_1(x) = -c_1\mathcal{L}^{-1}\big\{\mathcal{L}\{ y_0(x)\}\mathcal{L}\{k(x)\}\big\}$

$P^2 : \mathcal{L}\big(y_2(x)\big) = (1 + c_1)\mathcal{L}\{ y_1(x)\} - c_1\mathcal{L}\{ y_1(x)\}\mathcal{L}\{k(x)\} - c_2\mathcal{L}\{ y_0(x)\}\mathcal{L}\{k(x)\}$

$\Longrightarrow\ y_2(x) = \mathcal{L}^{-1}\big\{(1 + c_1)\mathcal{L}\{ y_1(x)\} - c_1\mathcal{L}\{ y_1(x)\}\mathcal{L}\{k(x)\} - c_2\mathcal{L}\{ y_0(x)\}\mathcal{L}\{k(x)\}\big\}$

$P^k : \mathcal{L}\big(y_k(x)\big) = (1 + c_1)\mathcal{L}\{ y_{k-1}(x)\} - \sum_{j=2}^{k-1} c_j\, \mathcal{L}\big\{ y_{k-j}(x)\big\} - \sum_{i=1}^{k} c_i\mathcal{L}\{ y_{k-i}(x)\}\mathcal{L}\{k(x)\}$

$$\Longrightarrow\ y_k(x) = \mathcal{L}^{-1}\big\{(1 + c_1)\mathcal{L}\{ y_{k-1}(x)\} - \sum_{j=2}^{k-1} c_j\, \mathcal{L}\big\{ y_{k-j}(x)\big\} - \sum_{i=1}^{k} c_i\mathcal{L}\{ y_{k-i}(x)\}\mathcal{L}\{k(x)\}\big\} \tag{16}$$

To find $c_1, c_2, c_3 \dots$, we used

$$y^n(x,1,c_j) = y_0(x,t) + \sum_{k=1}^{n} y_k(x,c_j)\ ,\ \ j = 1,2, \dots, n \tag{17}$$

Using Eqs. (17) and (10), we have

$$R(x,c_j) = \mathcal{L}^{-1}\left\{\mathcal{L}\{y^{(n)}(x)\}^n - \mathcal{L}\{f(x)\} - \mathcal{L}\{ y(x)\}^n\mathcal{L}\{k(x)\}\right\} \tag{18}$$

Finally, we can use least square method.

4. APPLICATIONS

Three applications of IDEs will be presented to show the efficiency of the new proposed method.

Application 4.1. Apply LT-OHAM to the second order IDE

$$y''(x) = e^x - x + \int_0^1 xty(t)dt, \tag{19}$$

with IC: $y(0) = 1, y'(0) = 1$ and exact solution $y(x) = e^x$.

By performing the LT on both sides, yields

$$\mathcal{L}\{y''(x)\} = \mathcal{L}\{ e^x - x\} - \mathcal{L}\left\{\int_0^1 xty(t)dt \ \right\} \tag{20}$$

Thus, we get

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) = -\frac{1}{s^2} + \frac{1}{s-1} + \frac{1}{s^2}\mathcal{L}\left\{\int_0^1 ty(t)dt \ \right\} \tag{21}$$

Substituting IC into Eq. (21), it holds that

$$s^2\mathcal{L}\{y\} - s - 1 = -\frac{1}{s^2} + \frac{1}{s-1} + \frac{1}{s^2}\mathcal{L}\left\{\int_0^1 ty(t)dt\right\} \quad (22)$$

i.e.,

$$\mathcal{L}\{y\} - \left(\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{(s-1)s^2}\right) - \frac{1}{s^4}\mathcal{L}\left\{\int_0^1 ty(t)dt\right\} = 0 \quad (23)$$

