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SOME DISCUSSIONS ON A KIND OF IMPROPER INTEGRALS

FENG QI[1,2,3,*] AND VIERA ČERŇANOVÁ[4]

Abstract. In the paper, the improper integral

$$I(a,b;\lambda,\eta)=\int_a^b \frac{1}{\sqrt{(t-a)(b-t)}}\frac{\ln^\lambda t}{t^\eta}\,\mathrm{d}\,t$$

for $b>a>0$ and $\lambda,\eta\in\mathbb{R}$ is discussed, some explicit formulas for special cases of $I(a,b;\lambda,\eta)$ are presented, and several identities of $I(a,b;k,\eta)$ for $k\in\mathbb{N}$ are established.

1. Motivation

The motivation of this paper origins from investigating central Delanoy numbers in [11]. For proving the main result [11, Theorem 1.4], we need [11, Lemmas 2.4 and 2.5]. Lemma 2.4 in [11] states that, for $b>a$ and $z\in\mathbb{C}\setminus(-\infty,-a]$, the principal branch of the function $\frac{1}{\sqrt{(z+a)(z+b)}}$ can be represented as

$$\frac{1}{\sqrt{(z+a)(z+b)}}=\frac{1}{\pi}\int_a^b \frac{1}{\sqrt{(t-a)(b-t)}}\frac{1}{t+z}\,\mathrm{d}\,t, \tag{1.1}$$

where $\mathbb{C}$ denotes the complex plane. When taking $z=0$, the integral representation (1.1) becomes

$$\boxed{\int_a^b \frac{1}{\sqrt{(t-a)(b-t)}}\frac{1}{t}\,\mathrm{d}\,t=\frac{\pi}{\sqrt{ab}},\quad b>a>0.} \tag{1.2}$$

Lemma 2.5 in [11] reads that the improper integral

$$\int_{1/\alpha}^\alpha \frac{1}{\sqrt{(t-1/\alpha)(\alpha-t)}}\frac{\ln^{2k-1} t}{t^\beta}\,\mathrm{d}\,t\begin{cases}<0, & \beta>\frac12\\ =0, & \beta=\frac12\\ >0, & \beta<\frac12\end{cases} \tag{1.3}$$

for all $k\in\mathbb{N}$, where $\alpha>1$ and $\beta\in\mathbb{R}$.

Motivated by the above results, we naturally introduce the improper integral

$$\begin{aligned}I(a,b;\lambda,\eta)&=\int_a^b \frac{1}{\sqrt{(t-a)(b-t)}}\frac{\ln^\lambda t}{t^\eta}\,\mathrm{d}\,t\\ &=\int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{\ln^\lambda[(b-a)s+a]}{[(b-a)s+a]^\eta}\,\mathrm{d}\,s\end{aligned}$$

for $b>a>0$ and $\lambda,\eta\in\mathbb{R}$ and consider a problem: how to compute the improper integral $I(a,b;\lambda,\eta)$?

2. Explicit formulas for special cases of $I(a,b;\lambda,\eta)$

In this section, we present several explicit formulas for special cases of the improper integral $I(a,b;\lambda,\eta)$.

In the monograph [4], we do not find such a kind of integrals $I(a,b;\lambda,\eta)$ for general $b>a>0$ and $\lambda,\eta\in\mathbb{R}$.

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2.1. From (1.1) or (1.2), it follows that

$$\boxed{I(a,b;0,1)=\frac{\pi}{\sqrt{ab}},\quad b>a>0.} \tag{2.1}$$

2.2. From (1.3), it follows that

$$\boxed{I\biggl(a,\frac{1}{a};2k-1,\frac{1}{2}\biggr)=0,\quad 0<a<1,\quad k\in\mathbb{N}.}$$

2.3. It is straightforward by using Euler's substitution that

$$\boxed{I(a,b;0,0)=\int_0^1\frac{1}{\sqrt{s(1-s)}}\,\mathrm{d}\,s=\pi,\quad b>a>0.}$$

