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The New Dagum-X Family of Distributions: Properties and Applications

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Abstract. Various statistical distributions are still being used extensively over the previous decades for modeling data in numerous areas such as engineering, sciences, and finance. Nonetheless, in a lot of applied areas, there is a continuous need for expanded forms of these distributions. However, many common distributions do not fit the data well. Thus, new distributions have been constructed in literature. The purpose of this article is to present a new family of distributions using the Dagum distribution as a generator and to study its properties such as hazard rate functions, moments, quantile function, ordered statistics and Renyi entropy. Moreover, one sub model called Dagum-Frechet distribution is discussed with some of its properties. The maximum likelihood estimation is employed to estimate the parameters of the proposed distribution, and the confidence intervals are obtained. Finally, two real data sets are analyzed to illustrate the performance of the purposed distribution.

1. Introduction

Statistical literature is abounding with many statistical distributions that are used for data modeling in various areas of applied life, such as engineering, actuarial sciences, education, demography, economics, finance, insurance, environmental, medical, and biological studies. The quality of statistical distribution is based on fitting the assumed probability distribution to the data. However, there are various issues where any of these distributions do not fit the data appropriately, especially in the areas of engineering, finance, medicine and environmental hazards. Therefore, a significant effort has been made in developing different families of distributions. Recently, there has been a growing interest of generating wide families of distributions from existing families of distributions by adding one or more additional parameter(s) to the baseline distribution. There are a lot of well-known family of distributions, such as Beta-G by [1], KumaraSwamy-G by [2], Exponentiated generalized-G by [3], Gamma-X

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family by [4] and Logistic-X by [5]. Moreover, [6] and [7] introduced the odd Lomax-G family of distributions and the Lomax Gumbel distribution respectively. The Zubair-G family of distributions was studied by [8] and the Zubair-Weibull distribution is obtained. [9] proposed the exponentiated Gumbel family of distributions and evolved three separate models.

The transformed transformer (T-X) method is considered as one of the most important ways to generalize distributions, which many have relied on in their researches. It is introduced by [10] for generating families of continuous distributions.

Let $r(t)$ be the probability density function (PDF) of a random variable T, where $\mathrm{T} \in [a, b]$, for $-\infty \leq a \leq b \leq \infty$. Assume W(G($x$)) be a function of the cumulative density function (CDF) G(x) of any random variable X, where W(G(x)) satisfies the following:

i. $\mathrm{W}(\mathrm{G}(x)) \in [a, b]$.
ii. W(G(x)) is differentiable and monotonically nondecreasing.
iii. $\mathrm{W}(\mathrm{G}(x)) \rightarrow a$ as $x \rightarrow -\infty$ and $\mathrm{W}(\mathrm{G}(x)) \rightarrow b$ as $x \rightarrow \infty$.

The CDF and PDF of the T-X family of distributions are given respectively as:

$$F(x) = \int_a^{W(G(x))} r(t)dt, \tag{1.1}$$

$$f(x) = \left[\frac{d}{dx} W(G(x))\right] r[W(G(x))]. \tag{1.2}$$

The definition of W(G(x)) depends on the support of the random variable T as follows:

(1) When the support of T is bounded: W(G(x)) can be defined as $G(x)$ or $G(x)^{\alpha}$.
(2) When the support of T is $[a,\infty)$, for $a \geq 0$: W(G(x)) can be defined as $-log(1 - G(x))$ or $G(x)/(1 - G(x))$ or $-log(1 - G(x)^{\alpha})$.
(3) When the support of T is $(-\infty,\infty)$: W(G(x)) can be defined as $log[-log(1 - G(x))]$ or $log[G(x)/(1 - G(x))]$ ([10]).

The main purpose of this article is to introduce a new family of distributions, called the Dagum-X family of distributions that are more adaptable to data in a wide range of applications. This article is organized as follows: In Section 2, Dagum-X family of distribution is defined. Also, its propability and cumulative distribution functions are introduced. Some special models of this family are presented in Section 3. Section 4 shows some mathematical properties of the Dagum-X family of distributions including survival, hazard function, rth moments, quantile function, Renyi entropy and order statistics. The characterizations of one sub-model of this family are studied in Section 5. The maximum likelihood method for parameter estimation is discussed in section 6. In section 7 simulation studies will be conducted to show the performance of the maximum likelihood estimation (MLE) method. Section 8 provides a real data application to show the flexibility of the Dagum-X family. Finally, concluding remarks are presented in Section 9.

2. The Dagum - X Family

The importance of a statistical model lies in the fitness of the probability distribution to the data. Thus, different families of probability distributions have been developed for fitting different types of data. However, there are still several constraints that affect on fitting the formed distributions to the data appropriately especially in particular applications.

