PROPERTIES OF WEIGHTED COMPOSITION OPERATORS ON SOME WEIGHTED HOLOMORPHIC FUNCTION CLASSES IN THE UNIT BALL

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ABSTRACT. In this paper, we introduce \( N_K \)-type spaces of holomorphic functions in the unit ball of \( \mathbb{C}^n \) by the help of a non-decreasing function \( K : (0, \infty) \to [0, \infty) \). Several important properties of these spaces in the unit ball are provided. The results are applied to characterize boundedness and compactness of weighted composition operators \( W_{a,\phi} \) from \( N_K(\mathbb{B}) \) spaces into Beurling-type classes. We also find the essential norm estimates for \( W_{a,\phi} \) from \( N_K(\mathbb{B}) \) spaces into Beurling-type classes.

1. Introduction

Through this paper, \( \mathbb{B} \) is the unit ball of the \( n \)-dimensional complex Euclidean space \( \mathbb{C}^n \), \( S \) is the boundary of \( \mathbb{B} \). We denote the class of all holomorphic functions, with the compact-open topology on the unit ball \( \mathbb{B} \) by \( \mathcal{H}(\mathbb{B}) \). For any \( z = (z_1, z_2, \ldots, z_n), w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n \), the inner product is defined by \( \langle z, w \rangle = z_1 \overline{w}_1 + \ldots + z_n \overline{w}_n \), and write \( |z| = \sqrt{\langle z, z \rangle} \).

Two quantities \( A_f \) and \( B_f \), both depending on a function \( f \in \mathcal{H}(\mathbb{B}) \), are said to be equivalent, written as \( A_f \approx B_f \), if there exists a finite positive constant \( M \) not depending on \( f \), such that

\[
\frac{1}{M} B_f \leq A_f \leq MB_f
\]

for every \( f \in \mathcal{H}(\mathbb{B}) \). If the quantities \( A_f \) and \( B_f \) are equivalent, then in particular we have \( A_f < \infty \) if and only if \( B_f < \infty \). As usual, the letter \( M \) will denote a positive constant, possibly different on each occurrence.

Given a point \( a \in \mathbb{B} \), we can associate with it the following automorphism \( \Phi_a(z) \in Aut(\mathbb{B}) \):

\[
\Phi_a(z) = \frac{a - P_a(z) - S_a Q_a(z)}{1 - \langle z, a \rangle}, \quad z \in \mathbb{B},
\]

where \( S_a = \sqrt{1 - |a|^2} P_a(z) \) is the orthogonal projection of \( \mathbb{C}^n \) on a subspace \([a]\) generated by \( a \), that is

\[
P_a(z) = \begin{cases} 
0, & \text{if } a = 0; \\
\frac{a(z,a)}{|a|^2}, & \text{if } a \neq 0,
\end{cases}
\]

and \( Q_a = I - P_a \) the projection on orthogonal complement \([a]\) (see, for example,[8] or [10]). The map \( \Phi_a \) has the following properties that \( \Phi_a(0) = a, \Phi_a(a) = 0, \Phi_a = \Phi_a^{-1} \) and

\[
1 - \langle \Phi_a(z), \Phi_a(w) \rangle = \frac{(1 - |a|^2)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)},
\]

where \( z \) and \( w \) are arbitrary points in \( \mathbb{B} \). In particular,

\[
1 - |\Phi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a \rangle|^2}.
\]

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Let $V$ be the Lebesgue volume measure on $\mathbb{C}^n$, normalized so that $V(\mathbb{B}) \equiv 1$ and $\sigma$ be the normalized surface measure on $\mathbb{S}$, so that $\sigma(\mathbb{B}) \equiv 1$. Let

$$d\tau(z) = \frac{dV(z)}{(1 - |z|^2)^{n+1}},$$

which is Möbius invariant, that is for any $\psi \in Aut(\mathbb{B})$, $f \in L^1(\mathbb{B})$, we have

$$\int_{\mathbb{B}} f(z)d\tau(z) = \int_{\mathbb{B}} f \circ \psi(z)d\tau(z).$$

For $a \in \mathbb{B}$, the Möbius invariant Green function in $\mathbb{B}$ denoted by $G(z, a) = g(\Phi_a(z))$ where $g(z)$ is defined by:

$$g(z) = \frac{n + 1}{2n} \int_{|z|}^1 (1 - t^2)^{n-1}t^{1/2}dt.$$

Let $H^\infty(\mathbb{B})$ denote the Banach space of bounded functions in $\mathcal{H}(\mathbb{B})$ with the norm

$$\|f\|_\infty = \sup_{z \in \mathbb{B}} |f(z)|.$$

For $\alpha > 0$, the Beurling-type space (sometimes also called the Bers-type space) $H^\infty_\alpha(\mathbb{B})$ in the unit ball $\mathbb{B}$ consists of those functions $f \in \mathcal{H}(\mathbb{B})$ for which

$$\|f\|_{H^\infty_\alpha(\mathbb{B})} = \sup_{z \in \mathbb{B}} |f(z)|(1 - |z|)^\alpha < \infty.$$

The Bergman space $A^2(\mathbb{B})$ consists of those functions $f \in \mathcal{H}(\mathbb{B})$ for which

$$\|f\|_{A^2(\mathbb{B})}^2 = \int_{\mathbb{B}} |f(z)|^2dV(z) < \infty.$$

Let $K : (0, \infty) \to [0, \infty)$ be a right-continuous, non-decreasing function and is not equal to zero identically. The $\mathcal{N}_K(\mathbb{B})$ space consists of all functions $f \in \mathcal{H}(\mathbb{B})$ such that

$$\|f\|_{\mathcal{N}_K(\mathbb{B})}^2 = \sup_{a \in \mathbb{B}} \int_{\mathbb{B}} |f(z)|^2K(G(z, a))d\tau(z) < \infty.$$

Moreover, $f \in \mathcal{H}(\mathbb{B})$ is said to belong to $\mathcal{N}_{K,0}(\mathbb{B})$ if

$$\lim_{|a| \to 1} \int_{\mathbb{B}} |f(z)|^2K(G(z, a))d\tau(z) = 0.$$

Clearly, if $K(t) = t^p$, then $\mathcal{N}_K(\mathbb{B}) = \mathcal{N}_p(\mathbb{B})$; since $G(z, a) \approx (1 - |\varphi_a(z)|^2)$ (see [6]). For $K(t) = 1$ it gives the Bergman space $A^2(\mathbb{B})$. If $\mathcal{N}_K(\mathbb{B})$ consists of just the constant functions, we say that it is trivial. Several important properties of the $\mathcal{N}_K(\mathbb{B})$ spaces in the unit disk in the complex plane have been characterized in [1], [4] and [9]. We assume from now that all $K : (0, \infty) \to [0, \infty)$ to appear in this paper are right-continuous, non-decreasing function and not equal to zero identically.

