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The Exact Norm of Modified Hardy Operator in Power-Weighted Lebesgue Spaces

Pebrudal Zanu[1,2,*], Wono Setya Budhi[2], Yudi Soeharyadi[2]

[1]*Doctoral Program at Institut Teknologi Bandung, Jl.Ganesa No.10, Bandung, Indonesia*
[2]*Institut Teknologi Bandung, Ganesa No.10, Bandung, Indonesia*

[*]*Corresponding author:* pebrudal@gmail.com

Abstract. We consider the necessary and sufficient condition for boundedness of modified Hardy operators from a power-weighted Lebesgue space to another. We also compute the exact norm of modified n-dimensional Hardy operators from those spaces.

1. Introduction

Let $\beta \in \mathbb{R}$ and f be measurable functions. The Hardy operator with parameter β is defined in [7] as follows

$$H^{\beta} f(x) = \frac{1}{|x|^{n-\beta}} \int_{|y| \leq |x|} \frac{f(y)}{|y|^{\beta}}\, dy,$$

where $x \in \mathbb{R}^n \setminus \{0\}$. Whenever $\beta = 0$, the operator H^0 is the classical Hardy operator [4], which is the average of function f at the ball $B(0, |x|)$.

In functional analysis, there is a quite extensive references on the boundedness and the norm of integral operators from a normed spaces X to another normed spaces Y. We cite [4,5,7] for the case of the classical Hardy operators. Let $\mathbb{S}^{n-1}$ denote the surface of unit sphere with its area $|\mathbb{S}^{n-1}|$. Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $\beta < \frac{n}{p'}$. As a consequence of Theorem 6.4 in [6], Karapetiants obtained an explicit expression for the norm of H^{β}, that is

$$\|H^{\beta}\|_{L^p \to L^p} = \frac{|\mathbb{S}^{n-1}|}{\frac{n}{p'} - \beta}.$$

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Our modification of the Hardy operators is as follows. Let $0 \le \gamma < n$, $\beta \in \mathbb{R}$, $0 < \beta + \gamma < n$ and f be measurable functions. We define

$$H_\gamma^\beta f(x) = \frac{1}{|x|^{n-\beta-\gamma}} \int_{|y|\le|x|} \frac{f(y)}{|y|^\beta}\, dy,$$

where $x \in \mathbb{R}^n \setminus \{0\}$. Our motivation for such modification to the Hardy operator is to describe the singularity in a more general way.

We propose a different method from the existing literature to compute the exact norm of the modified Hardy operator. We extend the idea from Muckenhoupt [1] by computing the minimum value of the constant of the boundedness, which is dependent on ε. Using its idea we are able to compute the upper bound of the norm of the operator H_γ^β.

Let X^*, Y^* be dual of X, Y respectively, and $\widetilde{T}$ be dual of operator T. If T is bounded from X to Y, then $\widetilde{T}$ is also bounded from Y^* to X^*. It satisfies

$$\|\widetilde{T}\|_{Y^* \to X^*} = \|T\|_{X \to Y}. \tag{1.1}$$

Throughout this article, the notation $A \lesssim B$, for $A, B > 0$, means there exists $C > 0$ such that $A \le CB$. The operator $\widetilde{T}$ is defined by

$$\int_{\mathbb{R}^n} (Tf)(x) g(x)\, dx = \int_{\mathbb{R}^n} f(x) (\widetilde{T}g)(x)\, dx$$

for every $f \in X$ and $g \in X^*$. Using this definition, we can prove that the dual of modified Hardy operators is

$$\widetilde{H}_\gamma^\beta f(x) = \frac{1}{|x|^\beta} \int_{|y|>|x|} \frac{f(y)}{|y|^{n-\beta-\gamma}}\, dy.$$