OHAM of Eq. (23) is

$L(y(x,p)) = \mathcal{L}\{y\}$, $\quad N(y(x,p)) = -\frac{1}{s^4}\mathcal{L}\left\{\int_0^1 ty(t)dt\right\}$, $\quad g(x) = \frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{(s-1)s^2}$,

which satisfies

$$(1-p)\left[\mathcal{L}\{y\} - \left(\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{(s-1)s^2}\right)\right] = H(p)\left[\mathcal{L}\{y\} - \left(\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{(s-1)s^2}\right) - \frac{1}{s^4}\mathcal{L}\left\{\int_0^1 ty(t)dt\right\}\right] \quad (24)$$

Next, one can use Eqs. (14) and (16), yields

$P^0 : \mathcal{L}(y_0(x)) = \mathcal{L}\left\{\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{(s-1)s^2}\right\} \Longrightarrow y_0(x) = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{(s-1)s^2}\right\} = 1 + x - \frac{1}{6}x^3 - 1 - x + e^x = e^x - \frac{1}{6}x^3$

$P^1 : \mathcal{L}(y_1(x)) = -c_1\left\{\frac{1}{s^4}\mathcal{L}\left\{\int_0^1 ty_0(t)dt\right\}\right\} \Longrightarrow y_1(x) = -c_1\mathcal{L}^{-1}\left\{\left\{\frac{1}{s^4}\mathcal{L}\left\{\int_0^1 ty_0(t)dt\right\}\right\}\right\} = -c_1\frac{29}{180}x^3$

$P^2 : \mathcal{L}(y_2(x)) = (1+c_1)\mathcal{L}\left\{-c_1\frac{29}{180}x^3\right\} - c_1\mathcal{L}\left\{\frac{1}{s^4}\left\{\int_0^1 t\left(-c_1\frac{29}{180}t^3\right)dt\right\}\right\} - c_2\mathcal{L}\left\{\frac{1}{s^4}\left\{\int_0^1 t\left(e^t - \frac{1}{6}t^3\right)dt\right\}\right\}$

$$\Longrightarrow y_2(x) = \mathcal{L}^{-1}\left\{(1+c_1)\mathcal{L}\left\{-c_1\frac{29}{180}x^3\right\} - c_1\mathcal{L}\left\{\frac{1}{s^4}\left\{\int_0^1 t\left(-c_1\frac{29}{180}t^3\right)dt\right\}\right\} - c_2\mathcal{L}\left\{\frac{1}{s^4}\left\{\int_0^1 t\left(e^t - \frac{1}{6}t^3\right)dt\right\}\right\}\right\} = \frac{-29}{5400}(30c_1 + 29c_1^2 + 30c_2) \quad (25)$$

Second order LT-OHAM solution given by

$$y^2(x) = y_0(x) + y_1(x) + y_2(x) = e^x - \frac{1}{6}x^3 - c_1\frac{29}{180}x^3 - \frac{29}{5400}(30c_1 + 29c_1^2 + 30c_2) \quad (26)$$

Using Eqs. (26) and (23), one has

$$R(x, c_1, c_2) = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{e^x - \frac{1}{6}x^3 - c_1\frac{29}{180}x^3 - \frac{29}{5400}(30c_1 + 29c_1^2 + 30c_2)\right\} - \left(\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{(s-1)s^2}\right) - \frac{1}{s^4}\mathcal{L}\left\{\int_0^1 t\left(e^t - \frac{1}{6}t^3 - c_1\frac{29}{180}t^3 - \frac{29}{5400}(30c_1 + 29c_1^2 + 30c_2)\right)dt\right\}\right\} \quad (27)$$

$c_1 = \frac{-30}{29}$ and $c_2 = 0$ are calculated by the following Eqs. (28) - (29)

$$J(c_1, c_2) = \int_0^1 R^2(x, c_1, c_2)\,dx \quad (28)$$

and

$$\frac{\partial J}{c_1}(c_1, c_2) = \frac{\partial J}{c_2}(c_1, c_2) = 0 \quad (29)$$

Therefore the solution becomes

$$y^2(x) = e^x \quad (30)$$

Application 4.2. Apply LT-OHAM to the first order IDE

$$y'(x) = 1-\frac{1}{3}x+\int_0^1 xty(t)dt, \tag{31}$$

with IC: $y(0)=0$ and exact solution $y(x)=x$.