Given $u \in \mathcal{H}(\mathbb{B})$ and $\phi$ a holomorphic self-map of $\mathbb{B}$. The weighted composition operator $W_{u, \phi} : \mathcal{H}(\mathbb{B}) \to \mathcal{H}(\mathbb{B})$ is defined by

$$W_{u, \phi}(f)(z) = u(z)(f \circ \phi)(z), \quad z \in \mathbb{B}.$$
Lemma 1.1. For \( z, w \in \mathbb{B} \), if \( \rho(z, w) \leq \frac{1}{2} \), Then
\[
(1.6) \quad \frac{1}{6} \leq \frac{1 - |z|^2}{1 - |w|^2} \leq 6.
\]

The following proposition was proved as part of Lemma 2.3 in [6], and hence, we omit the details.

Proposition 1.1. For \( z, w \in \mathbb{B} \), if \( \rho(z, w) \leq \frac{1}{2} \), Then
\[
|f(z) - f(w)| \leq 2\sqrt{n}|f(\Phi_n(z))|\rho(z, w).
\]

Recall that a linear operator \( T : X \rightarrow Y \) is said to be bounded if there exists a constant \( M > 0 \) such that \( \|T(f)\|_Y \leq M\|f\|_X \) for all maps \( f \in X \). Moreover, \( T : X \rightarrow Y \) is said to be compact if it takes bounded sets in \( X \) to sets in \( Y \) which have compact closure. For a Complex Banach spaces \( X \) and \( Y \) of \( \mathcal{H}(\mathbb{B}) \), \( T : X \rightarrow Y \) is compact (respectively weakly compact) if it maps the closed unit ball of \( X \) onto a relatively compact (respectively relatively weakly compact) set in \( Y \).

In this paper, we introduce \( \mathcal{N}_K(\mathbb{B}) \) spaces, in terms of the right continuous and non-decreasing function \( K : (0, \infty) \rightarrow (0, \infty) \) on the unit ball \( \mathbb{B} \). We prove that \( \mathcal{N}_K(\mathbb{B}) \) contained in Beurling-type space \( H^{\infty}_{B^\alpha}(\mathbb{B}) \), \( \alpha = \frac{n+1}{2} \). A sufficient and necessary condition for \( \mathcal{N}_K(\mathbb{B}) \) non-trivial is given. We discuss the nesting property of \( \mathcal{N}_K(\mathbb{B}) \). We obtain the complete characterizations of the boundedness and compactness of weighted composition operators from \( \mathcal{N}_K(\mathbb{B}) \) spaces into Beurling-type classes. We also find the essential norm estimates for these operators. Our results contain the results in the unit disk as particular cases (for example [4], [6] and [9]).

2. \( \mathcal{N}_K(\mathbb{B}) \) spaces in the unit ball

The following results play an important role in the proof of our main result. They also have their own interest.

Proposition 2.1. Let \( K : (0, \infty) \rightarrow [0, \infty) \) be non-decreasing function. Then \( \mathcal{N}_K(\mathbb{B}) \subset H^{\infty}_{B^{\alpha/n+1}}(\mathbb{B}) \).

Proof. For \( a \in \mathbb{B} \), let \( \mathbb{B}_{\frac{1}{2}} = \{ z \in \mathbb{B} : |z| < \frac{1}{2} \} \). Without loss of generality, assume that \( K(\frac{1}{4}) > 0 \). If \( f \in \mathcal{N}_K(\mathbb{B}) \), then
\[
\|f\|^2_{\mathcal{N}_K(\mathbb{B})} \geq K(3/4) \int_{\mathbb{B}_{\frac{1}{2}}} |f(z)|^2\,d\tau(z).
\]

By the subharmonicity of \( |f(z)|^2 \) and hence by ([7], Theorem 2.1.4), we have
\[
|f(0)|^2 \leq \frac{1}{V(\mathbb{B}_{\frac{1}{2}})} \int_{\mathbb{B}_{\frac{1}{2}}} |f(z)|^2\,d\tau(z) = 4^n \int_{\mathbb{B}_{\frac{1}{2}}} |f(z)|^2\,d\tau(z).
\]

Thus
\[
|f(0)|^2 \leq \frac{4^n}{K(3/4)} \|f\|^2_{\mathcal{N}_K(\mathbb{B})}, \quad f \in \mathcal{N}_K(\mathbb{B}).
\]

For every fixed \( z \in \mathbb{B} \), we put
\[
(2.1) \quad F(w) = \frac{(1 - |z|^2)^{\frac{n+1}{4}} |f(\Phi_n(w))|}{(1 - \langle w, z \rangle)^{n+1}}, \quad w \in \mathbb{B},
\]

which is clearly \( F \in \mathcal{H}(\mathbb{B}) \).

