SOME DISCUSSIONS ON A KIND OF IMPROPER INTEGRALS

FENG QI¹, ², ³, ∗ AND VIERA ČERŇANOVÁ⁴

ABSTRACT. In the paper, the improper integral

\[ I(a, b; \lambda, \eta) = \int_a^b \frac{1}{\sqrt{(t-a)(b-t)}} \ln^\lambda t \frac{1}{t^\eta} dt \]

for \( b > a > 0 \) and \( \lambda, \eta \in \mathbb{R} \) is discussed, some explicit formulas for special cases of \( I(a, b; \lambda, \eta) \) are presented, and several identities of \( I(a, b; k, \eta) \) for \( k \in \mathbb{N} \) are established.

1. Motivation

The motivation of this paper origins from investigating central Delanoy numbers in [11]. For proving the main result [11, Theorem 1.4], we need [11, Lemmas 2.4 and 2.5]. Lemma 2.4 in [11] states that, for \( b > a > 0 \) and \( z \in \mathbb{C} \setminus (-\infty, -a] \), the principal branch of the function \( \frac{1}{\sqrt{(z+a)(z+b)}} \) can be represented as

\[ \frac{1}{\sqrt{(z+a)(z+b)}} = \frac{1}{\pi} \int_a^b \frac{1}{\sqrt{(t-a)(b-t)}} \frac{1}{t+z} dt, \]

(1.1)

where \( \mathbb{C} \) denotes the complex plane. When taking \( z = 0 \), the integral representation (1.1) becomes

\[ \int_a^b \frac{1}{\sqrt{(t-a)(b-t)}} \frac{1}{t} dt = \frac{\pi}{\sqrt{ab}}, \quad b > a > 0. \]

(1.2)

Lemma 2.5 in [11] reads that the improper integral

\[ \int_{1/\alpha}^\alpha \frac{1}{\sqrt{(t-1/\alpha)(\alpha-t)}} \frac{\ln^{2k-1} t}{t^\beta} dt \begin{cases} < 0, & \beta > \frac{1}{2} \\ = 0, & \beta = \frac{1}{2} \\ > 0, & \beta < \frac{1}{2} \end{cases} \]

(1.3)

for all \( k \in \mathbb{N} \), where \( \alpha > 1 \) and \( \beta \in \mathbb{R} \).

Motivated by the above results, we naturally introduce the improper integral

\[ I(a, b; \lambda, \eta) = \int_a^b \frac{1}{\sqrt{(t-a)(b-t)}} \frac{\ln^\lambda t}{t^\eta} dt = \int_0^1 \frac{1}{\sqrt{s(1-s)}} \frac{\ln^\lambda [(b-a)s + a]}{[(b-a)s + a]^\eta} ds \]

for \( b > a > 0 \) and \( \lambda, \eta \in \mathbb{R} \) and consider a problem: how to compute the improper integral \( I(a, b; \lambda, \eta) \)?

2. Explicit formulas for special cases of \( I(a, b; \lambda, \eta) \)

In this section, we present several explicit formulas for special cases of the improper integral \( I(a, b; \lambda, \eta) \).

In the monograph [4], we do not find such a kind of integrals \( I(a, b; \lambda, \eta) \) for general \( b > a > 0 \) and \( \lambda, \eta \in \mathbb{R} \).

2010 Mathematics Subject Classification. Primary 40C10; Secondary 26A39, 26A42, 33C05, 33C20, 33E05, 33E20, 40A10, 97I50.

Key words and phrases. improper integral; explicit formula; identity; hypergeometric function; generalized hypergeometric series.

©2016 Authors retain the copyrights of their papers, and all open access articles are distributed under the terms of the Creative Commons Attribution License.
2.1. From (1.1) or (1.2), it follows that
\[
I(a, b; 0, 1) = \frac{\pi}{\sqrt{ab}}, \quad b > a > 0.
\] (2.1)

2.2. From (1.3), it follows that
\[
I\left(a, \frac{1}{a}; 2k - 1, \frac{1}{2}\right) = 0, \quad 0 < a < 1, \quad k \in \mathbb{N}.
\]

2.3. It is straightforward by using Euler’s substitution that
\[
I(a, b; 0, 0) = \int_0^1 \frac{1}{\sqrt{s(1-s)}} \, ds = \pi, \quad b > a > 0.
\]

