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Characterizations of Almost (τ_1, τ_2)-Continuous Functions

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Abstract. This paper is concerned with the concept of almost (τ_1, τ_2)-continuous functions. Moreover, some characterizations of almost (τ_1, τ_2)-continuous functions are investigated.

1. Introduction

In 1968, Singal and Singal [20] introduced the concept of almost continuous functions as a generalization of continuity. Popa [19] defined almost quasi-continuous functions as a generalization of almost continuity [20] and quasi-continuity [14]. In 1981, Munshi and Bassan [16] studied the notion of almost semi-continuous functions. Maheshwari et al. [13] introduced the concept of almost feebly continuous functions as a generalization of almost continuity [20]. Noiri [18] introduced and investigated the concept of almost α-continuous functions. In 1997, Nasef and Noiri [17] introduced two classes of functions, namely almost precontinuous functions and almost β-continuous functions by utilizing the notions of preopen sets and β-open sets due to Mashhour et al. [15] and Abd El-Monsef et al. [1], respectively. The class of almost precontinuity is a generalization of each of almost feeble continuity and almost α-continuity. The class of almost β-continuity is a generalization of almost quasi-continuity and almost semi-continuity. In 2009, Keskin and Noiri [11] introduced the concept of almost b-continuous functions by utilizing the notion of b-open sets due to Andrijević [2]. The class of almost b-continuity is a generalization of almost precontinuity and almost semi-continuity. The class of almost β-continuity is a generalization of almost b-continuity. Viriyapong and Boonpok [21] introduced and studied the notion of (Λ, sp)-continuous functions. Moreover, some characterizations of almost (Λ, s)-continuous functions were presented in [3]. In [4], the authors introduced and investigated the concept of weakly (Λ, p)-continuous functions.

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In 2010, Boonpok [8] introduced and investigated the concept of (i, j)-almost M-continuous functions in bimininmal structure spaces. Duangphui et al. [9] introduced and studied the notions of $(\mu, \mu')^{(m,n)}$-continuous functions, almost $(\mu, \mu')^{(m,n)}$-continuous functions and weakly $(\mu, \mu')^{(m,n)}$-continuous functions in bigeneralized topological spaces. Furthermore, several characterizations of $g_{(m,n)}$-continuous functions were established in [10]. In [5], the authors investigated several characterizations of almost weakly (τ_1, τ_2)-continuous multifunctions. Laprom et al. [12] introduced and studied the concept of almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. In this paper, we introduce the concept of almost (τ_1, τ_2)-continuous functions. In particular, some characterizations of almost (τ_1, τ_2)-continuous functions are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [7] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [7] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [7] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [22] (resp. $(\tau_1, \tau_2)s$-*open* [6], $(\tau_1, \tau_2)p$-*open* [6], $(\tau_1, \tau_2)\beta$-*open* [6]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))))$. The complement of a $(\tau_1, \tau_2)r$-open (resp. $(\tau_1, \tau_2)s$-open, $(\tau_1, \tau_2)p$-open, $(\tau_1, \tau_2)\beta$-open) set is called $(\tau_1, \tau_2)r$-*closed*, $(\tau_1, \tau_2)s$-*closed*, $(\tau_1, \tau_2)p$-*closed*, $(\tau_1, \tau_2)\beta$-*closed*.

3. Characterizations of almost (τ_1, τ_2)-continuous functions

In this section, we introduce the notion of almost (τ_1, τ_2)-continuous functions. Moreover, several characterizations of almost (τ_1, τ_2)-continuous functions are discussed.

Definition 3.1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be almost (τ_1, τ_2)-continuous if f has this property at each point of X.*

Theorem 3.1. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is almost (τ_1, τ_2)-continuous at $x \in X$;*
(2) *$x \in \tau_1\tau_2$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$;*
(3) *$x \in \tau_1\tau_2$-Int$(f^{-1}(V))$ for every $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$;*
(4) *for each $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $x \in U \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ and hence $x \in \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$.