We can prove that \( \|F\|^2_{\mathcal{N}_K(\mathbb{B})} \leq \|f\|^2_{\mathcal{N}_K(\mathbb{B})} \), and so \( F \in \mathcal{N}_K(\mathbb{B}) \). Then, we have
\[
|f(a)|^2(1 - |a|^2)^{n+1} = |F(0)|^2 \leq \frac{4^n}{K(3/4)} \|f\|^2_{\mathcal{N}_K(\mathbb{B})},
\]

for all \( z \in \mathbb{B} \), which implies that:
\[
\|f\|^2_{H^{\infty}_{B^{\alpha/n+1}}(\mathbb{B})} = \sup_{a \in \mathbb{B}} |f(a)|^2(1 - |a|^2)^{n+1} \leq \frac{4^n}{K(3/4)} \|f\|^2_{\mathcal{N}_K(\mathbb{B})},
\]

That is, \( \mathcal{N}_K(\mathbb{B}) \subset H^{\infty}_{B^{\alpha/n+1}}(\mathbb{B}) \).
Proposition 2.2. For \( z, w \in \mathbb{B} \) and \( f \in \mathscr{N}_K(\mathbb{B}) \), we have
\[
|f(z) - f(w)| \leq M \|f\|_{\mathscr{N}_K(\mathbb{B})} \max\{(1 - |z|^2)^{-\frac{n+1}{4}}, (1 - |w|^2)^{-\frac{n+1}{4}}\} \rho(z, w).
\]
Here, \( M = \frac{6^{\frac{n+1}{2}} 2^{(n+1)} \sqrt{n(3+2\sqrt{3})}}{K(3/4)^4} \).

Proof. We consider two cases:

Case 1: \( \rho(z, w) \geq \frac{1}{4} \).
Since \( |f(z) - f(w)| \leq |f(z)| + |f(w)| \), by Proposition 2.1, we have
\[
\min\{(1 - |z|^2)^{-\frac{n+1}{4}}, (1 - |w|^2)^{-\frac{n+1}{4}}\}|f(z) - f(w)|
\leq (1 - |z|^2)^{-\frac{n+1}{4}} |f(z)| + (1 - |w|^2)^{-\frac{n+1}{4}} |f(w)|
\leq 2 \|f\|_{H_{\frac{n+1}{2}}(\mathbb{B})} \leq \frac{2^{2n+1} \|f\|_{\mathscr{N}_K(\mathbb{B})}}{K(3/4)}
\leq \frac{2^{2n+3} \rho(z, w)}{K(3/4)} \|f\|_{\mathscr{N}_K(\mathbb{B})}.
\]
Which implies that
\[
|f(z) - f(w)| \leq \frac{2^{2n+3} \|f\|_{\mathscr{N}_K(\mathbb{B})}}{K(3/4)} \max\{(1 - |z|^2)^{-\frac{n+1}{4}}, (1 - |w|^2)^{-\frac{n+1}{4}}\} \rho(z, w).
\]

Case 2: \( \rho(z, w) > \frac{1}{4} \).
Take and fix \( w \in \mathbb{B} \), from \( \rho(\Phi_w(z), w) = |z| \) it follows that if \( z \in \mathbb{B}_2 \setminus \mathbb{B} \), then \( \rho(\Phi_w(z), w) < \frac{1}{2} \).
In this case, by Proposition 2.1 and Lemma 1.1, we have
\[
|f(\Phi_w(z))| \leq \frac{\|f\|_{H_{\frac{n+1}{2}}(\mathbb{B})}}{(1 - |\Phi_w(z)|^2)^{\frac{n+1}{4}}}
\leq \frac{2^{2n+1} \|f\|_{\mathscr{N}_K(\mathbb{B})}}{K(3/4)} \frac{1}{(1 - |\Phi_w(z)|^2)^{\frac{n+1}{4}}}
= \frac{2^{2n+1} \|f\|_{\mathscr{N}_K(\mathbb{B})}}{K(3/4)} \frac{1}{(1 - |w|^2)^{\frac{n+1}{4}}}
\leq \frac{6^{\frac{n+1}{2}} 2^{2n+1} \sqrt{n(3+2\sqrt{3})}}{K(3/4)} \frac{\|f\|_{\mathscr{N}_K(\mathbb{B})}}{(1 - |w|^2)^{\frac{n+1}{4}}} \rho(z, w).
\]
By Proposition 1.1, we have
\[
|f(z) - f(w)| \leq \frac{6^{\frac{n+1}{2}} 2^{2(n+1)} \sqrt{n(3+2\sqrt{3})}}{K(3/4)} \frac{\|f\|_{\mathscr{N}_K(\mathbb{B})}}{(1 - |w|^2)^{\frac{n+1}{4}}} \rho(z, w).
\]
Combining the results of the two cases yields
\[
|f(z) - f(w)| \leq M \|f\|_{\mathscr{N}_K(\mathbb{B})} \max\{(1 - |z|^2)^{-\frac{n+1}{4}}, (1 - |w|^2)^{-\frac{n+1}{4}}\} \rho(z, w),
\]
where \( M = \frac{6^{\frac{n+1}{2}} 2^{2(n+1)} \sqrt{n(3+2\sqrt{3})}}{K(3/4)^4} \).

Lemma 2.1. For \( a \in \mathbb{B}, 0 < \delta < 1 \) and \( f \in \mathscr{N}_K(\mathbb{B}) \), we have
\[
|f(a) - f(\delta a)| \leq \frac{M}{(1 - |\delta a|^2)^{\frac{n+1}{4}}} \|f\|_{\mathscr{N}_K(\mathbb{B})}.
\]
Consequently, for any \( 0 < r < 1 \), we have
\[
\sup_{|a| \leq r} |f(a) - f(\delta a)| \leq \frac{M}{(1 - r^2)^{\frac{n+1}{4}}} \|f\|_{\mathscr{N}_K(\mathbb{B})}.
\]
Here, \( M \) is the constant from Proposition 2.2.
Proof. Proposition 2.2 shows that
\[ |f(a) - f(\delta a)| \leq M\|f\|_{\mathcal{N}_K(B)} \max\{(1 - |a|^2)^{-\frac{n+1}{2}}, (1 - |\delta a|^2)^{-\frac{n+1}{2}}\} \rho(a, \delta a). \]
The well-known formula
\[ 1 - |\rho(a, \delta a)|^2 = 1 - |\Phi_a(\delta a)|^2 = \frac{(1 - |a|^2)(1 - |\delta a|^2)}{1 - \langle a, \delta a \rangle^2}, \]
together with simple calculations gives
\[ \rho(a, \delta a) = \frac{(1 - |a|^2)|a|}{1 - |\delta a|^2} \leq 1. \]

On the other hand, \((1 - |\delta a|^2)^{-\frac{n+1}{2}} \leq (1 - |a|^2)^{-\frac{n+1}{2}}\). The inequalities in (2.3) now follow.