2.4. When \(\lambda = 0, \eta \neq 0,\) and \(2a > b > a > 0,\) we have
\[
I(a, b; 0, \eta) = \frac{1}{a^n} \int_0^1 \frac{[1 + (b/a - 1)s]^{-\eta}}{\sqrt{s(1-s)}} \, ds
\]
\[
= \frac{1}{a^n} \int_0^1 (1-s)^{-1/2} \sum_{\ell=0}^{\infty} (-\eta)_\ell \left(\frac{b}{a} - 1\right)^\ell \frac{s^{\ell-1/2}}{\ell!} \, ds
\]
\[
= \frac{1}{a^n} \sum_{\ell=0}^{\infty} \frac{(-\eta)_\ell}{\ell!} \left(\frac{b}{a} - 1\right)^\ell \int_0^1 (1-s)^{-1/2} s^{\ell-1/2} \, ds
\]
\[
= \frac{1}{a^n} \sum_{\ell=0}^{\infty} \frac{(-\eta)_\ell}{\ell!} \left(\frac{b}{a} - 1\right)^\ell B\left(\frac{1}{2}, \ell + \frac{1}{2}\right)
\]
\[
= \frac{1}{a^n} \sum_{\ell=0}^{\infty} \frac{(\eta)_\ell (1/2)_\ell}{(1)_\ell \ell!} \left(1 - \frac{b}{a}\right)^\ell
\]
\[
= \frac{\pi}{a^n} 2F_1 \left(\eta, \frac{1}{2}; 1; 1 - \frac{b}{a}\right),
\]
where
\[
\langle x \rangle_n = \prod_{k=0}^{n-1} (x-k) = \begin{cases} x(x-1)\cdots(x-n+1), & n \geq 1 \\ 1, & n = 0 \end{cases}
\]
and
\[
(x)_\ell = \prod_{k=0}^{\ell-1} (x+k) = \begin{cases} x(x+1)(x+2)\cdots(x+\ell-1), & \ell \geq 1 \\ 1, & \ell = 0 \end{cases}
\]
are respectively called the falling and rising factorials of \(x \in \mathbb{R},\) the function \(B(x, y)\) denotes the classical beta function, and \(2F_1\) are the classical hypergeometric functions which are special cases of the generalized hypergeometric series
\[
_\rho F_\eta(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}
\]
for complex numbers \(a_i \in \mathbb{C}\) and \(b_i \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}\) and for positive integers \(p, q \in \mathbb{N}.\) This result
\[
I(a, b; 0, \eta) = \frac{\pi}{a^n} 2F_1 \left(\eta, \frac{1}{2}; 1; 1 - \frac{b}{a}\right), \quad \eta \neq 0, \quad 2a > b > a > 0
\]
can also be found in [3, p. xv, eq. (12)].
2.5. When \( \lambda = k \in \mathbb{N} \) and \( 2a > b > a > 0 \), the function \( \ln^k[(b-a)s + a] \) can be rewritten as

\[
\ln^k[(b-a)s + a] = \left( \ln a + \ln \left[ 1 + \left( \frac{b}{a} - 1 \right) s \right] \right)^k
\]

\[
= \sum_{\ell=0}^{k} \binom{k}{\ell} \ln^{k-\ell} a \ln^{\ell} \left[ 1 + \left( \frac{b}{a} - 1 \right) s \right]
\]

\[
= (\ln^k a) \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{(-1)^\ell}{\ln^\ell a} \left[ \sum_{m=1}^{\infty} \frac{1}{m} \left( 1 - \frac{b}{a} \right)^m s^m \right]^\ell
\]

\[
= (\ln^k a) \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{(-1)^\ell}{\ln^\ell a} s^\ell \left[ \sum_{m=0}^{\infty} \frac{1}{m+1} \left( 1 - \frac{b}{a} \right)^{m+1} s^m \right] \quad .
\]