(2) $\Rightarrow$ (3): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. Then, V is $\sigma_1\sigma_2$-open in Y and by (2),

$$x \in \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) = \tau_1\tau_2\text{-Int}(f^{-1}(V)).$$

(3) $\Rightarrow$ (4): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. Then by (3), $x \in \tau_1\tau_2\text{-Int}(f^{-1}(V))$ and there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq f^{-1}(V)$. Thus, $f(U) \subseteq V$.

(4) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $(\sigma_1, \sigma_2)r$-open in Y and by (4), there exists a $\tau_1\tau_2$-open set U of X containing x such that

$$f(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)).$$

This shows that f is almost (τ_1, τ_2)-continuous at x. □

Theorem 3.2. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is almost (τ_1, τ_2)-continuous;*
(2) *$f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(V))))$ for every $\sigma_1\sigma_2$-open set V of Y;*
(3) *$\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(K)))) \subseteq f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y;*
(4) *$\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(B))))) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;*
(5) *$f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(\sigma_1\sigma_2\text{-}Int(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Int(B)))))$ for every subset B of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. There exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$. Thus, $x \in \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$ and hence $f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$.

(2) $\Rightarrow$ (3): Let K be any $\sigma_1\sigma_2$-closed set of Y. By (2), we have

$$\begin{aligned} X - f^{-1}(K) &= f^{-1}(Y - K) \\ &\subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - K)))) \\ &= \tau_1\tau_2\text{-Int}(f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \\ &= X - \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \end{aligned}$$

and hence $\tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K)))) \subseteq f^{-1}(K)$.

(3) $\Rightarrow$ (4): The proof is obvious.

(4) $\Rightarrow$ (5): Let B be any subset of Y. By (4),

$$\begin{aligned} f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) &= X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &\subseteq X - \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(Y - B))))) \\ &= \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(B))))). \end{aligned}$$

(5) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then by (5),

$$x \in f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))).$$

By Theorem 3.1 (2), f is almost (τ_1, τ_2)-continuous at x. This shows that f is almost (τ_1, τ_2)-continuous. □

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*δ-open* if A is the union of $(\tau_1, \tau_2)r$-open sets of X. The complement of a $\tau_1\tau_2$-δ-open set is called $\tau_1\tau_2$-*δ-closed*. The union of all $\tau_1\tau_2$-δ-open sets of X contained in A is called the $\tau_1\tau_2$-*δ-interior* of A and is denoted by $\tau_1\tau_2$-δ-Int(A). The intersection of all $\tau_1\tau_2$-δ-closed sets of X containing A is called the $\tau_1\tau_2$-δ-*closure* of A and is denoted by $\tau_1\tau_2$-δ-Cl(A).

Theorem 3.3. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is almost (τ_1, τ_2)-continuous;*
(2) *$f^{-1}(V)$ is $\tau_1\tau_2$-open in X for every $(\sigma_1, \sigma_2)r$-open set V of Y;*
(3) *$f^{-1}(K)$ is $\tau_1\tau_2$-closed in X for every $(\sigma_1, \sigma_2)r$-closed set K of Y;*
(4) *$f(\tau_1\tau_2\text{-}Cl(A)) \subseteq \sigma_1\sigma_2\text{-}\delta\text{-}Cl(f(A))$ for every subset A of X;*
(5) *$\tau_1\tau_2\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}\delta\text{-}Cl(B))$ for every subset B of Y;*
(6) *$f^{-1}(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-δ-closed set K of Y;*
(7) *$f^{-1}(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-δ-open set V of Y.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, we have $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(V))) = V$ and by Theorem 3.2 (5),

$$\begin{aligned} f^{-1}(V) &= f^{-1}(\sigma_1\sigma_2\text{-Int}(V)) \\ &\subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(V))))) \\ &= \tau_1\tau_2\text{-Int}(f^{-1}(V)). \end{aligned}$$