If \(|a| \leq r\), then \(1 - |\delta a|^2 \geq 1 - r^2\). Taking supremum of (2.3) in \(a\) yields (2.4).

**Theorem 2.1.** If
\[ (2.5) \int_0^1 \frac{r^{2n-1}}{(1 - r^2)^{n+1}} K(g(r)) dr < \infty, \]
then \(\mathcal{N}_K(B)\) contains all polynomials; otherwise, \(\mathcal{N}_K(B)\) contains only constant functions.

**Proof.** First assume that (2.5) holds. Let \(f(z)\) be a polynomial i.e. (there exists a \(M > 0\) such that \(|f(z)|^2 \leq M\) for all \(z \in \mathbb{B}\). Then,
\[ \int_\mathbb{B} |f(z)|^2 K(G(z, a)) d\tau(z) = \int_\mathbb{B} |f(\Phi_a(z))|^2 K(g(z)) \frac{dV(z)}{(1 - |z|^2)^{n+1}} = 2n \int_0^1 \frac{r^{2n-1}}{1 - r^2} K(g(r)) dr \int_\mathbb{B} |f \circ \varphi_a(\zeta)|^2 d\sigma(\zeta) \leq 2nM \int_0^1 \frac{r^{2n-1}}{1 - r^2} K(g(r)) dr. \]
Since \(a\) is arbitrary, it follows that
\[ \|f\|_{\mathcal{N}_K(B)} \leq 2nM \int_0^1 \frac{r^{2n-1}}{(1 - r^2)^{n+1}} K(g(r)) dr < \infty. \]
Thus, \(f \in \mathcal{N}_K(B)\) and the first half of the theorem is proved.

Now, we assume that the integral in (2.5) is divergent. Let \(\alpha = (\alpha_1, \cdots, \alpha_n)\) is an \(n\)-tuple of non-negative integers, \(|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \geq 1\), \(f(z) = z^\alpha\). Then, we have \(|f(r \zeta)|^2 = r^{2|\alpha|} |\zeta^\alpha|^2\), and
\[ \int_\mathbb{B} |(r \zeta)^\alpha|^2 d\sigma(r \zeta) \geq r^{2|\alpha|} \frac{(n - 1)! \alpha!}{(n - 1 + |\alpha|)!} \geq M r^{2|\alpha|}. \]
Thus,
\[ (2.6) \|f\|_{\mathcal{N}_K(B)} \geq \frac{nM}{2^{|\alpha|-1}} \int_\frac{1}{2}^1 \frac{r^{2n-1}}{(1 - r^2)^{n+1}} K(g(r)) dr. \]
There exists \(a \in \mathbb{B}\) such that \(f(a) \neq 0\), by the subharmonicity of \(|f \circ \Phi_a(\zeta)|^2\),
\[ (2.7) \|f\|_{\mathcal{N}_K(B)} \geq \frac{3a}{2} |f(a)|^2 \int_0^{\frac{1}{2}} \frac{r^{2n-1}}{(1 - r^2)^{n+1}} K(g(r)) dr. \]
Combining (2.6) and (2.7), we see that (2.5) implies that \(\|f\|_{\mathcal{N}_K(B)} = \infty\). It is proved that \(f \notin \mathcal{N}_K(B)\) and, since \(\alpha\) is arbitrary, any non-constant polynomial is not contained in \(\mathcal{N}_K(B)\). We conclude that \(\mathcal{N}_K(B)\) contains only constant functions. The theorem is proved.
Lemma 2.2. For \( w \in \mathbb{B} \) we define the probe function in \( \mathcal{N}_K(\mathbb{B}) \) as
\[
h_w(z) = \frac{(1 - |w|^2)^{n+1}}{(1 - \langle z, w \rangle)^{\frac{n(n+1)}{2}}}
\]
Suppose that condition (2.5) is satisfied. Then \( h_w \in \mathcal{N}_K(\mathbb{B}) \) and \( \|h_w\|_{\mathcal{N}_K(\mathbb{B})} \leq 1 \).

Proof. Trivially \( h_w \in \mathcal{N}_K(\mathbb{B}) \). It is also easy to see that
\[
\|h_w\|^2_{\mathcal{N}_K(\mathbb{B})} = \sup_{a \in \mathbb{B}} \int_\mathbb{B} \left| \frac{(1 - |w|^2)^{n+1}}{(1 - \langle z, w \rangle)^{\frac{n(n+1)}{2}}} \right|^2 K(G(z, a)) \, dr(z) \leq 1,
\]
this by a change of variables and since condition (2.5) is satisfied.

Such \( h_w \) is a normalized reproducing kernel function in the Bergman space \( A^2(\mathbb{B}) \). Also note that
\[
h_w(w) = \left( \frac{1}{1 - |w|^2} \right)^{\frac{n+1}{2}}, \quad \forall w \in \mathbb{B}.
\]

In Section 4, we will discuss the estimation for the lower bounded of \( \|W_{u, \phi}\|_\epsilon \). We will make use of weakly convergent sequences in the Bergman space \( A^2(\mathbb{B}) \). The following lemma plays an important role.

Lemma 2.3. Suppose \( \{f_m\}_{m \geq 1} \in A^2(\mathbb{B}) \) is a sequence that converges weakly to zero in \( A^2(\mathbb{B}) \). Then \( \{f_m\}_{m \geq 1} \) converges weakly to zero in \( \mathcal{N}_K(\mathbb{B}) \) as well.