When \( 0 < a < b < 1 \) or \( 1 < a < b < a^2 \), if \( \lambda \in \mathbb{R} \), then

\[
\ln^\lambda[(b-a)s + a] = (\ln^\lambda a) \left( 1 + \frac{\ln[1+(b/a-1)s]}{\ln a} \right)^\lambda
\]

\[
= (\ln^\lambda a) \sum_{\ell=0}^{\infty} \frac{(\lambda)_\ell}{\ell!} \left( \frac{\ln[1+(b/a-1)s]}{\ln a} \right)^\ell
\]

\[
= (\ln^\lambda a) \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (\lambda)_\ell}{\ell! \ln^\ell a} \left[ \sum_{m=1}^{\infty} \frac{1}{m} \left( 1 - \frac{b}{a} \right)^m s^m \right]^\ell
\]

\[
= (\ln^\lambda a) \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (\lambda)_\ell}{\ell! \ln^\ell a} s^\ell \left[ \sum_{m=0}^{\infty} \frac{1}{m+1} \left( 1 - \frac{b}{a} \right)^{m+1} s^m \right]^\ell \quad .
\]

In [4, p. 18, 0.314], it was stated that

\[
\left( \sum_{k=0}^{\infty} a_k x^k \right)^n = \sum_{k=0}^{\infty} c_{n,k} x^k,
\]

where \( c_{n,0} = a_0^n \) and

\[
c_{n,m} = \frac{1}{m} \sum_{k=1}^{m} (kn - m + k) a_k c_{n,m-k}, \quad m \in \mathbb{N}.
\]

Hence, it follows that

\[
\left[ \sum_{m=0}^{\infty} \frac{1}{m+1} \left( 1 - \frac{b}{a} \right)^{m+1} s^m \right]^\ell = \sum_{m=0}^{\infty} c_{\ell,m} x^m,
\]

where \( c_{\ell,0} = (1 - \frac{b}{a})^\ell \) and

\[
c_{\ell,m} = \frac{1}{m} \sum_{k=1}^{m} k \ell - m + k \left( 1 - \frac{b}{a} \right)^k c_{\ell,m-k}
\]

\[
= \frac{1}{m} \left( 1 - \frac{b}{a} \right)^\ell \sum_{p=0}^{m} \left( \frac{m \ell - (\ell + 1)p}{m - p + 1} \left( 1 - \frac{b}{a} \right)^{-p} c_{\ell,p}
\]

for \( m \in \mathbb{N} \). Let \( C_{\ell,m} = (1 - \frac{b}{a})^{-m} c_{\ell,m} \), the above recursive formula becomes

\[
C_{\ell,m} = \frac{1}{m} \sum_{p=0}^{m-1} \frac{m \ell - p(\ell + 1)}{m - p} C_{\ell,p}
\] (2.2)
with $C_{\ell,0} = c_{\ell,0}$. Starting out from these points, it is much possible to find explicit formulas for computing the integral $I(a, b; \lambda, \eta)$. For example, when $\lambda \neq 0$ and $\eta = 1$,

$$I(a, b; \lambda, 1) = \frac{1}{(\lambda + 1)(b - a)} \int_0^1 \frac{1}{\sqrt{s(1 - s)}} \frac{d \ln^{\lambda + 1}[(b - a)s + a]}{d s} ds$$

$$= \frac{\ln^{\lambda + 1} a}{(\lambda + 1)(b - a)} \sum_{\ell = 0}^\infty \frac{(-1)^\ell (\lambda + 1)_\ell}{\ell! \ln^{\ell} a} \int_0^1 \frac{1}{\sqrt{s(1 - s)}} d s \left[ \sum_{m = 0}^\infty \frac{1}{m} \left( 1 - \frac{b}{a} \right)^m s^m \right] \ell ds$$

$$= \frac{\ln^{\lambda + 1} a}{(\lambda + 1)(b - a)} \sum_{\ell = 0}^\infty \frac{(-1)^\ell (\lambda + 1)_\ell}{\ell! \ln^{\ell} a} \int_0^1 (1 - s)^{-1/2} s^{m-1/2} ds$$

$$= \frac{\pi \ln^{\lambda + 1} a}{(\lambda + 1)(b - a)} \sum_{\ell = 0}^\infty \frac{(-1)^\ell (\lambda + 1)_\ell}{\ell! \ln^{\ell} a} \sum_{m = 0}^\infty (m + 1)c_{\ell, m + 1} B\left( \frac{1}{2}, m + \frac{1}{2} \right).$$

Hence, it would be important to derive a general formula for the recursive relation (2.2).