Thus, $f^{-1}(V)$ is $\tau_1\tau_2$-open in X.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (4): Let A be any subset of Y and K be any $\sigma_1\sigma_2$-δ-closed set of Y containing $f(A)$. Observe that $K = \sigma_1\sigma_2\text{-}\delta\text{-Cl}(K) = \cap\{F \mid K \subseteq F$ and F is $(\sigma_1, \sigma_2)r$-closed$\}$ and so

$$f^{-1}(K) = \cap\{f^{-1}(F) \mid K \subseteq F \text{ and } F \text{ is } (\sigma_1, \sigma_2)r\text{-closed}\}.$$

Now, by (3), we have $f^{-1}(K)$ is $\tau_1\tau_2$-closed and $A \subseteq f^{-1}(K)$. Thus, $\tau_1\tau_2\text{-Cl}(A) \subseteq f^{-1}(K)$ and hence $f(\tau_1\tau_2\text{-Cl}(A)) \subseteq K$. Since this is true for any $\sigma_1\sigma_2$-δ-closed set K containing $f(A)$, we have

$$f(\tau_1\tau_2\text{-Cl}(A)) \subseteq \sigma_1\sigma_2\text{-}\delta\text{-Cl}(f(A)).$$

(4) ⇒ (5): Let B be any subset of Y. By (4),

$$\begin{aligned} f(\tau_1\tau_2\text{-Cl}(f^{-1}(B))) &\subseteq \sigma_1\sigma_2\text{-}\delta\text{-Cl}(f(f^{-1}(B))) \\ &\subseteq \sigma_1\sigma_2\text{-}\delta\text{-Cl}(B). \end{aligned}$$

Thus, $\tau_1\tau_2$-Cl$(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2$-$\delta$-Cl$(B))$.

(5) ⇒ (6): Let K be any $\sigma_1\sigma_2$-δ-closed set of Y. Using (5), we have

$$\tau_1\tau_2\text{-Cl}(f^{-1}(K)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}\delta\text{-Cl}(K)) = f^{-1}(K)$$

and hence $f^{-1}(K)$ is $\tau_1\tau_2$-closed in X.

(6) ⇒ (7): This is obvious.

(7) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Since $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ is $(\sigma_1, \sigma_2)r$-open in Y, we have $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$ is $\sigma_1\sigma_2$-δ-open and by (7), $f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$ is $\tau_1\tau_2$-open in X. Put $U = f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$. Then, U is a $\tau_1\tau_2$-open set of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. Thus, f is almost (τ_1, τ_2)-continuous at x and hence f is almost (τ_1, τ_2)-continuous. □

Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $(\tau_1, \tau_2)s$-closed sets of X containing A is called the *$(\tau_1, \tau_2)s$-closure* [6] of A and is denoted by (τ_1, τ_2)-sCl(A). The union of all $(\tau_1, \tau_2)s$-open sets of X contained in A is called the *$(\tau_1, \tau_2)s$-interior* [6] of A and is denoted by (τ_1, τ_2)-sInt(A).

Lemma 3.1. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties of hold:*

(1) *(τ_1, τ_2)-sCl$(A) = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)) \cup A$ [6].*
(2) *If A is $\tau_1\tau_2$-open in X, then (τ_1, τ_2)-sCl$(A) = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ [6].*
(3) *If A is $(\tau_1, \tau_2)p$-open in X, then (τ_1, τ_2)-sCl$(A) = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$.*

Theorem 3.4. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is almost (τ_1, τ_2)-continuous;*
(2) *for each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$;*
(3) *for each $x \in X$ and each $(\sigma_1, \sigma_2)r$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$;*
(4) *for each $x \in X$ and each $\sigma_1\sigma_2$-δ-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$.*

Proof. (1) ⇒ (2): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (1), there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))$. Since V is $(\sigma_1, \sigma_2)p$-open, by Lemma 3.1, $f(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$.

(2) ⇒ (3): Let $x \in X$ and V be any $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$. Then, V is $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq (\sigma_1, \sigma_2)$-sCl$(V)$. Since V is $\sigma_1\sigma_2$-open, by Lemma 3.1, $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)) = V$.