Let $\alpha \in \mathbb{R}$, $1 \le p < \infty$. The power-weighted Lebesgue spaces $|x|^{\alpha p}$, denoted by $L_\alpha^p(\mathbb{R})$, is the set of measurable function f with the norm satisfying

$$\|f\|_{L_\alpha^p} := \left(\int_{\mathbb{R}^n} |x|^{\alpha p} |f(x)|^p\, dx \right)^{\frac{1}{p}} < \infty.$$

For $\alpha = 0$, we are back to the regular Lebesgue spaces, $L_\alpha^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. For $1 < p < \infty$ and $\alpha \in \mathbb{R}$, the dual of $L_\alpha^p(\mathbb{R}^n)$ is $L_{-\alpha}^{p'}(\mathbb{R}^n)$.

This article is written in three sections. In the first section,we introduce the novelty of our research. In the second section, we consider the necessary boundedness H_γ^β from $L_{\alpha_1}^{p_1}(\mathbb{R}^n)$ to $L_{\alpha_2}^{p_2}(\mathbb{R}^n)$. In the third section, we consider sufficient condition for boundedness H_γ^β from $L_{\alpha_1}^{p_1}(\mathbb{R}^n)$ to $L_{\alpha_2}^{p_2}(\mathbb{R}^n)$ and compute the exact norm.

2. The Necessary Conditions

Let $1 < p_1 < \infty$, $\gamma < \frac{n}{p_1}$. The well-known identity of necessary conditions for boundedness of fractional integral operators from $L^{p_1}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$ is $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\gamma}{n}$, see in [2, Ch.5]. In the following result, we propose the necessary conditions of modified Hardy operators from $L_{\alpha_1}^{p_1}(\mathbb{R}^n)$ to $L_{\alpha_2}^{p_2}(\mathbb{R}^n)$.

Theorem 2.1. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $1 \le p_1, p_2 < \infty$. If the operator H_γ^β is bounded from $L_{\alpha_1}^{p_1}(\mathbb{R}^n)$ to $L_{\alpha_2}^{p_2}(\mathbb{R}^n)$, then*

$$\frac{1}{p_2} + \frac{\alpha_2}{n} = \frac{1}{p_1} + \frac{\alpha_1}{n} - \frac{\gamma}{n}.$$

Proof. We prove the necessary condition similar to the one for fractional integral operators in Lebesgue spaces [2, Ch. 5]. Let $t > 0$, $f \in L_{\alpha_1}^{p_1}(\mathbb{R}^n)$, and f not identically zero. Define the dilation of function f by $\delta_t f(x) = f(tx)$. By calculating the norm $\delta_t f$ in $L_{\alpha_1}^{p_1}(\mathbb{R}^n)$ we obtain

$$\begin{aligned} \|\delta_t f\|_{L_{\alpha_1}^{p_1}} &= \left(\int_{\mathbb{R}^n} |x|^{\alpha_1 p_1} |f(tx)|^{p_1} \, dx \right)^{\frac{1}{p_1}} = t^{\alpha_1 - \frac{n}{p_1}} \left(\int_{\mathbb{R}^n} |y|^{\alpha_1 p_1} |f(y)|^{p_1} \, dy \right)^{\frac{1}{p_1}} \\ &= t^{\alpha_1 - \frac{n}{p_1}} \|f\|_{L_{\alpha_1}^{p_1}}. \end{aligned} \tag{2.1}$$

We compute the $H_\gamma^\beta f(x)$ by substitution $z = ty$

$$\begin{aligned} H_\gamma^\beta \delta_t f(x) &= \frac{1}{|x|^{n-\beta-\gamma}} \int_{|y| \le |x|} \frac{f(ty)}{|y|^\beta} \, dy \\ &= \frac{t^{n-\beta-\gamma}}{|tx|^{n-\beta-\gamma}} \int_{|z| \le |tx|} \frac{f(z)}{|t^{-1}z|^\beta} t^{-n} \, dz \\ &= t^{-\gamma} H_\gamma^\beta f(tx). \end{aligned}$$