2.6. For $k \geq 0$, differentiating with respect to $z$ on both sides of (1.1) gives

$$\frac{d^k}{dz^k} \frac{1}{\sqrt{(z + a)(z + b)}} = (-1)^k k! \int_a^b \frac{1}{\sqrt{(t - a)(b - t)}} \frac{1}{(t + z)^{k+1}} dt.$$  \hspace{1cm} (2.3)

By the Faà di Bruno formula

$$\frac{d^n}{dt^n} f \circ h(t) = \sum_{k = 0}^n f^{(k)}(h(t))B_n,k(h'(t), h''(t), \ldots, h^{(n-k+1)}(t)), \quad n \geq 0$$

in [2, p. 139, Theorem C], where

$$B_n,k(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{1 \leq \ell_1, \ell_2, \ldots, \ell_n \in \mathbb{N} \cup \{0\}} \frac{n!}{\prod_{i=1}^{n} \ell_i ! \prod_{i=1}^{n} \ell_i !} \prod_{i=1}^{n} \left( \frac{x_i}{\ell_i} \right), \quad n \geq k \geq 0$$

is called [2, p. 134, Theorem A] the Bell polynomials of the second kind, we obtain

$$\frac{d^k}{dz^k} \frac{1}{\sqrt{(z + a)(z + b)}} = \sum_{\ell = 0}^k \left[ \frac{1}{\sqrt{a}} \right]^{(\ell)} B_{k,\ell}(u'(z), u''(z), 0, \ldots, 0)$$

$$= \frac{1}{2} \sum_{\ell = 0}^k \left[ \frac{1}{\sqrt{a}} \right]^{(\ell)} B_{k,\ell}(2z + a + b, 2, 0, \ldots, 0)$$

$$= \frac{1}{2} \sum_{\ell = 0}^k \left[ \frac{1}{\sqrt{a}} \right]^{(\ell)} B_{k,\ell}(2z + a + b, 2, 0, \ldots, 0)$$

as $z \to 0$, where $u = u(z) = (z + a)(z + b)$. Recall from [2, p. 135] that

$$B_n,k(abx_1, ab^2x_2, \ldots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_n,k(x_1, x_2, \ldots, x_{n-k+1}),$$  \hspace{1cm} (2.4)
where \(a\) and \(b\) are any complex numbers and \(n \geq k \geq 0\). Recall from [5, Theorem 4.1], [17, Theorem 3.1],
and [18, Lemma 2.5] that
\[
B_{n,k}(x,1,0,\ldots,0) = \frac{(n-k)!}{2^{n-k}} \binom{n}{k} \left(\frac{k}{n-k}\right)^{x^{2k-n}}, \quad n \geq k \geq 0.
\] (2.5)
Accordingly, by (2.4) and (2.5), it follows that
\[
\lim_{z \to 0} \frac{d^k}{dz^k} \frac{1}{\sqrt{(z+a)(z+b)}} = \sum_{\ell=0}^{k} \left(\frac{-1}{2}\right) \frac{1}{\ell (ab)^{\ell+1/2}} 2^\ell B_{k,\ell} \left(\frac{a+b}{2}, 1, 0, \ldots, 0\right)
\]
\[
= \sum_{\ell=0}^{k} \left(\frac{-1}{2}\right) \frac{1}{\ell (ab)^{\ell+1/2}} 2^\ell (k-\ell)! \binom{k}{\ell} \left(\frac{a+b}{2}\right)^{2\ell-k}.
\]
Letting \(z \to 0\) on both sides of (2.3), employing the above result, and simplifying lead to
\[
\int_{a}^{b} \frac{1}{\sqrt{(t-a)(b-t)}} \frac{1}{t^{k+1}} dt = \frac{(-1)^k \pi}{(a+b)^k \sqrt{ab}} \sum_{\ell=0}^{k} \left(\frac{-1}{2}\right) 2^\ell \frac{(2\ell-1)!}{(2\ell)!} \binom{k}{\ell} \left(\frac{a+b}{2}\right)^{\ell} \left(\frac{1/a + 1/b}{2}\right)^{\ell},
\]
that is,
\[
I(a, b; 0, k+1) = \frac{\pi}{G(a, b)} \frac{(-1)^k}{[2A(a, b)]^k} \sum_{\ell=0}^{k} \left(\frac{-1}{2}\right) 2^\ell \frac{(2\ell-1)!}{(2\ell)!} \binom{k}{\ell} \left[A(a, b)\right]^\ell \left[H(a, b)\right]^{-\ell},
\] (2.6)
for \(b > a > 0\) and \(k \geq 0\), where \(\binom{p}{q} = 0\) for \(q > p \geq 0\), the double factorial of negative odd integers
\((-2n+1)!!\) is defined by
\[
(-2n-1)!! = \frac{(-1)^n}{(2n-1)!!} = \frac{(-1)^n 2^n n!}{(2n)!}, \quad n = 0, 1, \ldots,
\]
and the quantities
\[
A(a, b) = \frac{a+b}{2}, \quad G(a, b) = \sqrt{ab}, \quad \text{and} \quad H(a, b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}
\]
are respectively the well-known arithmetic, geometric, and harmonic means of two positive numbers \(a\) and \(b\).