(3) ⇒ (4): Let $x \in X$ and V be any $\sigma_1\sigma_2$-δ-open set of Y containing $f(x)$. Then, there exists a $\sigma_1\sigma_2$-open set W of Y containing $f(x)$ such that $W \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(W)) \subseteq V$. Since $\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(W))$ is a $(\sigma_1, \sigma_2)r$-open set of Y containing $f(x)$, by (3), there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(W)) \subseteq V$.

(4) ⇒ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, $\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl(V)) is a $\sigma_1\sigma_2$-δ-open set V of Y containing $f(x)$. By (4), there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)). Thus, f is almost (τ_1, τ_2)-continuous at x. This shows that f is almost (τ_1, τ_2)-continuous. □

Theorem 3.5. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is almost (τ_1, τ_2)-continuous;*
(2) *$f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y;*
(3) *$f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(K)))$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-closed set K of Y.*

Proof. (1) ⇒ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$. Then, we have $f(x) \in \sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)) and $\sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl(V)) is $(\sigma_1, \sigma_2)r$-open in Y. Since f is almost (τ_1, τ_2)-continuous, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq \sigma_1\sigma_2$-Int($\sigma_1\sigma_2$-Cl($V$)). Thus, $x \in U \subseteq f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$ and hence $x \in \tau_1\tau_2$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$. This shows that $f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))) \subseteq \tau_1\tau_2$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$. Therefore, $f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V)))$ is $\tau_1\tau_2$-open in X.

(2) ⇒ (3): The proof is obvious.

(3) ⇒ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, K is a $\sigma_1\sigma_2$-closed set of Y. By (3), $f^{-1}(\sigma_1\sigma_2$-Cl$(\sigma_1\sigma_2$-Int$(K)))$ is $\tau_1\tau_2$-closed in X. Since K is $(\sigma_1, \sigma_2)r$-closed, we have

$$f^{-1}(\sigma_1\sigma_2\text{-Cl}(\sigma_1\sigma_2\text{-Int}(K))) = f^{-1}(K)$$

and hence $f^{-1}(K)$ is $\tau_1\tau_2$-closed in X. Thus, by Theorem 3.3, f is almost (τ_1, τ_2)-continuous. □

Theorem 3.6. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is almost (τ_1, τ_2)-continuous;*
(2) *$\tau_1\tau_2$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)\beta$-open set V of Y;*
(3) *$f^{-1}(\sigma_1\sigma_2$-Int$(K)) \subseteq \tau_1\tau_2$-Int$(f^{-1}(K))$ for every $(\sigma_1, \sigma_2)\beta$-closed set K of Y;*
(4) *$f^{-1}(\sigma_1\sigma_2$-Int$(K)) \subseteq \tau_1\tau_2$-Int$(f^{-1}(K))$ for every $(\sigma_1, \sigma_2)s$-closed set K of Y;*
(5) *$\tau_1\tau_2$-Cl$(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(V))$ for every $(\sigma_1, \sigma_2)s$-open set V of Y;*
(6) *$f^{-1}(V) \subseteq \tau_1\tau_2$-Int$(f^{-1}(\sigma_1\sigma_2$-Int$(\sigma_1\sigma_2$-Cl$(V))))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y.*

Proof. (1) ⇒ (2): Let V be any $(\sigma_1, \sigma_2)\beta$-open set of Y. Then, $\sigma_1\sigma_2$-Cl(V) is $(\sigma_1, \sigma_2)r$-closed, by Theorem 3.3 (3), $\tau_1\tau_2$-Cl$(f^{-1}(\sigma_1\sigma_2$-Cl$(V))) = f^{-1}(\sigma_1\sigma_2$-Cl$(V))$. Thus,

$$\tau_1\tau_2\text{-Cl}(f^{-1}(V)) \subseteq \tau_1\tau_2\text{-Cl}(f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))) = f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)).$$