2.4. When $\lambda=0$, $\eta\ne0$, and $2a>b>a>0$, we have

$$\begin{aligned}
I(a,b;0,\eta)&=\frac{1}{a^\eta}\int_0^1\frac{[1+(b/a-1)s]^{-\eta}}{\sqrt{s(1-s)}}\,\mathrm{d}\,s\\
&=\frac{1}{a^\eta}\int_0^1(1-s)^{-1/2}\sum_{\ell=0}^\infty\langle-\eta\rangle_\ell\biggl(\frac{b}{a}-1\biggr)^\ell\frac{s^{\ell-1/2}}{\ell!}\,\mathrm{d}\,s\\
&=\frac{1}{a^\eta}\sum_{\ell=0}^\infty\frac{\langle-\eta\rangle_\ell}{\ell!}\biggl(\frac{b}{a}-1\biggr)^\ell\int_0^1(1-s)^{-1/2}s^{\ell-1/2}\,\mathrm{d}\,s\\
&=\frac{1}{a^\eta}\sum_{\ell=0}^\infty\frac{\langle-\eta\rangle_\ell}{\ell!}\biggl(\frac{b}{a}-1\biggr)^\ell B\biggl(\frac{1}{2},\ell+\frac{1}{2}\biggr)\\
&=\frac{1}{a^\eta}\sum_{\ell=0}^\infty(\eta)_\ell\frac{\Gamma(1/2)\Gamma(\ell+1/2)}{\Gamma(\ell+1)}\frac{1}{\ell!}\biggl(1-\frac{b}{a}\biggr)^\ell\\
&=\frac{\pi}{a^\eta}\sum_{\ell=0}^\infty\frac{(\eta)_\ell(1/2)_\ell}{(1)_\ell}\frac{1}{\ell!}\biggl(1-\frac{b}{a}\biggr)^\ell\\
&=\frac{\pi}{a^\eta}\,{}_2F_1\biggl(\eta,\frac{1}{2};1;1-\frac{b}{a}\biggr),
\end{aligned}$$

where

$$\langle x\rangle_n=\prod_{k=0}^{n-1}(x-k)=\begin{cases}x(x-1)\cdots(x-n+1), & n\ge1\\ 1, & n=0\end{cases}$$

and

$$(x)_\ell=\prod_{k=0}^{\ell-1}(x+k)=\begin{cases}x(x+1)(x+2)\cdots(x+\ell-1), & \ell\ge1\\ 1, & \ell=0\end{cases}$$

are respectively called the falling and rising factorials of $x\in\mathbb{R}$, the function $B(x,y)$ denotes the classical beta function, and ${}_2F_1$ are the classical hypergeometric functions which are special cases of the generalized hypergeometric series

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{n=0}^\infty\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{z^n}{n!}$$

for complex numbers $a_i\in\mathbb{C}$ and $b_i\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$ and for positive integers $p,q\in\mathbb{N}$. This result

$$\boxed{I(a,b;0,\eta)=\frac{\pi}{a^\eta}\,{}_2F_1\biggl(\eta,\frac{1}{2};1;1-\frac{b}{a}\biggr),\quad \eta\ne0,\quad 2a>b>a>0}$$

can also be found in [3, p. xv, eq. (12)].

2.5. When $\lambda = k \in \mathbb{N}$ and $2a > b > a > 0$, the function $\ln^k[(b-a)s+a]$ can be rewritten as

$$\begin{aligned}\ln^k[(b-a)s+a] &= \left(\ln a + \ln\left[1+\left(\frac{b}{a}-1\right)s\right]\right)^k \\ &= \sum_{\ell=0}^{k}\binom{k}{\ell}\ln^{k-\ell} a \ln^{\ell}\left[1+\left(\frac{b}{a}-1\right)s\right] \\ &= \left(\ln^k a\right)\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^\ell}{\ln^\ell a}\left[\sum_{m=1}^{\infty}\frac{1}{m}\left(1-\frac{b}{a}\right)^m s^m\right]^\ell \\ &= \left(\ln^k a\right)\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^\ell}{\ln^\ell a}s^\ell\left[\sum_{m=0}^{\infty}\frac{1}{m+1}\left(1-\frac{b}{a}\right)^{m+1} s^m\right]^\ell.\end{aligned}$$