When \(k = 0\) in (2.6), the integral (1.2) or (2.1) is recovered.

In fact, the above argument implies that
\[
\int_{a}^{b} \frac{1}{\sqrt{(t-a)(b-t)}} \frac{1}{t^{k+1}} dt = \frac{(-1)^k}{[2A(z+a, z+b)]^k} \frac{\pi}{G(z+a, z+b)}
\]
\[
\times \sum_{\ell=0}^{k} \left(\frac{-1}{2}\right) 2^\ell \frac{(2\ell-1)!}{(2\ell)!} \binom{k}{\ell} \left[A(z+a, z+b)\right]^\ell \left[H(z+a, z+b)\right]^{-\ell}
\]
for \(b > a > 0\) and \(k \geq 0\). This is equivalent to (2.6).

By the way, the ratio \(\frac{(2n)!!}{(2n)!!}\) is called the Wallis ratio. For more information, please refer to the paper [7] and plenty of references cited therein.

Alternatively differentiating with respect to \(z\) on both sides of (1.1) leads to
\[
\frac{d^k}{dz^k} \frac{1}{\sqrt{(z+a)(z+b)}} = \frac{d^k}{dz^k} \left(\frac{1}{\sqrt{z+a} \sqrt{z+b}}\right)
\]
\[
= \sum_{\ell=0}^{k} \binom{k}{\ell} \left(\frac{1}{\sqrt{z+a}}\right)^{\ell} \left(\frac{1}{\sqrt{z+b}}\right)^{(k-\ell)}
\]
more explicit formulas for the improper integral \( I \) under different conditions from those discussed above on \( 2.7 \).

Substituting this into (2.3) and taking the limit \( z \to 0 \) result in

\[
I(a, b; 0, k + 1) = \frac{\pi}{\sqrt{ab}} \frac{1}{b^k} \sum_{\ell=0}^{k} \left( 2\ell - 1 \right)! [2(k - \ell) - 1]!! \left( \frac{b}{a} \right)^{\ell}
\]

for \( b > a > 0 \) and \( k \geq 0 \). This is an alternative expression for \( I(a, b; 0, k + 1) \).

2.7. Under different conditions from those discussed above on \( b > a > 0 \) and \( \lambda, \eta \in \mathbb{R} \), can one discover more explicit formulas for the improper integral \( I(a, b; \lambda, \eta) \)?

3. Identities for \( I(a, b; k, \eta) \)

In this section, we present several identities for the improper integral \( I(a, b; k, \eta) \).

3.1. Substituting \( s = \frac{1}{k} \) into \( I(a, b; k, \eta) \) yields

\[
I(a, b; k, \eta) = (-1)^k \sqrt{ab} I \left( \frac{1}{b}, \frac{1}{a}; k, 1 - \eta \right)
\]

for \( k \geq 0 \), \( \eta \in \mathbb{R} \), and \( a, b > 0 \) with \( a \neq b \). In particular, it can be derived that

\[
I(a, b; 0, 1) = \frac{1}{\sqrt{ab}} I \left( \frac{1}{b}, \frac{1}{a}; 0, 0 \right)
\]

and

\[
I \left( \frac{1}{b}, b; k, \eta \right) = (-1)^k I \left( \frac{1}{b}, b; k, 1 - \eta \right).
\]

3.2. Substituting \( s = \frac{1}{a} \) into \( I(a, b; k, \eta) \) gives

\[
I(a, b; k, \eta) = \frac{1}{a^\eta} \left[ (\ln k) a I \left( 1, \frac{b}{a}; 0, \eta \right) + I \left( 1, \frac{b}{a}; k, \eta \right) \right]
\]

for \( k \in \mathbb{N} \), \( \eta \in \mathbb{R} \), and \( a, b > 0 \) with \( a \neq b \). In particular,

\[
I(a, 1; k, \eta) = \frac{1}{a^\eta} \left[ (\ln k) a I \left( 1, \frac{1}{a}; 0, \eta \right) + I \left( 1, \frac{1}{a}; k, \eta \right) \right].
\]