Computing $H_\gamma^\beta \delta_t f$ in the norm $L_{\alpha_2}^{p_2}(\mathbb{R}^n)$

$$\|H_\gamma^\beta \delta_t f\|_{L_{\alpha_2}^{p_2}} = t^{-\gamma} \left(\int_{\mathbb{R}^n} |x|^{\alpha_2 p_2} |H_\gamma^\beta f(tx)|^{p_2} \, dx \right)^{\frac{1}{p_2}} = t^{-\alpha_2 - \gamma - \frac{n}{p_2}} \|H_\gamma^\beta f\|_{L_{\alpha_2}^{p_2}}. \tag{2.2}$$

By boundedness of H_γ^β from $L_{\alpha_1}^{p_1}(\mathbb{R}^n)$ to $L_{\alpha_2}^{p_2}(\mathbb{R}^n)$, (2.1), and (2.2)

$$\|H_\gamma^\beta f\|_{L_{\alpha_2}^{p_2}} \le t^{\alpha_2 + \gamma + \frac{n}{p_2}} \|H_\gamma^\beta \delta_t(f)\|_{L_{\alpha_2}^{p_2}} \lesssim t^{\alpha_2 + \gamma + \frac{n}{p_2} - \alpha_1 - \frac{n}{p_1}} \|f\|_{L_{\alpha_1}^{p_1}},$$

for every $t > 0$. If $\alpha_2 + \gamma + \dfrac{n}{p_2} - \alpha_1 - \dfrac{n}{p_1} > 0$, then $t^{\alpha_2 + \gamma + \frac{n}{p_2} - \alpha_1 - \frac{n}{p_1}} \to \infty$ when $t \to \infty$. If $\alpha_2 + \gamma + \dfrac{n}{p_2} - \alpha_1 - \dfrac{n}{p_1} < 0$, then $t^{\alpha_2 + \gamma + \frac{n}{p_2} - \alpha_1 - \frac{n}{p_1}} \to \infty$ when $t \to 0^+$. It implies

$$\alpha_2 + \gamma + \frac{n}{p_2} - \alpha_1 - \frac{n}{p_1} = 0.$$

□

The following is the necessary condition for parameter β.

Theorem 2.2. *Let $1 < p_1, p_2 < \infty$, $0 \le \gamma < \frac{n}{p_1}$, and*

$$\frac{1}{p_2} + \frac{\alpha_2}{n} = \frac{1}{p_1} + \frac{\alpha_2}{n} - \frac{\gamma}{n}.$$

If H_γ^β is bounded from $L_{\alpha_1}^{p_1}(\mathbb{R}^n)$ to $L_{\alpha_2}^{p_2}(\mathbb{R}^n)$, then $\beta + \alpha_1 < \frac{n}{p_1'}$.

Proof. Using contradiction argument, let $\beta+\alpha_1 \geq \dfrac{n}{p_1'}$. We choose $f(x) = |x|^{-\alpha_1}\chi_{\{x:|x|\leq 1\}}(x)$, where α_1, p_1 as the hyphotesis of this theorem. Note that $f \in L^{p_1}_{\alpha_1}(\mathbb{R}^n)$. If $\beta+\alpha_1$ and $1 \leq |x| < \infty$, then

$$\begin{aligned} H^\beta_\gamma f(x) &= \frac{1}{|x|^{n-\beta-\gamma}}\left(\int_{|y|\leq 1} |y|^{-\alpha_1-\beta}\,dy\right) \\ &= \frac{|\mathbb{S}^{n-1}|}{|x|^{n-\beta-\gamma}}\int_{r\leq 1} r^{-\alpha_1-\beta+n-1} = \infty. \end{aligned}$$