When $0 < a < b < 1$ or $1 < a < b < a^2$, if $\lambda \in \mathbb{R}$, then

$$\begin{aligned}\ln^\lambda[(b-a)s+a] &= \left(\ln^\lambda a\right)\left(1+\frac{\ln[1+(b/a-1)s]}{\ln a}\right)^\lambda \\ &= \left(\ln^\lambda a\right)\sum_{\ell=0}^{\infty}\frac{\langle\lambda\rangle_\ell}{\ell!}\left(\frac{\ln[1+(b/a-1)s]}{\ln a}\right)^\ell \\ &= \left(\ln^\lambda a\right)\sum_{\ell=0}^{\infty}\frac{(-1)^\ell\langle\lambda\rangle_\ell}{\ell!\ln^\ell a}\left[\sum_{m=1}^{\infty}\frac{1}{m}\left(1-\frac{b}{a}\right)^m s^m\right]^\ell \\ &= \left(\ln^\lambda a\right)\sum_{\ell=0}^{\infty}\frac{(-1)^\ell\langle\lambda\rangle_\ell}{\ell!\ln^\ell a}s^\ell\left[\sum_{m=0}^{\infty}\frac{1}{m+1}\left(1-\frac{b}{a}\right)^{m+1} s^m\right]^\ell.\end{aligned}$$

In [4, p. 18, 0.314], it was stated that

$$\left(\sum_{k=0}^{\infty}a_k x^k\right)^n = \sum_{k=0}^{\infty}c_{n,k}x^k,$$

where $c_{n,0} = a_0^n$ and

$$c_{n,m} = \frac{1}{ma_0}\sum_{k=1}^{m}(kn-m+k)a_k c_{n,m-k}, \quad m \in \mathbb{N}.$$

Hence, it follows that

$$\left[\sum_{m=0}^{\infty}\frac{1}{m+1}\left(1-\frac{b}{a}\right)^{m+1} s^m\right]^\ell = \sum_{m=0}^{\infty}c_{\ell,m}x^m,$$

where $c_{\ell,0} = \left(1-\frac{b}{a}\right)^\ell$ and

$$\begin{aligned}c_{\ell,m} &= \frac{1}{m}\sum_{k=1}^{m}\frac{k\ell-m+k}{k+1}\left(1-\frac{b}{a}\right)^k c_{\ell,m-k} \\ &= \frac{1}{m}\left(1-\frac{b}{a}\right)^m\sum_{p=0}^{m-1}\frac{m\ell-(\ell+1)p}{m-p+1}\left(1-\frac{b}{a}\right)^{-p}c_{\ell,p}\end{aligned}$$

for $m \in \mathbb{N}$. Let $C_{\ell,m} = \left(1-\frac{b}{a}\right)^{-m}c_{\ell,m}$, the above recursive formula becomes

$$C_{\ell,m} = \frac{1}{m}\sum_{p=0}^{m-1}\frac{m\ell-p(\ell+1)}{m-p+1}C_{\ell,p} \tag{2.2}$$

with $C_{\ell,0} = c_{\ell,0}$. Starting out from these points, it is much possible to find explicit formulas for computing the integral $I(a,b;\lambda,\eta)$. For example, when $\lambda \ne 0$ and $\eta = 1$,

$$\begin{aligned}
I(a,b;\lambda,1) &= \frac{1}{(\lambda+1)(b-a)}\int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{\mathrm{d}\ln^{\lambda+1}[(b-a)s+a]}{\mathrm{d}\, s}\,\mathrm{d}\, s \\
&= \frac{\ln^{\lambda+1} a}{(\lambda+1)(b-a)}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell\langle\lambda+1\rangle_\ell}{\ell!\ln^\ell a} \\
&\quad\times\int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{\mathrm{d}}{\mathrm{d}\, s}\left[\sum_{m=0}^{\infty}\frac{1}{m}\left(1-\frac{b}{a}\right)^m s^m\right]^\ell \mathrm{d}\, s \\
&= \frac{\ln^{\lambda+1} a}{(\lambda+1)(b-a)}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell\langle\lambda+1\rangle_\ell}{\ell!\ln^\ell a}\int_0^1 \frac{1}{\sqrt{s(1-s)}}\frac{\mathrm{d}}{\mathrm{d}\, s}\sum_{m=0}^{\infty} c_{\ell,m}s^m\,\mathrm{d}\, s \\
&= \frac{\ln^{\lambda+1} a}{(\lambda+1)(b-a)}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell\langle\lambda+1\rangle_\ell}{\ell!\ln^\ell a}\sum_{m=0}^{\infty}(m+1)c_{\ell,m+1}\int_0^1 (1-s)^{-1/2}s^{m-1/2}\,\mathrm{d}\, s \\
&= \frac{\ln^{\lambda+1} a}{(\lambda+1)(b-a)}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell\langle\lambda+1\rangle_\ell}{\ell!\ln^\ell a}\sum_{m=0}^{\infty}(m+1)c_{\ell,m+1}B\left(\frac{1}{2},m+\frac{1}{2}\right) \\
&= \frac{\pi\ln^{\lambda+1} a}{(\lambda+1)(b-a)}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell\langle\lambda+1\rangle_\ell}{\ell!\ln^\ell a}\sum_{m=0}^{\infty}(m+1)c_{\ell,m+1}\frac{(1/2)_m}{(1)_m}.
\end{aligned}$$