As mentioned earlier, Dagum distribution has received interest from researchers because of its competition with other models. Different extensions that include Dagum distribution have been proposed and developed using different approaches in attempt to provide more flexibility in fitting data. Using the kurtosis diagram provided by [11] and [12], [13] presented the log-Dagum distribution and examined the changes in the kurtosis. More structural properties and parameter estimates for the log-Dagum distribution were addressed by [14]. [15] proposed a new class of distributions called Mc-Dagum distribution. Several distributions, including the beta-Dagum, beta-Burr III, beta-Fisk, Dagum, Burr III, and Fisk distributions, are included in this class of distributions as special cases. They obtained the properties of the model and the maximum likelihood estimates of the model parameters. [16] proposed a new class of weighted Dagum and related distributions and discussed this class in detail. [17] studied a new five-parameter model called the extended Dagum distribution and discussed the features of the model. The proposed model contains as special cases the log-logistic and Burr III distributions among others. [18] proposed a new four parameter distribution called the Dagum-Poisson (DP) distribution by compounding Dagum and Poisson distributions. The structural properties and the maximum likelihood estimates (MLEs) of the parameters are obtained. [19] proposed the exponentiated generalized exponential Dagum distribution. There are several sub-models in this family of distributions, including the Dagum distribution, Burr III distribution, exponentiated generalized Dagum distribution, Fisk distribution, and exponentiated generalized exponential Burr III distribution. [20] introduced a new model called a power log-Dagum distribution. The model consists of many new sub-models such as: linear log-Dagum, power logistic, log-Dagum distributions and linear logistic among them. Three distinct estimating procedures are given along with the model's properties. The odd Dagum-G family, which [21] introduced, is a new family of continuous distributions with three additional shape parameters. The properties of the suggested family and the model parameters estimates are attained.

Using the T-X approach, several new distributions have been introduced in the literature. We generalized the Dagum distribution using T-X method by [10]. The new family of Dagum distribution, called Dagum-X, can be defined as follows: Let $F(t)$ and $f(t)$ be the CDF and the PDF for a Dagum random variable $\mathrm{T} \in [0, \infty)$, given by

$$F(t;\lambda,\delta,\beta) = (1+\lambda t^{-\delta})^{-\beta} \quad , t>0, \quad \lambda,\delta,\beta>0, \tag{2.1}$$

and

$$f(t;\lambda,\delta,\beta) = \beta\lambda\delta t^{-\delta-1}(1+\lambda t^{-\delta})^{-\beta-1} \quad , t>0, \quad \lambda,\delta,\beta>0, \tag{2.2}$$

where $\geq$ is a scale parameter and $\circ$ and $\odot$ are shape parameters.

By replacing t in Equation (2.1) by the $W(G(X)) = \frac{G(x;\underline{\theta})}{\overline{G}(x;\underline{\theta})}$, we obtained the CDF of a new family namely, Dagum-X family, where $G(x;\underline{\theta})$ and $\overline{G}(x;\underline{\theta})$=1-$G(x;\underline{\theta})$ are the baseline CDF and survival function (SF) depending on a parameter vector $\underline{\theta}$.

$$F(x;\lambda,\delta,\beta,\underline{\theta}) = \left(1+\lambda\left[\frac{G(x;\underline{\theta})}{\overline{G}(x;\underline{\theta})}\right]^{-\delta}\right)^{-\beta}, \tag{2.3}$$

The PDF is obtained by differentiating Equation (2.3) with respect to (w. r. t.) x as follows:

$$f(x;\lambda,\delta,\beta,\underline{\theta}) = \beta\lambda\delta g(x)\frac{[G(x)]^{-\delta-1}}{[1-G(x)]^{-\delta+1}}\left[1+\lambda\left(\frac{G(x)}{1-G(x)}\right)^{-\delta}\right]^{-\beta-1}, \quad t>0, \quad \lambda,\delta,\beta>0. \tag{2.4}$$

3. Special Models

One of the main reasons for the desire to generate different families of distributions is to provide different extensions of appropriate distributions that are more flexible to use with data in various applications. In this section, some Dagum-X special distributions are introduced, such as Dagum-Weibull(D-W), Dagum-exponential(D-exp), Dagum-Rayleigh(D-R) and Dagum-Fréchet(D-Fr).

3.1. **The Dagum- Weibull distribution.**

The Weibull distribution is one of the lifetime distributions that is most frequently used in different areas, such as economics, biology, hydrology and engineering sciences due to its simplicity and versatility. It generalizes the exponential model to include non constant failure rate functions. In particular, it encompasses both increasing and decreasing failure rate functions. As it is well known that Weibull distribution (with scale and shape parameters a, $b>0$) has CDF and PDF given by:

$$G(x;a,b) = 1-e^{-(\frac{x}{a})^b}, \quad x>0, \quad a,b>0 \tag{3.1}$$

and

$$g(x;a,b) = \frac{b}{a^b}x^{b-1}e^{-(\frac{x}{a})^b}, \quad x>0, \quad a,b>0 \tag{3.2}$$