Taking the LT on both sides, we get

$$\mathcal{L}\{y'(x)\} = \mathcal{L}\left\{1-\frac{1}{3}x\right\}+\mathcal{L}\left\{\int_0^1 xty(t)dt\right\} \tag{32}$$

One has

$$s\mathcal{L}\{y\}-y(0) = \frac{1}{s}-\frac{1}{3s^2}+\frac{1}{s^2}\mathcal{L}\left\{\int_0^1 ty(t)dt\right\} \tag{33}$$

Substituting IC into Eq. (33), it holds that

$$\mathcal{L}\{y\}-\left(\frac{1}{s^2}-\frac{1}{3s^3}\right)-\frac{1}{s^3}\mathcal{L}\left\{\int_0^1 ty(t)dt\right\}=0 \tag{34}$$

Using OHAM of Eq. (34)

$L\big(y(x,p)\big)=\mathcal{L}\{y\}$, $\quad N\big(y(x,p)\big)=-\frac{1}{s^3}\mathcal{L}\left\{\int_0^1 ty(t)dt\right\}$, $\quad g(x)=\frac{1}{s^2}-\frac{1}{3s^3}$,

which satisfies

$$(1-p)\left[\mathcal{L}\{y\}-\left(\frac{1}{s^2}-\frac{1}{3s^3}\right)\right] = H(p)\left[\mathcal{L}\{y\}-\left(\frac{1}{s^2}-\frac{1}{3s^3}\right)-\frac{1}{s^3}\mathcal{L}\left\{\int_0^1 ty(t)dt\right\}\right] \tag{35}$$

Using Eqs. (14) and (16), yields

$$P^0: \mathcal{L}\big(y_0(x)\big)=\mathcal{L}\left\{\frac{1}{s^2}-\frac{1}{3s^3}\right\} \Rightarrow y_0(x)=\mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{3s^3}\right\}=x-\frac{1}{6}x^2$$

$$P^1: \mathcal{L}\big(y_1(x)\big)=-c_1\left\{\frac{1}{s^3}\mathcal{L}\left\{\int_0^1 ty_0(t)dt\right\}\right\} \Rightarrow y_1(x)=-c_1\mathcal{L}^{-1}\left\{\left\{\frac{1}{s^3}\mathcal{L}\left\{\int_0^1 t\left(t-\frac{1}{6}t^2\right)dt\right\}\right\}\right\} = -c_1\frac{7}{48}x^2$$

$$P^2: \mathcal{L}\big(y_2(x)\big)=(1+c_1)\mathcal{L}\left\{-c_1\frac{7}{48}x^2\right\}-c_1\mathcal{L}\left\{\frac{1}{s^3}\left\{\int_0^1 t\left(-c_1\frac{7}{48}t^2\right)dt\right\}\right\}-c_2\mathcal{L}\left\{\frac{1}{s^3}\left\{\int_0^1 t\left(t-\frac{1}{6}t^2\right)dt\right\}\right\}$$

$$\Rightarrow y_2(x)=\mathcal{L}^{-1}\left\{(1+c_1)\mathcal{L}\left\{-c_1\frac{7}{48}x^2\right\}-c_1\mathcal{L}\left\{\frac{1}{s^3}\left\{\int_0^1 t\left(-c_1\frac{7}{48}t^2\right)dt\right\}\right\}-c_2\mathcal{L}\left\{\frac{1}{s^3}\left\{\int_0^1 t\left(t-\frac{1}{6}t^2\right)dt\right\}\right\}\right\}=\frac{-7}{384}(8c_1+7c_1^2+8c_2)x^2 \tag{36}$$

Therefore LT-OHAM solution is

$$y^2(x)=y_0(x)+y_1(x)+y_2(x)=x-\frac{1}{6}x^2-c_1\frac{7}{48}x^2-\frac{7}{384}(8c_1+7c_1^2+8c_2)x^2 \tag{37}$$