If $\frac{n}{p_1'} \leq \alpha_1+\beta < n$ and $1 \leq |x| < \infty$, then

$$\begin{aligned} H^\beta_\gamma f(x) &= \frac{1}{|x|^{n-\beta-\gamma}}\int_{|y|\leq|x|} |y|^{-\alpha_1-\beta}\,dy \\ &= \frac{|\mathbb{S}^{n-1}|}{|x|^{n-\beta-\gamma}}\int_{r\leq 1} r^{-\alpha_1-\beta+n-1} \\ &= \frac{|\mathbb{S}^{n-1}|}{n-\beta-\alpha_1}|x|^{-n+\gamma+\beta}. \end{aligned}$$

Next, we estimate the norm $H^\beta_\gamma f$ in $L^{p_2}_{\alpha_2}(\mathbb{R}^n)$.

$$\begin{aligned} \|H^\beta_\gamma f\|_{L^{\alpha_2}_{p_2}} &\geq \frac{|\mathbb{S}^{n-1}|}{-n+\beta+\gamma+\alpha_2}\left(\int_{|x|\geq 1} |x|^{(-n+\beta+\gamma+\alpha_2)p_2}\,dx\right)^{\frac{1}{p_2}} \\ &= \frac{|\mathbb{S}^{n-1}|}{-n+\beta+\gamma+\alpha_2}\left(\int_{r\geq 1} r^{\left(-\frac{n}{p_1'}+\beta+\alpha_1\right)p_2-1}\right)^{\frac{1}{p_2}} = \infty. \end{aligned}$$

□

3. The Sufficient Conditions and the Exact Norm

In this section, we compute the exact norm of modified Hardy operators. This is the general result for the norm of Hardy operators.

Theorem 3.1. *Let $\alpha_1 \geq \alpha_2$, $1 < p_1 \leq p_2 < \infty$, $0 \leq \gamma < \frac{n}{p_1}$, and*

$$\frac{1}{p_2}+\frac{\alpha_2}{n} = \frac{1}{p_1}+\frac{\alpha_1}{n}-\frac{\gamma}{n}.$$

If $\beta+\alpha_1 < \frac{n}{p_1'}$, then H^β_γ is bounded from $L^{p_1}_{\alpha_1}(\mathbb{R}^n)$ to $L^{p_2}_{\alpha_2}(\mathbb{R}^n)$. The exact norm of H^β_γ is

$$\|H^\beta_\gamma\|_{L^{p_1}_{\alpha_1}\to L^{p_2}_{\alpha_2}} = |\mathbb{S}^{n-1}|^{\frac{1}{p_1'}+\frac{1}{p_2}}\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}}\left(\frac{1}{\frac{n}{p_1'}-(\beta+\alpha_1)}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}}.$$

Proof. Suppose $f \in L^{p_1}_{\alpha_1}(\mathbb{R}^n)$ with parameter in the theorem. Using similar technique in [1], we choose ε with $\beta + \alpha_1 < \varepsilon < \frac{n}{p_1'}$. By Hölder's inequality, we obtain

$$
\begin{aligned}
|H^{\beta}_{\gamma} f(x)| &\le |x|^{-n+\beta+\gamma} \int_{|y|\le|x|} \frac{|f(y)|}{|y|^{\beta-\varepsilon}} |y|^{-\varepsilon}\, dy \\
&\le |x|^{-n+\beta+\gamma} \left(\int_{|y|\le|x|} \frac{|f(y)|^{p_1}}{|y|^{(\beta-\varepsilon)p_1}}\, dy \right)^{\frac{1}{p_1}} \left(\int_{|y|\le|x|} |y|^{-\varepsilon p_1'}\, dy \right)^{\frac{1}{p_1'}} \\
&= \left(\frac{|\mathbb{S}^{n-1}|}{-\varepsilon p_1' + n} \right)^{\frac{1}{p_1'}} |x|^{-\frac{n}{p_1}+\beta+\gamma-\varepsilon} \left(\int_{|y|\le|x|} \frac{|f(y)|^{p_1}}{|y|^{(\beta-\varepsilon)p_1}}\, dy \right)^{\frac{1}{p_1}} \\
&:= |\mathbb{S}^{n-1}|^{\frac{1}{p_1'}} c_1(\varepsilon) |x|^{-\frac{n}{p_1}+\beta+\gamma-\varepsilon} \left(\int_{|y|\le|x|} \frac{|f(y)|^{p_1}}{|y|^{(\beta-\varepsilon)p_1}}\, dy \right)^{\frac{1}{p_1}}. \qquad (3.1)
\end{aligned}
$$