Hence, it would be important to derive a general formula for the recursive relation (2.2).

2.6. For $k \ge 0$, differentiating with respect to z on both sides of (1.1) gives

$$\frac{\mathrm{d}^k}{\mathrm{d}\, z^k}\frac{1}{\sqrt{(z+a)(z+b)}} = (-1)^k\frac{k!}{\pi}\int_a^b \frac{1}{\sqrt{(t-a)(b-t)}}\frac{1}{(t+z)^{k+1}}\,\mathrm{d}\, t. \tag{2.3}$$

By the Faá di Bruno formula

$$\frac{\mathrm{d}^n}{\mathrm{d}\, t^n} f\circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t))\mathrm{B}_{n,k}\big(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\big), \quad n \ge 0$$

in [2, p. 139, Theorem C], where

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}) = \sum_{\substack{1\le i\le n,\ell_i\in\mathbb{N}\cup\{0\}\\ \sum_{i=1}^n i\ell_i=n\\ \sum_{i=1}^n \ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}, \quad n\ge k\ge 0$$

is called [2, p. 134, Theorem A] the Bell polynomials of the second kind, we obtain

$$\begin{aligned}
\frac{\mathrm{d}^k}{\mathrm{d}\, z^k}\frac{1}{\sqrt{(z+a)(z+b)}} &= \sum_{\ell=0}^{k}\left(\frac{1}{\sqrt{u}}\right)^{(\ell)}\mathrm{B}_{k,\ell}(u'(z),u''(z),0,\dots,0) \\
&= \sum_{\ell=0}^{k}\left\langle-\frac{1}{2}\right\rangle_\ell\frac{1}{u^{\ell+1/2}}\mathrm{B}_{k,\ell}(2z+a+b,2,0,\dots,0) \\
&= \sum_{\ell=0}^{k}\left\langle-\frac{1}{2}\right\rangle_\ell\frac{1}{[(z+a)(z+b)]^{\ell+1/2}}\mathrm{B}_{k,\ell}(2z+a+b,2,0,\dots,0) \\
&\to \sum_{\ell=0}^{k}\left\langle-\frac{1}{2}\right\rangle_\ell\frac{1}{(ab)^{\ell+1/2}}\mathrm{B}_{k,\ell}(a+b,2,0,\dots,0)
\end{aligned}$$

as $z\to 0$, where $u = u(z) = (z+a)(z+b)$. Recall from [2, p. 135] that

$$\mathrm{B}_{n,k}\big(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\big) = a^kb^n\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}), \tag{2.4}$$

where a and b are any complex numbers and $n\ge k\ge0$. Recall from [5, Theoem 4.1], [17, Theorem 3.1], and [18, Lemma 2.5] that

$$\mathrm{B}_{n,k}(x,1,0,\dots,0)=\frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n},\quad n\ge k\ge0. \tag{2.5}$$