3.3. From (3.1), it follows that

\[
I(a, 1; k, \eta) = (-1)^k \sqrt{a} I \left( 1, \frac{1}{a}; k, 1 - \eta \right)
\]

Substituting (3.3) into (3.2) leads to

\[
I \left( 1, \frac{1}{a}; k, \eta \right) = (-1)^k \frac{1}{a^\eta - 1/2} I \left( 1, \frac{1}{a}; k, 1 - \eta \right) - (\ln k) a I \left( 1, \frac{1}{a}; 0, \eta \right)
\]

for \( 1 \neq a > 0 \), \( k \in \mathbb{N} \), and \( \eta \in \mathbb{R} \). Consequently,

\[
I(1, b; k, \eta) = \frac{(-1)^k}{b^{1/2 - \eta}} I(1, b; k, 1 - \eta) + (\ln b) I(1, b; 0, \eta)
\]
for $1 \neq b > 0$, $k \in \mathbb{N}$, and $\eta \in \mathbb{R}$.

4. Remarks

By the way, we list two remarks on (1.1) and integral representations of the weighted geometric means.

Remark 4.1. The integral representation (1.1) can be generalized as follows. For $a_k < a_{k+1}$ and $w_k > 0$ with $\sum_{k=1}^{n} w_k = 1$, the principal branch of the reciprocal of the weighted geometric mean $\prod_{k=1}^{n} (z + a_k)^{w_k}$ on $\mathbb{C} \setminus (-\infty, -a_1)$ can be represented by

$$
\frac{1}{\prod_{k=1}^{n} (z + a_k)^{w_k}} = \frac{1}{\pi} \sum_{m=1}^{n-1} \sin \left( \frac{\pi}{n} \sum_{j=1}^{m} w_j \right) \int_{a_m}^{a_{m+1}} \prod_{k=1}^{n} |t - a_k|^{w_k - 1} \frac{1}{t + z} \, dt.
$$

Remark 4.2. Before getting the integral representation (1.1), the following integral representation for the weighted geometric mean $\prod_{k=1}^{n} (z + a_k)^{w_k}$ was obtained. Let $w_k > 0$ and $\sum_{k=1}^{n} w_k = 1$ for $1 \leq k \leq n$ and $n \geq 2$. If $a = (a_1, a_2, \ldots, a_n)$ is a positive and strictly increasing sequence, that is, $0 < a_1 < a_2 < \cdots < a_n$, then the principal branch of the weighted geometric mean

$$G_{w,n}(a + z) = \prod_{k=1}^{n} (a_k + z)^{w_k}, \quad z \in \mathbb{C} \setminus (-\infty, -a_1]$$

has the Lévy–Khintchine expression

$$G_{w,n}(a + z) = G_{w,n}(a) + z + \int_{0}^{\infty} m_{a,w,n}(u)(1 - e^{-zu}) \, du, \quad \text{(4.1)}$$

where the density

$$m_{a,w,n}(u) = \frac{1}{\pi} \sum_{m=1}^{n-1} \sin \left( \frac{\pi}{n} \sum_{j=1}^{m} w_j \right) \int_{a_m}^{a_{m+1}} \prod_{k=1}^{n} |a_k - t|^{w_k} e^{-ut} \, dt.$$\

For more detailed information, please refer to [1, 6, 8, 9, 12, 13, 14, 15, 16] and closely-related references therein.