From $\frac{1}{p_2} + \frac{\alpha_2}{n} = \frac{1}{p_1} + \frac{\alpha_1}{n} - \frac{\gamma}{n}$, Inequality (3.1) simpifies

$$
\begin{aligned}
\|H^{\beta}_{\gamma} f\|_{L^{p_2}_{\alpha_2}} &\le |\mathbb{S}^{n-1}|^{\frac{1}{p_1'}} c_1(\varepsilon) \left[\int_{\mathbb{R}^n} \left(|x|^{-\frac{n}{p_1}+\beta+\gamma-\varepsilon+\alpha_2} \left(\int_{|y|\le|x|} \frac{|f(y)|^{p_1}}{|y|^{(\beta-\varepsilon)p_1}}\, dy \right)^{\frac{1}{p_1}} \right)^{p_2} dx \right]^{\frac{1}{p_2}} \\
&\le |\mathbb{S}^{n-1}|^{\frac{1}{p_1'}} c_1(\varepsilon) \left[\int_{\mathbb{R}^n} \left(\int_{|y|\le|x|} \frac{|f(y)|^{p_1}}{|y|^{(\beta-\varepsilon)p_1}} |x|^{\left(-\frac{n}{p_1}+\beta+\gamma-\varepsilon+\alpha_2\right)p_1}\, dy \right)^{\frac{p_2}{p_1}} dx \right]^{\frac{p_1}{p_2}\cdot\frac{1}{p_1}}
\end{aligned}
$$

Since $p_1 \le p_2$ and by Minkowski's inequality

$$
\begin{aligned}
\|H^{\beta}_{\gamma} f\|_{L^{p_2}_{\alpha_2}} &\le |\mathbb{S}^{n-1}|^{\frac{1}{p_1'}} c_1(\varepsilon) \left[\int_{\mathbb{R}^n} \left(\int_{|x|\ge|y|} \frac{|f(y)|^{p_2}}{|y|^{(\beta-\varepsilon)p_2}} |x|^{\left(-\frac{n}{p_2}+\beta-\varepsilon+\alpha_1\right)p_2}\, dx \right)^{\frac{p_1}{p_2}} dy \right]^{\frac{1}{p_1}} \\
&= |\mathbb{S}^{n-1}|^{\frac{1}{p_1'}} c_1(\varepsilon) \left[\int_{\mathbb{R}^n} \frac{|f(y)|^{p_1}}{|y|^{(\beta-\varepsilon)p_1}} \left(\int_{|x|\ge|y|} |x|^{\left(-\frac{n}{p_2}+\beta-\varepsilon+\alpha_1\right)p_2}\, dx \right)^{\frac{p_1}{p_2}} dy \right]^{\frac{1}{p_1}} \\
&= |\mathbb{S}^{n-1}|^{\frac{1}{p_1'}} c_1(\varepsilon) \left(\frac{|\mathbb{S}^{n-1}|}{p_2(\varepsilon-\beta-\alpha_1)} \right)^{\frac{1}{p_2}} \|f\|_{L^{p_1}_{\alpha_1}} \\
&= |\mathbb{S}^{n-1}|^{\frac{1}{p_1'}+\frac{1}{p_2}} c_1(\varepsilon) \left(\frac{1}{p_2(\varepsilon-\beta-\alpha_1)} \right)^{\frac{1}{p_2}} \|f\|_{L^{p_1}_{\alpha_1}} \\
&:= |\mathbb{S}^{n-1}|^{\frac{1}{p_1'}+\frac{1}{p_2}} c_1(\varepsilon) c_2(\varepsilon) \|f\|_{L^{p_1}_{\alpha_1}}.
\end{aligned}
$$