(2) ⇒ (3): Let K be any $(\sigma_1, \sigma_2)\beta$-closed set of Y. Then $Y - K$ is $(\sigma_1, \sigma_2)\beta$-open in Y. By (2), $\tau_1\tau_2$-Cl$(f^{-1}(Y - K)) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(Y - K))$ and $\tau_1\tau_2$-Cl$(X - f^{-1}(K)) \subseteq f^{-1}(Y - \sigma_1\sigma_2$-Cl$(K))$. Thus, $X - \tau_1\tau_2$-Int$(f^{-1}(K)) \subseteq X - f^{-1}(\sigma_1\sigma_2$-Int$(K))$ and hence $f^{-1}(\sigma_1\sigma_2$-Int$(K)) \subseteq \tau_1\tau_2$-Int$(f^{-1}(K))$.

(3) ⇒ (4): It is obvious since every $(\sigma_1, \sigma_2)s$-open set is $(\sigma_1, \sigma_2)\beta$-open.

(4) $\Rightarrow$ (5): Let Let V be any $(\sigma_1, \sigma_2)s$-open set of Y. Then, $Y - V$ is $(\sigma_1, \sigma_2)s$-closed in Y. By (4), $f^{-1}(\sigma_1\sigma_2\text{-Int}(Y - V)) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(Y - V))$ and $f^{-1}(Y - \sigma_1\sigma_2\text{-Cl}(V)) \subseteq \tau_1\tau_2\text{-Int}(X - f^{-1}(V))$. Thus, $X - f^{-1}(\sigma_1\sigma_2\text{-Cl}(V)) \subseteq X - \tau_1\tau_2\text{-Cl}(f^{-1}(V))$ and hence $\tau_1\tau_2\text{-Cl}(f^{-1}(V)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(V))$.

(5) $\Rightarrow$ (1): Let K be any $(\sigma_1, \sigma_2)r$-closed set of Y. Then, K is $(\sigma_1, \sigma_2)s$-open in Y and by (5),

$$\tau_1\tau_2\text{-Cl}(f^{-1}(K)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(K)) = f^{-1}(K).$$

Thus, $f^{-1}(K)$ is $\tau_1\tau_2$-closed in X and hence f is almost (τ_1, τ_2)-continuous by Theorem 3.3 (3).

(1) $\Rightarrow$ (6): Let V be any $(\sigma_1, \sigma_2)p$-open set of Y. Then, we have $V \subseteq \sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ and $\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))$ is $(\sigma_1, \sigma_2)r$-open. By Theorem 3.3 (2), $f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))$ is $\tau_1\tau_2$-open in X. Thus, $f^{-1}(V) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))) = \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V))))$.

(6) $\Rightarrow$ (1): Let V be any $(\sigma_1, \sigma_2)r$-open set of Y. Then, V is $(\sigma_1, \sigma_2)p$-open and by (6),

$$f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(\sigma_1\sigma_2\text{-Int}(\sigma_1\sigma_2\text{-Cl}(V)))) = \tau_1\tau_2\text{-Int}(f^{-1}(V)).$$

Thus, $f^{-1}(V)$ is $\tau_1\tau_2$-open in X. It follows from Theorem 3.3 (2) that f is almost (τ_1, τ_2)-continuous. □

Corollary 3.1. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is almost (τ_1, τ_2)-continuous;*

(2) *$f^{-1}(V) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}((\sigma_1, \sigma_2)\text{-}sCl(V)))$ for every $(\sigma_1, \sigma_2)p$-open set V of Y;*

(3) *$\tau_1\tau_2\text{-}Cl(f^{-1}(\sigma_1\sigma_2\text{-}Cl(\sigma_1\sigma_2\text{-}Cl(K)))) \subseteq f^{-1}(K)$ for every $(\sigma_1, \sigma_2)p$-closed set K of Y;*

(4) *$\tau_1\tau_2\text{-}Cl(f^{-1}((\sigma_1, \sigma_2)\text{-}sInt(K))) \subseteq f^{-1}(K)$ for every $(\sigma_1, \sigma_2)p$-closed set K of Y.*

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