Using Eqs. (37) and (34), we have

$$R(x,c_1,c_2)=\mathcal{L}^{-1}\left\{\mathcal{L}\left\{x-\frac{1}{6}x^2-c_1\frac{7}{48}x^2-\frac{7}{384}(8c_1+7c_1^2+8c_2)x^2\right\}-\left(\frac{1}{s^2}-\frac{1}{3s^3}\right)-\frac{1}{s^3}\mathcal{L}\left\{\int_0^1 t\left(t-\frac{1}{6}t^2-c_1\frac{7}{48}t^2-\frac{7}{384}(8c_1+7c_1^2+8c_2)t^2\right)dt\right\}\right\} \tag{38}$$

To find values of c_1 and c_2, we consider

$$J(c_1,c_2)=\int_0^1 R^2(x,c_1,c_2)\,dx$$

and

$$\frac{\partial J}{c_1}(c_1, c_2) = \frac{\partial J}{c_2}(c_1, c_2) = 0$$

Hence $c_1 = \frac{-8}{7}$ and $c_2 = 0$ and the solution is

$$y^2(x) = x \tag{39}$$

Application 4.3. Apply LT-OHAM to the third order IDE

$$y'''(x) = \sin x - x - \int_0^{\frac{\pi}{2}} xty'(t)dt, \tag{40}$$

with IC: $y(0) = 1, y'(0) = 0, y''(x) = -1$ and exact solution $y(x) = \cos x$.

Taking the LT on both sides, we get

$$\mathcal{L}\{y'''(x)\} = \mathcal{L}\{\sin x - x\} - \mathcal{L}\left\{\int_0^{\frac{\pi}{2}} xty'(t)dt\right\} \tag{41}$$

We have

$$s^3\mathcal{L}\{y\} - s^2y(0) - sy'(0) - y''(0) = \frac{1}{s^2+1} - \frac{1}{s^2} - \frac{1}{s^2}\mathcal{L}\left\{\int_0^{\frac{\pi}{2}} ty'(t)dt\right\} \tag{42}$$

i.e.,

$$\mathcal{L}\{y\} - \left(\frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^3(s^2+1)} - \frac{1}{s^5}\right) + \frac{1}{s^5}\mathcal{L}\left\{\int_0^{\frac{\pi}{2}} ty'(t)dt\right\} = 0 \tag{43}$$

Using OHAM of Eq. (43), yields

$$L\big(y(x,p)\big) = \mathcal{L}\{y\}, \qquad N\big(y(x,p)\big) = \frac{1}{s^5}\mathcal{L}\left\{\int_0^{\frac{\pi}{2}} ty'(t)dt\right\}, \qquad g(x) = \frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^3(s^2+1)} - \frac{1}{s^5},$$

which satisfies

$$(1-p)\left[\mathcal{L}\{y\} - \left(\frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^3(s^2+1)} - \frac{1}{s^5}\right)\right] = H(p)\left[\mathcal{L}\{y\} - \left(\frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^3(s^2+1)} - \frac{1}{s^5}\right) + \frac{1}{s^5}\mathcal{L}\left\{\int_0^{\frac{\pi}{2}} ty'(t)dt\right\}\right] \tag{44}$$

Using Eqs. (14) and (16), yields

$$P^0 : \mathcal{L}\big(y_0(x)\big) = \mathcal{L}\left\{\frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^3(s^2+1)} - \frac{1}{s^5}\right\} \Longrightarrow y_0(x) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^3(s^2+1)} - \frac{1}{s^5}\right\} = \cos x - \frac{1}{24}x^4$$

$$P^1 : \mathcal{L}\big(y_1(x)\big) = c_1\left\{\frac{1}{s^5}\mathcal{L}\left\{\int_0^{\frac{\pi}{2}} ty_0'(t)dt\right\}\right\} \Longrightarrow y_1(x) = c_1\mathcal{L}^{-1}\left\{c_1\left\{\frac{1}{s^5}\mathcal{L}\left\{\int_0^{\frac{\pi}{2}} ty_0'(t)dt\right\}\right\}\right\} =$$