Let $s(\varepsilon) = c_1(\varepsilon)c_2(\varepsilon) = \left(\frac{1}{-\varepsilon p_1'+n}\right)^{\frac{1}{p_1'}} \left(\frac{1}{p_2(\varepsilon-\beta-\alpha_1)}\right)^{\frac{1}{p_2}}$. Note that the function s is differentiable in ε-interval $\beta + \alpha_1 < \varepsilon < \frac{n}{p_1'}$, where

$$
\begin{aligned}
s'(\varepsilon) &= (-\varepsilon p_1' + n)^{-1-\frac{1}{p_1'}} (p_2(\varepsilon-\beta-\alpha_1))^{-\frac{1}{p_2}} - (p_2(\varepsilon-\beta-\alpha_1))^{-1-\frac{1}{p_2}} (-\varepsilon p_1' + n)^{-\frac{1}{p_1'}} \\
&= (-\varepsilon p_1' + n)^{-\frac{1}{p_1'}} (p_2(\varepsilon-\beta-\alpha_1))^{-\frac{1}{p_2}} \left((-\varepsilon p_1' + n)^{-1} - (p_2(\varepsilon-\beta-\alpha_1))^{-1} \right).
\end{aligned}
$$

The stationary point of s is $\varepsilon_0 = \frac{n+\beta p_2+\alpha_1 p_2}{p_1'+p_2}$. Note that

$$\beta+\alpha_1-\varepsilon_0 = \beta+\alpha_1 - \frac{n+\beta p_2+\alpha_1 p_2}{p_1'+p_2} = \frac{(\beta+\alpha_1)p_1'-n}{p_1'+p_2} < 0,$$

and

$$\frac{n}{p_1'}-\varepsilon_0 = \frac{n}{p_1'} - \frac{n+\beta p_2+\alpha_1 p_2}{p_1'+p_2} = \frac{p_2(n-\beta p_1'-\alpha_1 p_1')}{p_1'+p_2} > 0.$$

Hence, we obtain $\beta+\alpha_1<\varepsilon_0<\frac{n}{p_1'}$.

The minimum value of s is obtained at $\varepsilon_0 = \frac{n+\beta p_2+\alpha_1 p_2}{p_1'+p_2}$

$$\begin{aligned} s_{\min} &= s\left(\frac{n+\beta p_2+\alpha_1 p_2}{p_1'+p_2}\right) = \left(\frac{p_1'+p_2}{(n-(\beta+\alpha_1)p_1')p_2}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}} \\ &= \left(\frac{1}{p_1'}+\frac{1}{p_2}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}}\left(\frac{1}{\frac{n}{p_1'}-(\beta+\alpha_1)}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}}. \end{aligned}$$

Hence, we obtain an inequality in the following direction

$$\begin{aligned} \|H_\gamma^\beta\|_{L^{p_1}_{\alpha_1}\to L^{p_2}_{\alpha_2}} &= \sup_{f\in L^{p_1}_{\alpha_1}\setminus\{0\}} \frac{\|H_\gamma^\beta f\|_{L^{p_2}_{\alpha_2}}}{\|f\|_{L^{p_1}_{\alpha_1}}} \\ &\le |\mathbb{S}^{n-1}|^{\frac{1}{p_1'}+\frac{1}{p_2}}\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}}\left(\frac{1}{\frac{n}{p_1'}-(\beta+\alpha_1)}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}}. \end{aligned}$$