Proof. Let \( \Gamma \in \overline{\mathcal{N}_K(\mathbb{B})} \) be a bounded linear functional on \( \mathcal{N}_K(\mathbb{B}) \). By the fact that \( \|f\|_{\mathcal{N}_K(\mathbb{B})} \leq \|f\|_{A^2(\mathbb{B})} \), then
\[
\|\Gamma\|_{\overline{\mathcal{N}_K(\mathbb{B})}} = \sup_{f \in A^2(\mathbb{B})} \frac{|\Gamma(f)|}{\|f\|_{A^2(\mathbb{B})}} \leq \sup_{f \in A^2(\mathbb{B})} \frac{|\Gamma(f)|}{\|f\|_{\mathcal{N}_K(\mathbb{B})}}
\]
\[
\leq \sup_{f \in \mathcal{N}_K(\mathbb{B})} \frac{|\Gamma(f)|}{\|f\|_{\mathcal{N}_K(\mathbb{B})}} = \|\Gamma\|_{\mathcal{N}_K(\mathbb{B})},
\]
which implies \( \Gamma \) is also a bounded linear functional on \( A^2(\mathbb{B}) \). Since \( f_m \to 0 \) weakly in \( A^2(\mathbb{B}) \), we conclude that \( \Gamma(f_m) \to 0 \). Therefore, \( f_m \to 0 \) weakly in \( \mathcal{N}_K(\mathbb{B}) \) as well.

Corollary 2.1. Let \( \{w_m\}_{m \in \mathbb{N}} \subset \mathbb{B} \) and \( |w_m| \to 1 \) as \( m \to \infty \), then \( \{h_{w_m}\} \) converges weakly to zero in \( \mathcal{N}_K(\mathbb{B}) \).

Proof. It is well known that \( h_{w_m} \to 0 \) weakly in \( A^2(\mathbb{B}) \) as \( m \to \infty \). Indeed, for any \( f \in A^2(\mathbb{B}) \), using the reproducing property, we have
\[
\langle f, h_{w_m} \rangle = (1 - |w_m|^2)^{\frac{n+1}{2}} f(w_m),
\]
which converges to zero as \( m \to \infty \), because the set of polynomials is dense in \( A^2(\mathbb{B}) \) (see [10], Proposition 2.6). The conclusion of the corollary follows immediately from Lemma 2.3.

3. Weighted composition operators from \( \mathcal{N}_K(\mathbb{B}) \) into \( H^\infty_\alpha(\mathbb{B}) \)

In this section, we will consider the operator \( W_{u, \phi} : \mathcal{N}_K(\mathbb{B}) \to H^\infty_\alpha(\mathbb{B}) \).

Theorem 3.1. Let \( \phi : \mathbb{B} \to \mathbb{B} \) be a holomorphic mapping and \( u \in \mathcal{H}(\mathbb{B}) \). For \( 0 < \alpha < \infty \), then \( W_{u, \phi} : H^\infty_\alpha(\mathbb{B}) \to \mathcal{N}_K(\mathbb{B}) \) is a bounded operator if and only if
\[
\sup_{\alpha \in \mathbb{B}} \frac{|u(z)|(1 - |z|^2)^{\alpha}}{(1 - |\phi(z)|^2)^{\frac{n+1}{2}}} < \infty.
\]

Proof. First assume that condition (3.1) holds, by Proposition 1.1, we have
\[
\|W_{u, \phi}(f)\|_{H^\infty_\alpha(\mathbb{B})} = \sup_{z \in \mathbb{B}} |u(z)||f(\phi(z))|(1 - |z|^2)^{\alpha}
\]
\[
\leq \|f\|_{H^\infty_\alpha}, \sup_{z \in \mathbb{B}} \frac{|u(z)|(1 - |z|^2)^{\alpha}}{(1 - |\phi(z)|^2)^{\frac{n+1}{2}}}
\]
\[
\leq C\|f\|_{\mathcal{N}_K(\mathbb{B})}.
\]
This implies that \( W_{u, \phi} : H^\infty_\alpha(\mathcal{B}) \to \mathcal{N}_K(\mathcal{B}) \) is a bounded operator.

Conversely, assume that \( W_{u, \phi} : \mathcal{N}_K(\mathcal{B}) \to H^\infty_\alpha(\mathcal{B}) \) is bounded, then \( \|W_{u, \phi}(f)\|_{H^\infty_\alpha(\mathcal{B})} \leq \|f\|_{\mathcal{N}_K(\mathcal{B})} \). Let \( h_w \) be the test function in Lemma 2.2 with \( w = \phi(z) \), then we get

\[
h_{\phi(z)}(\phi(z)) = \left( \frac{1}{1 - |\phi(z)|^2} \right)^{\frac{n+1}{2}}.
\]

Hence, there exist a positive constant \( M \) such that:

\[
M \geq \|h_w\|_{\mathcal{N}_K(\mathcal{B})} \geq \|W_{u, \phi}(h_w)\|_{H^\infty_\alpha(\mathcal{B})} \geq \frac{|u(z)|(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{n+1}{2}}}.
\]

This completes the proof of the theorem.

Using the standard arguments similar to those outlined in proposition 3.11 of [2], we have the following lemma:

**Lemma 3.1.** Let \( \phi : \mathcal{B} \to \mathcal{B} \) be a holomorphic mapping and \( u \in \mathcal{H}(\mathcal{B}) \). For \( 0 < \alpha < \infty \), then \( W_{u, \phi} : H^\infty_\alpha(\mathcal{B}) \to \mathcal{N}_K(\mathcal{B}) \) is compact if and only if

\[
\lim_{m \to \infty} \|W_{u, \phi}(f_m)\|_{\mathcal{N}_K(\mathcal{B})} = 0,
\]

for every bounded sequence \( \{f_m\} \subset \mathcal{N}_K(\mathcal{B}) \) which converges to 0 uniformly on any compact subsets of \( \mathcal{B} \) as \( m \to \infty \).

**Theorem 3.2.** Let \( \phi : \mathcal{B} \to \mathcal{B} \) be a holomorphic mapping and \( u \in \mathcal{H}(\mathcal{B}) \). For \( 0 < \alpha < \infty \), then \( W_{u, \phi} : \mathcal{N}_K(\mathcal{B}) \to H^\infty_\alpha(\mathcal{B}) \) is compact if and only if

\[
\lim_{r \to 1^-} \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{n+1}{2}}} = 0.
\]

**Proof.** First assume that \( W_{u, \phi} : \mathcal{N}_K(\mathcal{B}) \to H^\infty_\alpha(\mathcal{B}) \) is compact, then it is bounded. By Theorem 3.1, we have

\[
L = \sup_{\alpha \in \mathbb{B}} \frac{|u(z)|(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{n+1}{2}}} < \infty,
\]

Note that \( \lim_{r \to 1^-} L(r) \) always exists, where:

\[
L(r) = \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{n+1}{2}}}.
\]

Now, we show that (3.4) holds.