Accordingly, by (2.4) and (2.5), it follows that

$$\begin{aligned}\lim_{z\to0}\frac{\mathrm{d}^k}{\mathrm{d}\,z^k}\frac{1}{\sqrt{(z+a)(z+b)}}&=\sum_{\ell=0}^{k}\left\langle-\frac12\right\rangle_\ell\frac{1}{(ab)^{\ell+1/2}}2^\ell\mathrm{B}_{k,\ell}\left(\frac{a+b}{2},1,0,\dots,0\right)\\&=\sum_{\ell=0}^{k}\left\langle-\frac12\right\rangle_\ell\frac{1}{(ab)^{\ell+1/2}}2^\ell\frac{(k-\ell)!}{2^{k-\ell}}\binom{k}{\ell}\binom{\ell}{k-\ell}\left(\frac{a+b}{2}\right)^{2\ell-k}.\end{aligned}$$

Letting $z\to0$ on both sides of (2.3), employing the above result, and simplifying lead to

$$\begin{aligned}\int_a^b&\frac{1}{\sqrt{(t-a)(b-t)}}\frac{1}{t^{k+1}}\,\mathrm{d}\,t\\&=\frac{(-1)^k\pi}{(a+b)^k\sqrt{ab}}\sum_{\ell=0}^{k}(-1)^\ell2^{2\ell}\frac{(2\ell-1)!!}{(2\ell)!!}\binom{\ell}{k-\ell}\left(\frac{a+b}{2}\right)^\ell\left(\frac{1/a+1/b}{2}\right)^\ell,\end{aligned}$$

that is,

$$\boxed{I(a,b;0,k+1)=\frac{\pi}{G(a,b)}\frac{(-1)^k}{[2A(a,b)]^k}\sum_{\ell=0}^{k}(-1)^\ell2^{2\ell}\frac{(2\ell-1)!!}{(2\ell)!!}\binom{\ell}{k-\ell}\left[\frac{A(a,b)}{H(a,b)}\right]^\ell} \tag{2.6}$$

for $b>a>0$ and $k\ge0$, where $\binom{p}{q}=0$ for $q>p\ge0$, the double factorial of negative odd integers $-(2n+1)$ is defined by

$$(-2n-1)!!=\frac{(-1)^n}{(2n-1)!!}=(-1)^n\frac{2^nn!}{(2n)!},\quad n=0,1,\dots,$$

and the quantities

$$A(a,b)=\frac{a+b}{2},\quad G(a,b)=\sqrt{ab}\,,\quad\text{and}\quad H(a,b)=\frac{2}{\frac1a+\frac1b}$$

are respectively the well-known arithmetic, geometric, and harmonic means of two positive numbers a and b.

When $k=0$ in (2.6), the integral (1.2) or (2.1) is recovered.

In fact, the above argument implies that

$$\begin{aligned}\int_a^b\frac{1}{\sqrt{(t-a)(b-t)}}\frac{1}{(t+z)^{k+1}}\,\mathrm{d}\,t&=\frac{(-1)^k}{[2A(z+a,z+b)]^k}\frac{\pi}{G(z+a,z+b)}\\&\quad\times\sum_{\ell=0}^{k}(-1)^\ell2^{2\ell}\frac{(2\ell-1)!!}{(2\ell)!}\binom{\ell}{k-\ell}\left[\frac{A(z+a,z+b)}{H(z+a,z+b)}\right]^\ell\end{aligned}$$

for $b>a>0$ and $k\ge0$. This is equivalent to (2.6).

By the way, the ratio $\frac{(2\ell-1)!!}{(2\ell)!}$ is called the Wallis ratio. For more information, please refer to the paper [7] and plenty of references cited therein.

Alternatively differentiating with respect to z on both sides of (1.1) leads to

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}\,z^k}\frac{1}{\sqrt{(z+a)(z+b)}}&=\frac{\mathrm{d}^k}{\mathrm{d}\,z^k}\left(\frac{1}{\sqrt{z+a}}\frac{1}{\sqrt{z+b}}\right)\\&=\sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{1}{\sqrt{z+a}}\right)^{(\ell)}\left(\frac{1}{\sqrt{z+b}}\right)^{(k-\ell)}\end{aligned}$$

$$= \sum_{\ell=0}^{k} \binom{k}{\ell} \left\langle -\frac{1}{2} \right\rangle_{\ell} \frac{1}{(z+a)^{\ell+1/2}} \left\langle -\frac{1}{2} \right\rangle_{k-\ell} \frac{1}{(z+b)^{k-\ell+1/2}}$$

$$= \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} \frac{(2\ell-1)!!}{2^{\ell}} \frac{1}{(z+a)^{\ell+1/2}} (-1)^{k-\ell} \frac{[2(k-\ell)-1]!!}{2^{k-\ell}} \frac{1}{(z+b)^{k-\ell+1/2}}$$

$$= \frac{(-1)^k}{2^k} \frac{1}{(z+a)^{1/2}} \frac{1}{(z+b)^{k+1/2}} \sum_{\ell=0}^{k} \binom{k}{\ell} (2\ell-1)!![2(k-\ell)-1]!! \left(\frac{z+b}{z+a}\right)^{\ell}.$$