For the other side of in equality, let $0<\delta<\frac{n}{p_1'}-(\beta+\alpha_1)$. Write $C(p_1,p_2) = \frac{(p_1)^{\frac{1}{p_1}}(p_2)^{\frac{1}{p_2'}}}{p_2^2\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}}}$. Choose

$$f(x) = \begin{cases} |x|^{-\delta-\frac{n}{p_1}-\alpha_1} & ,|x|\ge 1 \\ 0 & ,|x|<1 \end{cases} \quad\text{and}\quad g(x) = \begin{cases} |x|^{-\frac{\delta p_2}{p_2'}-\frac{n}{p_2'}+\alpha_2} & ,|x|\ge \min\left\{1, C(p_1,p_2)^{\frac{p_2'}{\delta p_2}}\right\} \\ 0 & ,|x|<\min\left\{1, C(p_1,p_2)^{\frac{p_2'}{\delta p_2}}\right\}. \end{cases}$$

Note that

$$\|f\|_{L^{p_1}_{\alpha_1}} = \left(\frac{|\mathbb{S}^{n-1}|}{\delta p_1}\right)^{\frac{1}{p_1}} \quad\text{and}\quad \|g\|_{L^{p_2'}_{-\alpha_2}} = \left(\frac{|\mathbb{S}^{n-1}|}{\delta p_2}\right)^{\frac{1}{p_2'}} \min\{1, C(p_1,p_2)\}. \tag{3.2}$$

Assuming $|x|\ge 1$ yields

$$\begin{aligned} H_\gamma^\beta f(x) &= |x|^{-n+\beta+\gamma}\int_{1\le|y|\le|x|}|y|^{-\delta-\frac{n}{p_1}-\beta-\alpha_1}\,dy \\ &= \frac{|\mathbb{S}^{n-1}|}{\frac{n}{p_1'}-(\beta+\alpha_1)-\delta}\left(|x|^{-\delta-\frac{n}{p_2}-\alpha_2}-|x|^{-n+\beta+\gamma}\right). \end{aligned} \tag{3.3}$$

For $\min\left\{1, C(p_1,p_2)^{\frac{p_2'}{\delta p_2}}\right\} \le |x| \le 1$, we obtain $H_\gamma^\beta f(x) = 0$. Hence

$$\begin{aligned}\int_{|x|\ge 1}(H_\gamma^\beta f(x))g(x)\,dx &= \frac{|\mathbb{S}^{n-1}|^2}{\frac{n}{p_1'}-(\beta+\alpha_1)-\delta}\left(\frac{1}{\delta p_2}-\frac{1}{\delta(p_2-1)+\frac{n}{p_1'}-(\beta+\alpha_1)}\right)\\ &= \frac{|\mathbb{S}^{n-1}|^2}{\frac{n}{p_1'}-(\beta+\alpha_1)-\delta}\left(\frac{\frac{n}{p_1'}-(\beta+\alpha_1)-\delta}{(\delta p_2)\left[\delta(p_2-1)+\frac{n}{p_1'}-(\beta+\alpha_1)\right]}\right)\\ &= \frac{|\mathbb{S}^{n-1}|^2}{(\delta p_2)\left[\delta(p_2-1)+\frac{n}{p_1'}-(\beta+\alpha_1)\right]}. \end{aligned} \tag{3.4}$$

By Hölder's inequality, boundedness of H_γ^β from $L^{p_1}_{\alpha_1}(\mathbb{R}^n)$ to $L^{p_2}_{\alpha_2}(\mathbb{R}^n)$, and duality, we obtain

$$\int_{|x|\ge 1}(H_\gamma^\beta f(x))g(x)\,dx \le \|H_\gamma^\beta\|_{L^{p_1}_{\alpha_1}\to L^{p_2}_{\alpha_2}}\|f\|_{L^{p_1}_{\alpha_1}}\|g\|_{L^{p_2'}_{-\alpha_2}}.$$