Assume on the contrary that there exists \( \varepsilon_0 > 0 \) such that

\[
\lim_{r \to 1^-} \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{n+1}{2}}} = \varepsilon_0.
\]

There exists an \( r_0 \in (0, 1) \) such that \( r_0 < r < 1 \), we have \( L(r) > \frac{\varepsilon_0}{2} \). Then, by the standard diagonal process, we can construct a sequence \( \{z_m\} \subset \mathcal{B} \) such that \( |\phi(z)| \to 1 \) as \( m \to \infty \), and also for each \( m \in \mathbb{N} \),

\[
\frac{|u(z)|(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{n+1}{2}}} \geq \frac{\varepsilon_0}{4}.
\]

Clearly, we can assume that \( w_n = \phi(z_m) \) tends to \( w_0 \in \partial \mathcal{B} \) as \( m \to \infty \). Let \( h_{w_m} = \frac{1 - |w_m|^2}{(1 - |w_m|^2)^{\frac{n+1}{2}}} \) be the function in Lemma 2.2 with \( w_m = \phi(z_m) \). Then \( h_{w_m} \to h_{w_0} \) with respect to the compact-open topology. Define \( f_m = h_{w_m} - h_{w_0} \). Then \( \|f_m\|_{\mathcal{N}_K(\mathcal{B})} \leq 1 \) and \( f_m \to 0 \) uniformly on compact subsets of \( \mathcal{B} \). Thus, \( f_m \circ \phi \to 0 \) in \( H^\infty_\alpha(\mathcal{B}) \) by assumption. But, for \( m \) big enough,

\[
\|W_{u, \phi}(f_m)\|_{H^\infty_\alpha(\mathcal{B})} \geq \frac{|u(z_m)|(1 - |z_m|^2)^\alpha}{(1 - |\phi(z_m)|^2)^{\frac{n+1}{2}}} \geq \frac{\varepsilon_0}{4},
\]

which is a contradiction.

Conversely, if (3.4) holds, we assume that \( \{f_m\} \) is a bounded sequence in \( \mathcal{N}_K(\mathcal{B}) \) norm which converges
to zero uniformly on every compact subset of $mathbb{B}$, then for all $\varepsilon > 0$ there exists $\delta \in (0,1)$ and $m_\varepsilon < m$ such that for $\delta < r < 1$, we have

$$
\|W_{u,\phi}(f_n)\|_{H_\alpha^\infty}(B) \leq \sup_{|\phi(z)| > r} |u(z)||f_m(\phi(z))|(1 - |z|^2)^\alpha \\
+ \sup_{|\phi(z)| \leq r} |f_m(\phi(z))|(1 - |z|^2)^\alpha \leq \varepsilon.
$$

From this $\|W_{u,\phi}(f_n)\|_{H_\alpha^\infty}(B) \to 0$ as $m \to \infty$, it follows that $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H_\alpha^\infty(\mathbb{B})$ is a compact operator. This completes the proof of the theorem.

As a corollary of Theorems 3.1 and 3.2, we have:

**Corollary 3.1.** Let $\phi : \mathbb{B} \to \mathbb{B}$ be a holomorphic mapping and $0 < \alpha < \infty$. Then composition operator $C_{\phi} : \mathcal{N}_K(\mathbb{B}) \to H_\alpha^\infty(\mathbb{B})$

- is bounded if and only if
  
  $$
  \sup_{z \in \mathbb{B}} \frac{(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{\alpha + 1}{\alpha}}} < \infty;
  $$

- is compact if and only if
  
  $$
  \lim_{r \to 1^-} \sup_{|\phi(z)| > r} \frac{(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{\alpha + 1}{\alpha}}} = 0.
  $$

4. **Essential norms of weighted composition operators from $\mathcal{N}_K(\mathbb{B})$ into $H_\alpha^\infty(\mathbb{B})$**

In this section, we study the essential norm of weighted composition operator $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H_\alpha^\infty(\mathbb{B})$. Let us denote by $\mathcal{C} := \mathcal{C}(\mathcal{N}_K(\mathbb{B}), H_\alpha^\infty(\mathbb{B}))$ the set of all compact operators acting from $\mathcal{N}_K(\mathbb{B})$ into $H_\alpha^\infty(\mathbb{B})$. Then the essential norm of $W_{u,\phi}$ is defined as follows:

$$
\|W_{u,\phi}\|_e = \inf_{\mathcal{O} \in \mathcal{C}} \{\|W_{u,\phi} - \mathcal{O}\|\}.
$$

Obviously, the essential norm of a compact operator is zero.

Note that by using the standard argument, it can be shown that a composition operator $C_{\phi} : \mathcal{N}_K(\mathbb{B}) \to \mathcal{N}_K(\mathbb{B})$ is compact if and only if for any bounded sequence $\{f_m\} \subset \mathcal{N}_K(\mathbb{B})$ converging to zero uniformly on every compact subset of $\mathbb{B}$, the sequence $\{|f_m \circ \phi\|\}$ converges to zero as $m \to \infty$.

**Lemma 4.1.** Suppose $\phi : \mathbb{B} \to \mathbb{B}$ is a holomorphic mapping such that $\|\phi\|_{\infty} < 1$, and $u \in \mathcal{H}(\mathbb{B})$. For $0 < \alpha < \infty$, then $W_{u,\phi} : H_\alpha^\infty(\mathbb{B}) \to \mathcal{N}_K(\mathbb{B})$ is compact if and only if

$$
\lim_{m \to \infty} \|W_{u,\phi}(f_m)\|_{\mathcal{N}_K(\mathbb{B})} = 0,
$$

**Proof.** Let $r = \|\phi\|_{\infty}$ and take an arbitrary $f \in \mathcal{N}_K(\mathbb{B})$. Then we have

$$
\|W_{u,\phi}(f)\|_{\mathcal{N}_K(\mathbb{B})} \leq \|f \circ \phi\|_{\infty}\|u\|_{\mathcal{N}_K(\mathbb{B})} \leq \left( \sup_{|z| \leq r} |f(z)| \right)\|u\|_{\mathcal{N}_K(\mathbb{B})} < \infty.
$$

This shows that $W_{u,\phi}$ maps $\mathcal{N}_K(\mathbb{B})$ into itself.