Substituting this into (2.3) and taking the limit $z \to 0$ result in

$$\boxed{I(a,b;0,k+1) = \frac{\pi}{\sqrt{ab}} \frac{1}{b^k} \sum_{\ell=0}^{k} \frac{(2\ell-1)!!}{(2\ell)!!} \frac{[2(k-\ell)-1]!!}{[2(k-\ell)]!!} \left(\frac{b}{a}\right)^{\ell}}$$

for $b > a > 0$ and $k \ge 0$. This is an alternative expression for $I(a,b;0,k+1)$.

2.7. Under different conditions from those discussed above on $b > a > 0$ and $\lambda, \eta \in \mathbb{R}$, can one discover more explicit formulas for the improper integral $I(a,b;\lambda,\eta)$?

3. Identities for $I(a,b;k,\eta)$

In this section, we present several identities for the improper integral $I(a,b;k,\eta)$.

3.1. Substituting $s = \frac{1}{t}$ into $I(a,b;k,\eta)$ yields

$$\boxed{I(a,b;k,\eta) = \frac{(-1)^k}{\sqrt{ab}} I\left(\frac{1}{b}, \frac{1}{a}; k, 1-\eta\right)} \tag{3.1}$$

for $k \ge 0$, $\eta \in \mathbb{R}$, and $a, b > 0$ with $a \ne b$. In particular, it can be derived that

$$\boxed{I(a,b;0,1) = \frac{1}{\sqrt{ab}} I\left(\frac{1}{b}, \frac{1}{a}; 0, 0\right)}$$

and

$$\boxed{I\left(\frac{1}{b}, b; k, \eta\right) = (-1)^k I\left(\frac{1}{b}, b; k, 1-\eta\right).}$$

3.2. Substituting $s = \frac{t}{a}$ into $I(a,b;k,\eta)$ gives

$$\boxed{I(a,b;k,\eta) = \frac{1}{a^{\eta}} \left[(\ln^k a) I\left(1, \frac{b}{a}; 0, \eta\right) + I\left(1, \frac{b}{a}; k, \eta\right) \right]}$$

for $k \in \mathbb{N}$, $\eta \in \mathbb{R}$, and $a, b > 0$ with $a \ne b$. In particular,

$$\boxed{I(a,1;k,\eta) = \frac{1}{a^{\eta}} \left[(\ln^k a) I\left(1, \frac{1}{a}; 0, \eta\right) + I\left(1, \frac{1}{a}; k, \eta\right) \right].} \tag{3.2}$$

3.3. From (3.1), it follows that

$$\boxed{I(a,1;k,\eta) = \frac{(-1)^k}{\sqrt{a}} I\left(1, \frac{1}{a}; k, 1-\eta\right)} \tag{3.3}$$

Substituting (3.3) into (3.2) leads to

$$\boxed{I\left(1, \frac{1}{a}; k, \eta\right) = \frac{(-1)^k}{a^{\eta-1/2}} I\left(1, \frac{1}{a}; k, 1-\eta\right) - (\ln^k a) I\left(1, \frac{1}{a}; 0, \eta\right)}$$

for $1 \ne a > 0$, $k \in \mathbb{N}$, and $\eta \in \mathbb{R}$. Consequently,

$$\boxed{I(1,b;k,\eta) = \frac{(-1)^k}{b^{1/2-\eta}} I(1,b;k,1-\eta) + (\ln^k b) I(1,b;0,\eta)}$$

for $1 \ne b > 0$, $k \in \mathbb{N}$, and $\eta \in \mathbb{R}$.

4. Remarks

By the way, we list two remarks on (1.1) and integral representations of the weighted geometric means.