From (3.2) and (3.4)

$$\begin{aligned}\|H_\gamma^\beta\|_{L^{p_1}_{\alpha_1}\to L^{p_2}_{\alpha_2}} &\ge \frac{|\mathbb{S}^{n-1}|^{\frac{1}{p_1'}+\frac{1}{p_2}}}{\min\{1, C(p_1,p_2)\}}\times\frac{(\delta p_1)^{\frac{1}{p_1}}(\delta p_2)^{\frac{1}{p_2'}}}{(\delta p_2)\left[\delta(p_2-1)+\frac{n}{p_1'}-(\beta+\alpha_1)\right]}.\\ &= |\mathbb{S}^{n-1}|^{\frac{1}{p_1'}+\frac{1}{p_2}}\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}}\frac{p_2^2}{(p_1)^{\frac{1}{p_1}}(p_2)^{\frac{1}{p_2'}}}\\ &\quad\times\frac{(\delta p_1)^{\frac{1}{p_1}}(\delta p_2)^{\frac{1}{p_2'}}}{(\delta p_2)\left[\delta(p_2-1)+\frac{n}{p_1'}-(\beta+\alpha_1)\right]}.\end{aligned}$$

As δ tends to $\frac{n}{p_1'}-(\beta+\alpha_1)$ from left, we obtain.

$$\begin{aligned}\|H_\gamma^\beta\|_{L^{p_1}_{\alpha_1}\to L^{p_2}_{\alpha_2}} &\ge \lim_{\delta\to\left(\frac{n}{p_1'}-(\beta+\alpha_1)\right)^-}|\mathbb{S}^{n-1}|^{\frac{1}{p_1'}+\frac{1}{p_2}}\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}}\frac{p_2^2}{(p_1)^{\frac{1}{p_1}}(p_2)^{\frac{1}{p_2'}}}\\ &\quad\times\frac{(\delta p_1)^{\frac{1}{p_1}}(\delta p_2)^{\frac{1}{p_2'}}}{(\delta p_2)\left[\delta(p_2-1)+\frac{n}{p_1'}-(\beta+\alpha_1)\right]}\\ &= |\mathbb{S}^{n-1}|^{\frac{1}{p_1'}+\frac{1}{p_2}}\left(\frac{1}{p_1'}+\frac{1}{p_2}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}}\left(\frac{1}{\frac{n}{p_1'}-(\beta+\alpha_1)}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}}.\end{aligned}$$

□

The following corollary is a direct consequences of duality in (1.1)

Corollary 3.1. *Let* $1 < p_1 < \infty$, $0 \le \gamma < \frac{n}{p_1}$, $p_1 \le p_2$, $\alpha_1 \ge \alpha_2$ *and* $\frac{1}{p_2} + \frac{\alpha_2}{n} = \frac{1}{p_1} + \frac{\alpha_1}{n} - \frac{\gamma}{n}$. *If* $\beta + \alpha_1 < \frac{n}{p_1'}$, *then* $\widetilde{H}_\gamma^\beta$ *is bounded from* $L_{-\alpha_2}^{p_2'}(\mathbb{R}^n)$ *to* $L_{-\alpha_1}^{p_1'}(\mathbb{R}^n)$. *The norm of* $\widetilde{H}_\gamma^\beta$ *is*

$$\|\widetilde{H}_\gamma^\beta\|_{L_{-\alpha_2}^{p_2'} \to L_{-\alpha_1}^{p_1'}} = |\mathbb{S}^{n-1}|^{\frac{1}{p_1'}+\frac{1}{p_2}} \left(\frac{1}{p_1'} + \frac{1}{p_2}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}} \left(\frac{p_1'}{n - (\beta+\alpha_1)p_1'}\right)^{\frac{1}{p_1'}+\frac{1}{p_2}}.$$

We remark that establishing the exact norm by our alternated approach is expected to become as our initial building block in establishing explicit norm for integral operators with various kernels, particularly for homogeneous kernels of degree $-n+\gamma$. Furthermore, we aim to explore the potential extension of these developments into other functional spaces. We believe that these findings hold promise for stimulating meaningful discussions among mathematicians.

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