Now suppose that $\{f_m\}$ is a bounded sequence in $\mathcal{N}_K(\mathbb{B})$ that converges to zero uniformly on every compact subset of $\mathbb{B}$. Applying the above estimate with $f = f_m$, we have

$$
\|W_{u,\phi}(f_m)\|_{\mathcal{N}_K(\mathbb{B})} \leq \left( \sup_{|z| \leq r} |f_m(z)| \right)\|u\|_{\mathcal{N}_K(\mathbb{B})} < \infty.
$$

Since the set $z : |z| \leq r$ is compact, the right-hand side of the last quantity converges to 0 as $m \to \infty$, hence so does the sequence $\{|W_{u,\phi}(f_m)|\}$. This means that $W_{u,\phi}$ is compact.

In the following theorem we formulate and prove an estimate for the upper bound of the essential norm of $W_{u,\phi} : \mathcal{N}_K(\mathbb{B}) \to H_\alpha^\infty(\mathbb{B})$. 

Proof. Since \( W_{u,\phi} \) is bounded, we see that \( u \in H^{\infty}_a(\mathbb{B}) \), and Theorems 3.1, 3.2 shows that

\[
\lim_{r \to 1^-} \sup_{|\phi(z)| > r} \left( \frac{|u(z)|}{1 - |\phi(z)|^2} \right)^{\frac{\alpha}{2}}
\]

exists and is a real number.

First, we prove that for any \( r \in [0, 1) \),

\[
\|W_{u,\phi}\|_e \leq M \sup_{|\phi(z)| > r} \left( \frac{|u(z)|}{1 - |\phi(z)|^2} \right)^{\frac{\alpha}{2}}.
\]

For each \( k \in \mathbb{N} \), set \( \phi_k(z) = \frac{k z}{k + 1} \) for all \( z \in \mathbb{B} \). By Lemma 4.1, \( C_{\phi_k} \) is compact on \( \mathcal{N}_K(\mathbb{B}) \), and hence, \( \mathcal{O} = W_{u,\phi} \circ C_{\phi_k} \in \mathcal{C} \) is compact acting from \( \mathcal{N}_K(\mathbb{B}) \) into \( H^{\infty}_a(\mathbb{B}) \). Then for any \( k \in \mathbb{N} \), we have

\[
\|W_{u,\phi}\|_e = \inf_{\mathcal{O} \in \mathcal{C}} \left\{ \|W_{u,\phi} - \mathcal{O}\| \right\} \leq \|W_{u,\phi} - W_{u,\phi} \circ C_{\phi_k}\|_e
\]

which implies that

\[
\|W_{u,\phi}\|_e \leq \inf_{k \in \mathbb{N}} \left\{ \sup_{\|f\|_{\mathcal{N}_K(\mathbb{B})} \leq 1} \|(W_{u,\phi} - W_{u,\phi} \circ C_{\phi_k})(f)\|_{H^{\infty}_a(\mathbb{B})} \right\}.
\]

For \( f \in \mathcal{N}_K(\mathbb{B}) \), we estimate

\[
\|(W_{u,\phi} - W_{u,\phi} \circ C_{\phi_k})(f)\|_{H^{\infty}_a(\mathbb{B})}
\]

\[
= \sup_{z \in \mathbb{B}} \left\{ |u(z)| \left| f(\phi(z)) - f\left( \frac{k \phi(z)}{k + 1} \right) \right| (1 - |\phi(z)|^2)^\alpha \right\}
\]

\[
\leq \sup_{|\phi(z)| > r} \left\{ |u(z)| \left| f(\phi(z)) - f\left( \frac{k \phi(z)}{k + 1} \right) \right| (1 - |\phi(z)|^2)^\alpha \right\}
\]

\[
+ \sup_{|\phi(z)| \leq r} \left\{ |u(z)| \left| f(\phi(z)) - f\left( \frac{k \phi(z)}{k + 1} \right) \right| (1 - |\phi(z)|^2)^\alpha \right\}.
\]

On the one hand, by Lemma 2.1, equation (2.3), we have

\[
\sup_{|\phi(z)| > r} \left\{ |u(z)| \left| f(\phi(z)) - f\left( \frac{k \phi(z)}{k + 1} \right) \right| (1 - |\phi(z)|^2)^\alpha \right\}
\]

\[
\leq \sup_{|\phi(z)| > r} \left| f(\phi(z)) - f\left( \frac{k \phi(z)}{k + 1} \right) \right| \sup_{z \in \mathbb{B}} \left\{ |u(z)| (1 - |\phi(z)|^2)^\alpha \right\}
\]

\[
\leq \left( \sup_{|\phi(z)| > r} \frac{M |u(z)| (1 - |\phi(z)|^2)^\alpha}{1 - |\phi(z)|^2} \right) \|f\|_{\mathcal{N}_K(\mathbb{B})}.
\]

On the other hand, by Lemma 2.1, equation (2.4), we have

\[
\sup_{|\phi(z)| \leq r} \left\{ |u(z)| \left| f(\phi(z)) - f\left( \frac{k \phi(z)}{k + 1} \right) \right| (1 - |\phi(z)|^2)^\alpha \right\}
\]

\[
\leq \left( \frac{Mr u\|_{H^{\infty}_a(\mathbb{B})}}{(k + 1)(1 - r^2)^\alpha} \right) \|f\|_{\mathcal{N}_K(\mathbb{B})}.
\]
Therefore, if \( \| f \|_{N_\alpha(\B)} \leq 1 \), then
\[
\|(W_{u,\phi} - W_{u,\phi} \circ \phi_k)(f)\|_{H_\alpha^\infty(\B)}
\leq \sup_{z \in \B} \left\{ |u(z)| \left| f(\phi(z)) - f\left(\frac{k\phi(z)}{k + 1}\right)\right| (1 - |z|^2)^\alpha \right\}
\leq M \left\{ \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)^\alpha}{(k + 1)(1 - r^2)^\alpha} + Mr \| u \|_{H_\alpha^\infty(\B)} \right\}.
\]