$$-c_1\left(\frac{1}{24} - \frac{\pi^5}{23040}\right)x^4$$

$$P^2 : \mathcal{L}\big(y_2(x)\big) = (1+c_1)\mathcal{L}\left\{-c_1\left(\frac{1}{24} - \frac{\pi^5}{23040}\right)x^4\right\} + c_1\mathcal{L}\left\{\frac{1}{s^5}\left\{\int_0^{\frac{\pi}{2}} t\left(-c_1\left(\frac{1}{24} - \frac{\pi^5}{23040}\right)t^4\right)' dt\right\}\right\} +$$

$$c_2\mathcal{L}\left\{\frac{1}{s^5}\left\{\int_0^{\frac{\pi}{2}} t\left(\cos t - \frac{1}{24}t^4\right)' dt\right\}\right\}$$

$$\Longrightarrow y_2(x) = \mathcal{L}^{-1}\left\{(1+c_1)\mathcal{L}\left\{-c_1\left(\frac{1}{24} - \frac{\pi^5}{23040}\right)x^4\right\} + c_1\mathcal{L}\left\{\frac{1}{s^5}\left\{\int_0^{\frac{\pi}{2}} t\left(-c_1\left(\frac{1}{24} - \frac{\pi^5}{23040}\right)t^4\right)' dt\right\}\right\} +\right.$$

$$\left. c_2\mathcal{L}\left\{\frac{1}{s^5}\left\{\int_0^{\frac{\pi}{2}} t\left(\cos t - \frac{1}{24}t^4\right)' dt\right\}\right\}\right\} = \frac{(960+\pi^5)}{22118400}(960c_1 + (960+\pi^5)c_1^2 + 960c_2)x^4 \tag{45}$$

The LT-OHAM solution is

$$y^2(x) = y_0(x) + y_1(x) + y_2(x) = \cos x - \frac{1}{24}x^4 - c_1\left(\frac{1}{24} - \frac{\pi^5}{23040}\right)x^4 + \frac{(960+\pi^5)}{22118400}(960c_1 + (960 + \pi^5)c_1^2 + 960c_2)x^4 \quad (46)$$

Now, we have

$$R(x, c_1, c_2) = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\cos x - \frac{1}{24}x^4 - c_1\left(\frac{1}{24} - \frac{\pi^5}{23040}\right)x^4 + \frac{(960+\pi^5)}{22118400}(960c_1 + (960 + \pi^5)c_1^2 + 960c_2)x^4\right\} - \left(\frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^3(s^2+1)} - \frac{1}{s^5}\right) + \frac{1}{s^5}\mathcal{L}\left\{\int_0^{\frac{\pi}{2}} t\left(\cos t - \frac{1}{24}t^4 - c_1\left(\frac{1}{24} - \frac{\pi^5}{23040}\right)t^4 + \frac{(960+\pi^5)}{22118400}(960c_1 + (960 + \pi^5)c_1^2 + 960c_2)t^4\right)' dt\right\}\right\} \quad (47)$$

Using $J(c_1, c_2) = \int_0^{\frac{\pi}{2}} R^2(x, c_1, c_2)\, dx$ and $\frac{\partial J}{c_1}(c_1, c_2) = \frac{\partial J}{c_2}(c_1, c_2) = 0$, yields $c_1 = \frac{-960}{960+\pi^5}$ and $c_2 = 0$ and the solution is

$$y^2(x) = \cos x \quad (48)$$

5. CONCLUSION

By combined optimal homotopy asymptotic method and Laplace transform, we introduced a new method namely LT-OHAM. They were applied to handle IDEs. To illustrate the efficiency of the method, some applications of IDEs were successfully tested by using our new hybrid method. Also, the results showed that the presented hybrid method is very powerful and effective. Cleary, this method no need of large computer memory and high computation time.

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