Remark 4.1. The integral representation (1.1) can be generalized as follows. For $a_k < a_{k+1}$ and $w_k > 0$ with $\sum_{k=1}^{n} w_k = 1$, the principal branch of the reciprocal of the weighted geometric mean $\prod_{k=1}^{n}(z+a_k)^{w_k}$ on $\mathbb{C} \setminus (-\infty, -a_1]$ can be represented by

$$\frac{1}{\prod_{k=1}^{n}(z+a_k)^{w_k}} = \frac{1}{\pi}\sum_{m=1}^{n-1}\sin\left(\pi\sum_{\ell=1}^{m} w_\ell\right)\int_{a_m}^{a_{m+1}} \frac{1}{\prod_{k=1}^{n}|t-a_k|^{w_k}}\frac{1}{t+z}\,\mathrm{d}\,t.$$

Remark 4.2. Before getting the integral representation (1.1), the following integral representation for the weight geometric mean $\prod_{k=1}^{n}(z+a_k)^{w_k}$ was obtained. Let $w_k > 0$ and $\sum_{k=1}^{n} w_k = 1$ for $1 \le k \le n$ and $n \ge 2$. If $a = (a_1, a_2, \dots, a_n)$ is a positive and strictly increasing sequence, that is, $0 < a_1 < a_2 < \cdots < a_n$, then the principal branch of the weighted geometric mean

$$G_{w,n}(a+z) = \prod_{k=1}^{n}(a_k+z)^{w_k}, \quad z \in \mathbb{C} \setminus (-\infty, -a_1]$$

has the Lévy–Khintchine expression

$$G_{w,n}(a+z) = G_{w,n}(a) + z + \int_0^\infty m_{a,w,n}(u)(1-e^{-zu})\,\mathrm{d}\,u, \tag{4.1}$$

where the density

$$m_{a,w,n}(u) = \frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\left(\pi\sum_{j=1}^{\ell} w_j\right)\int_{a_\ell}^{a_{\ell+1}}\prod_{k=1}^{n}|a_k-t|^{w_k}e^{-ut}\,\mathrm{d}\,t.$$

For more detailed information, please refer to [1, 6, 8, 9, 12, 13, 14, 15, 16] and closely-related references therein.

Remark 4.3. Letting $n = 2$ and $w_1 = w_2 = \frac{1}{2}$ in (4.1) or setting $n = 2$ in [14, Theorem 1.1] leads to

$$\begin{aligned}\sqrt{(z+a)(z+b)} &= \sqrt{ab} + z + \frac{1}{\pi}\int_0^\infty\left[\int_a^b\sqrt{(b-t)(t-a)}\,e^{-ut}\,\mathrm{d}\,t\right](1-e^{-zu})\,\mathrm{d}\,u\\ &= \sqrt{ab} + z + \frac{1}{\pi}\int_a^b\sqrt{(b-t)(t-a)}\left[\int_0^\infty e^{-ut}(1-e^{-zu})\,\mathrm{d}\,u\right]\mathrm{d}\,t\\ &= \sqrt{ab} + z + \frac{z}{\pi}\int_a^b\frac{\sqrt{(b-t)(t-a)}}{t}\frac{1}{t+z}\,\mathrm{d}\,t,\end{aligned}$$

that is,

$$\int_a^b\frac{\sqrt{(b-t)(t-a)}}{t}\frac{1}{t+z}\,\mathrm{d}\,t = \pi\left[\frac{\sqrt{(z+a)(z+b)}-\sqrt{ab}}{z}-1\right],$$

for $b > a > 0$. Taking the limit $z \to 0$ on both sides of (4.2) yields

$$\boxed{\int_a^b\frac{\sqrt{(b-t)(t-a)}}{t^2}\,\mathrm{d}\,t = \pi\left(\frac{a+b}{2\sqrt{ab}}-1\right) = \pi\left[\frac{A(a,b)}{G(a,b)}-1\right], \quad b > a > 0.} \tag{4.2}$$