Remark 4.3. Letting $n = 2$ and $w_1 = w_2 = \frac{1}{2}$ in (4.1) or setting $n = 2$ in [14, Theorem 1.1] leads to

$$\sqrt{(z + a)(z + b)} = \sqrt{ab} + z + \frac{1}{\pi} \int_{0}^{\infty} \left[ \int_{z}^{\sqrt{(b - t)(t - a)}} e^{-ut} \, dt \right] \left(1 - e^{-zu} \right) \, du$$

$$= \sqrt{ab} + z + \frac{1}{\pi} \int_{z}^{\sqrt{(b - t)(t - a)}} e^{-ut} \left(1 - e^{-zu} \right) \, du$$

$$= \sqrt{ab} + z + \frac{1}{\pi} \int_{z}^{\sqrt{(b - t)(t - a)}} \frac{1}{t + z} \, dt,$$

that is,

$$\int_{z}^{\sqrt{(b - t)(t - a)}} \frac{1}{t + z} \, dt = \pi \left[ \frac{\sqrt{(z + a)(z + b)} - \sqrt{ab}}{z} - 1 \right],$$

for $b > a > 0$. Taking the limit $z \to 0$ on both sides of (4.2) yields

$$\int_{a}^{b} \frac{\sqrt{(b - t)(t - a)}}{t^2} \, dt = \pi \left( \frac{a + b}{2\sqrt{ab}} - 1 \right) = \pi \left( \frac{A(a,b)}{G(a,b)} - 1 \right), \quad b > a > 0. \quad \text{(4.2)}$$

For $k \in \mathbb{N}$, differentiating $k$ times with respect to $z$ procures

$$\frac{1}{\pi} \int_{a}^{b} \frac{\sqrt{(b - t)(t - a)}}{t} \left( \frac{(-1)^k k!}{(t + z)^{k+1}} \right) \left[ \frac{\sqrt{(z + a)(z + b)} - \sqrt{ab}}{z} \right]^k \, dt$$

$$= \sqrt{ab} \left[ \frac{1}{z} \left( \sqrt{1 + \frac{a + b}{ab} z + \frac{1}{ab} z^2} - 1 \right) \right]^k.$$
\[
\sqrt{ab} \left[ \frac{1}{z} \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} \left( \frac{a + b}{ab} z + \frac{1}{ab} z^2 \right)^\ell \right]^{(k)}, \quad \left| \frac{a + b}{ab} z + \frac{1}{ab} z^2 \right| < 1
\]

\[
= \sqrt{ab} \sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{(2\ell-3)!!}{2^\ell} \frac{1}{\ell! (ab)^\ell} \sum_{q=0}^{k} \binom{k}{q} (z^{-1})^{(q)} [(a + b + z)^\ell]^{(k-q)}
\]

\[
\rightarrow \sqrt{ab} \sum_{\ell=1}^{k+1} (-1)^{\ell-1} \frac{(2\ell-3)!!}{2^\ell} \frac{1}{\ell! (ab)^\ell} \binom{k}{\ell-1} (\ell)(\ell-1)! \lim_{z \rightarrow 0} [(a + b + z)^\ell]^{(k-\ell+1)}
\]

\[
= \sqrt{ab} \sum_{\ell=1}^{k+1} (-1)^{\ell-1} \frac{(2\ell-3)!!}{2^\ell} \frac{1}{\ell! (ab)^\ell} \binom{k}{\ell-1} (\ell)(\ell-1)!(a + b + z)^{2\ell-k-1}
\]

\[
= \frac{1}{(a + b)^{k-1} \sqrt{ab}} \sum_{\ell=0}^{k} (-1)^{\ell} \frac{(2\ell-1)!!}{2^{\ell+1}} \frac{1}{\ell+1} \binom{k}{\ell+1} (a + b + z)^{2\ell - k}
\]

\[
= \frac{1}{(a + b)^{k-1} \sqrt{ab}} \sum_{\ell=0}^{k} (-1)^{\ell} \frac{(2\ell-1)!!}{2^{\ell+1}} \frac{1}{\ell+1} \binom{k}{\ell+1} (a + b + z)^{2\ell - k}
\]

as \(z \rightarrow 0\). As a result, we have

\[
\int_a^b \frac{\sqrt{(b-t)(t-a)}}{t^{k+2}} \, dt = \pi (-1)^k \frac{1}{(a + b)^k} \sum_{\ell=0}^{k} (-1)^{\ell} \frac{(2\ell-1)!!}{2^{\ell+1}} \frac{1}{\ell+1} (a + b)^{2\ell+1}
\]

for \(b > a > 0\) and \(k \in \mathbb{N}\).

**Remark 4.4.** This paper is a slightly modified version of the preprint [10].

**References**


1 Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China
2 College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China
3 Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China
4 Institute of Computer Science and Mathematics, Slovak University of Technology, Bratislava, Slovak Republic

*Corresponding author: qifeng618@gmail.com