It then follows that
\[
\inf_{k \in \mathbb{N}} \left\{ \sup_{\| f \|_{N_\alpha(\B)} \leq 1} \|(W_{u,\phi} - W_{u,\phi} \circ \phi_k)(f)\|_{H_\alpha^\infty(\B)} \right\}
\leq M \inf_{k \in \mathbb{N}} \left\{ \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)^\alpha}{(k + 1)(1 - r^2)^\alpha} + Mr \| u \|_{H_\alpha^\infty(\B)} \right\}
\leq M \left( \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{\alpha + 1}{2}}} \right).
\]

Combining (4.4) and (4.5), we obtain (4.3). Now letting \( r \to 1 \) in (4.3), we arrive at the desired inequality
\[
\|W_{u,\phi}\|_e \leq M \lim_{r \to 1} \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{\alpha + 1}{2}}}.
\]

This completes the proof of the theorem.

We now discuss the estimation for the lower bound of the essential norm of \( W_{u,\phi} : N_\alpha(\B) \to H_\alpha^\infty(\B) \).

Theorem 4.2. Let \( \phi : \B \to \B \) be a holomorphic mapping and \( u \in H(\B) \). For \( 0 < \alpha < \infty \), suppose that \( W_{u,\phi} : N_\alpha(\B) \to H_\alpha^\infty(\B) \) is a bounded operator. Then
\[
\|W_{u,\phi}\|_e \geq \lim_{r \to 1} \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{\alpha + 1}{2}}}.
\]

Proof. The case \( \| \phi \|_\infty < 1 \) is obvious since the right hand side is zero. Now assume that \( \| \phi \|_\infty = 1 \). For any \( r \in (0, 1) \), the set \( S_r := \{ z \in \B : |\phi(z)| > r \} \) is not empty. For each \( z \in \B \), consider the probe function \( h_w \) in Lemma 2.2 with \( w = \phi(z) \). Then for any compact operator \( \mathcal{O} \in \mathcal{C} \) we have
\[
\|W_{u,\phi} - \mathcal{O}\| \geq \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{\alpha + 1}{2}}}.
\]

which is equivalent to
\[
\|W_{u,\phi} - \mathcal{O}\| + \|\mathcal{O}(h_w(z))\|_{H_\alpha^\infty(\B)} \geq \frac{|u(z)|(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{\alpha + 1}{2}}}.
\]

Taking the supremum on \( z \) over the set \( S_r \) on both sides of (4.8) yields
\[
\|W_{u,\phi} - \mathcal{O}\| + \sup_{z \in S_r} \|\mathcal{O}(h_w(z))\|_{H_\alpha^\infty(\B)} \geq \sup_{z \in S_r} \frac{|u(z)|(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{\alpha + 1}{2}}}.
\]

which is
\[
\|W_{u,\phi} - \mathcal{O}\| + \sup_{|\phi(z)| > r} \|\mathcal{O}(h_w(z))\|_{H_\alpha^\infty(\B)} \geq \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^{\frac{\alpha + 1}{2}}}.
\]

Denote \( H(r) = \sup_{|\phi(z)| > r} \|\mathcal{O}(h_w(z))\|_{H_\alpha^\infty(\B)} \). Since \( H(r) \) decreases as \( r \) increases, \( \lim_{r \to 1} H(r) \) exists. We claim that this limit is necessarily zero. For the purpose of obtaining a contradiction, assume that
\[ \lim H(r) = L > 0. \] Then there is a sequence \( \{ z_m \} \subset \mathbb{B} \) satisfying \(|\phi(z_m)| \to 1\) as \( m \to \infty \), and for each \( m \in \mathbb{N} \),

\[ (4.10) \quad \|\mathcal{O}(h_{\phi(z)})\|_{H_{\infty}^2(\mathbb{B})} \geq \frac{1}{2} L. \]

By Corollary 3.1, \( \{ h_{\phi(z_m)} \} \) converges weakly to zero in \( N_K(\mathbb{B}) \). Since \( \mathcal{O} \) is compact, we have \( \|\mathcal{O}(h_{\phi(z)})\|_{H_{\infty}^2(\mathbb{B})} \) converges to zero as \( m \to \infty \), which contradicts (4.10). Therefore,

\[ \lim_{r \to 1^-} \sup_{|\phi(z)| > r} \|\mathcal{O}(h_{\phi(z)})\|_{H_{\infty}^2(\mathbb{B})} = 0. \]

Letting \( r \to 1^- \) on both sides of (4.9), we conclude that for any compact operator \( \mathcal{O} \in \mathcal{C} \),

\[ \|W_{u,\phi} - \mathcal{O}\| \geq \lim_{r \to 1^-} \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)\alpha}{(1 - |\phi(z)|^2)^{\frac{\alpha}{2}}}. \]

From this, it follows that

\[ \|W_{u,\phi}\|_e = \inf_{\mathcal{O} \in \mathcal{C}} \|W_{u,\phi} - \mathcal{O}\| \geq \lim_{r \to 1^-} \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)\alpha}{(1 - |\phi(z)|^2)^{\frac{\alpha}{2}}}. \]

This completes the proof of the theorem.

In conclusion, combining Theorems 4.1 and 4.2, we obtain a full description of the essential norm of \( W_{u,\phi} \).

**Theorem 4.3.** Let \( \phi : \mathbb{B} \to \mathbb{B} \) be a holomorphic mapping and \( u \in \mathcal{H}(\mathbb{B}) \). For \( 0 < \alpha < \infty \), suppose that \( W_{u,\phi} : N_K(\mathbb{B}) \to H_{\infty}^2(\mathbb{B}) \) is a bounded operator. Then

\[ (4.11) \quad \|W_{u,\phi}\|_e \approx \lim_{r \to 1^-} \sup_{|\phi(z)| > r} \frac{|u(z)|(1 - |z|^2)\alpha}{(1 - |\phi(z)|^2)^{\frac{\alpha}{2}}}. \]

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### References


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