For $k \in \mathbb{N}$, differentiating k times with respect to z procures

$$\begin{aligned}\frac{1}{\pi}\int_a^b\frac{\sqrt{(b-t)(t-a)}}{t}\frac{(-1)^k k!}{(t+z)^{k+1}}\,\mathrm{d}\,t &= \left[\frac{\sqrt{(z+a)(z+b)}-\sqrt{ab}}{z}\right]^{(k)}\\ &= \sqrt{ab}\left[\frac{1}{z}\left(\sqrt{1+\frac{a+b}{ab}z+\frac{1}{ab}z^2}-1\right)\right]^{(k)}\end{aligned}$$

$$= \sqrt{ab}\left[\frac{1}{z}\sum_{\ell=1}^{\infty}\left\langle\frac{1}{2}\right\rangle_{\ell}\frac{1}{\ell!}\left(\frac{a+b}{ab}z+\frac{1}{ab}z^2\right)^{\ell}\right]^{(k)}, \quad \left|\frac{a+b}{ab}z+\frac{1}{ab}z^2\right|<1$$

$$= \sqrt{ab}\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\frac{(2\ell-3)!!}{2^{\ell}}\frac{1}{\ell!}\frac{1}{(ab)^{\ell}}\left[z^{\ell-1}(a+b+z)^{\ell}\right]^{(k)}$$

$$= \sqrt{ab}\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\frac{(2\ell-3)!!}{2^{\ell}}\frac{1}{\ell!}\frac{1}{(ab)^{\ell}}\sum_{q=0}^{k}\binom{k}{q}\left(z^{\ell-1}\right)^{(q)}\left[(a+b+z)^{\ell}\right]^{(k-q)}$$

$$\to \sqrt{ab}\sum_{\ell=1}^{k+1}(-1)^{\ell-1}\frac{(2\ell-3)!!}{2^{\ell}}\frac{1}{\ell!}\frac{1}{(ab)^{\ell}}\binom{k}{\ell-1}(\ell-1)!\lim_{z\to0}\left[(a+b+z)^{\ell}\right]^{(k-\ell+1)}$$

$$= \sqrt{ab}\sum_{\ell=1}^{k+1}(-1)^{\ell-1}\frac{(2\ell-3)!!}{2^{\ell}}\frac{1}{\ell}\frac{1}{(ab)^{\ell}}\binom{k}{\ell-1}\langle\ell\rangle_{k-\ell+1}(a+b)^{2\ell-k-1}$$

$$= \frac{1}{(a+b)^{k-1}\sqrt{ab}}\sum_{\ell=0}^{k}(-1)^{\ell}\frac{(2\ell-1)!!}{2^{\ell+1}}\frac{1}{\ell+1}\binom{k}{\ell}\langle\ell+1\rangle_{k-\ell}\frac{(a+b)^{2\ell}}{(ab)^{\ell}}$$

$$= \frac{1}{(a+b)^{k-1}\sqrt{ab}}\sum_{\ell=0}^{k}(-1)^{\ell}\frac{(2\ell-1)!!}{2^{\ell+1}}\frac{1}{\ell+1}\binom{k}{\ell}\frac{(\ell+1)!}{(2\ell-k+1)!}\frac{(a+b)^{2\ell}}{(ab)^{\ell}}$$

$$= \frac{k!}{(a+b)^{k-1}\sqrt{ab}}\sum_{\ell=0}^{k}(-1)^{\ell}\frac{(2\ell-1)!!}{[2(\ell+1)]!!}\binom{\ell+1}{k-\ell}\frac{(a+b)^{2\ell}}{(ab)^{\ell}}$$

as $z\to0$. As a result, we have

$$\boxed{\int_a^b\frac{\sqrt{(b-t)(t-a)}}{t^{k+2}}\,\mathrm{d}\,t=\pi\frac{(-1)^k}{(a+b)^k}\sum_{\ell=0}^{k}(-1)^{\ell}\frac{(2\ell-1)!!}{[2(\ell+1)]!!}\binom{\ell+1}{k-\ell}\left(\frac{a+b}{\sqrt{ab}}\right)^{2\ell+1}}$$

for $b>a>0$ and $k\in\mathbb{N}$.

Remark 4.4. This paper is a slightly modified version of the preprint [10].

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[1]Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

[2]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China

[3]Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China

[4]Institute of Computer Science and Mathematics, Slovak University of Technology, Bratislava, Slovak Republic

[*]Corresponding author: qifeng618@gmail.com