The CDF and PDF of Dagum-Weibull distribution can be obtained by substituting Equations (3.1) and Equations (3.2) in Equations (2.3) and Equations (2.4) as follows:

$$F(x;\lambda,\delta,\beta,a,b) = \left(1+\lambda\left[\frac{1-e^{-(\frac{x}{a})^b}}{e^{-(\frac{x}{a})^b}}\right]^{-\delta}\right)^{-\beta}, \quad x>0, \quad \lambda,\delta,\beta,a,b>0, \tag{3.3}$$

and

$$f(x;\lambda,\delta,\beta,a,b)=\left(\frac{\beta\lambda\delta b}{a^{b}}\right)x^{b-1}\frac{\left(1-e^{-\left(\frac{x}{a}\right)^{b}}\right)^{-\delta-1}}{\left(e^{-\left(\frac{x}{a}\right)^{b}}\right)^{-\delta}}\left(1+\lambda\left[\frac{1-e^{-\left(\frac{x}{a}\right)^{b}}}{\left(e^{-\left(\frac{x}{a}\right)^{b}}\right)}\right]^{-\delta}\right)^{-\beta-1}, \tag{3.4}$$

$$x>0,\quad \lambda,\delta,\beta,a,b>0.$$

3.2. **The Dagum-exponential distribution.** The exponential distribution is one of the common distributions in reliability analysis. It is a particular case of the gamma distribution and often used to model the time elapsed between events. The exponential distribution (with parameter $a>0$) has CDF and PDF given by:

$$G(x;a)=1-e^{-\left(\frac{x}{a}\right)},\quad x>0,\quad a>0 \tag{3.5}$$

and

$$g(x;a)=\frac{1}{a}e^{-\left(\frac{x}{a}\right)},\quad x>0,\quad a>0 \tag{3.6}$$

The CDF and PDF of Dagum-exponential distribution can be obtained by substituting Equations (3.5) and Equations (3.6) in Equations (2.3) and Equations (2.4) as follows:

$$F(x;\lambda,\delta,\beta,a)=\left(1+\lambda\left[\frac{1-e^{-\left(\frac{x}{a}\right)}}{e^{-\left(\frac{x}{a}\right)}}\right]^{-\delta}\right)^{-\beta},\quad x>0,\quad \lambda,\delta,\beta,a>0, \tag{3.7}$$

and

$$f(x;\lambda,\delta,\beta,a)=\left(\frac{\beta\lambda\delta}{a}\right)\frac{\left(1-e^{-\left(\frac{x}{a}\right)}\right)^{-\delta-1}}{\left(e^{-\left(\frac{x}{a}\right)}\right)^{-\delta}}\left(1+\lambda\left[\frac{1-e^{-\left(\frac{x}{a}\right)}}{\left(e^{-\left(\frac{x}{a}\right)}\right)}\right]^{-\delta}\right)^{-\beta-1},\quad x>0,\quad \lambda,\delta,\beta,a>0. \tag{3.8}$$

3.3. **The Dagum-Rayleigh distribution.** The Rayleigh distribution is a continuous probability distribution introduced by [22]. It is a special case of the Weibull distribution with a scale parameter of 2. It plays an essential role in modeling and analyzing lifetime data such as survival and reliability analysis, theory of communication, physical sciences, technology, diagnostic imaging and applied statistics. The Rayleigh distribution (with scale parameter $a>0$) has CDF and PDF given by:

$$G(x;a)=1-e^{-\left(\frac{x}{a}\right)^{2}},\quad x>0,\quad a>0 \tag{3.9}$$

and

$$g(x;a)=\frac{2}{a^{2}}xe^{-\left(\frac{x}{a}\right)^{2}},\quad x>0,\quad a>0 \tag{3.10}$$

The CDF and PDF of Dagum-Rayleigh distribution can be obtained by substituting Equations (3.9) and Equations (3.10) in Equations (2.3) and Equations (2.4) as follows:

$$F(x;\lambda,\delta,\beta,a)=\left(1+\lambda\left[\frac{1-e^{-\left(\frac{x}{a}\right)^{2}}}{e^{-\left(\frac{x}{a}\right)^{2}}}\right]^{-\delta}\right)^{-\beta},\quad x>0,\quad \lambda,\delta,\beta,a>0 \tag{3.11}$$

and

$$f(x;\lambda,\delta,\beta,a)=\left(\frac{2\beta\lambda\delta}{a^2}\right)\ (x)\ \frac{\left(1-e^{-(\frac{x}{a})^2}\right)^{-\delta-1}}{\left(e^{-(\frac{x}{a})^2}\right)^{-\delta}}\left(1+\lambda\left[\frac{1-e^{-(\frac{x}{a})^2}}{\left(e^{-(\frac{x}{a})^2}\right)}\right]^{-\delta}\right)^{-\beta-1}, \tag{3.12}$$

$x>0, \quad \lambda,\delta,\beta,a>0.$

3.4. **The Dagum-Fréchet distribution.** The Fréchet (Fr) Distribution was developed in the 1920s by French mathematician Maurice René Fréchet to model maximum values in a data set that came from different phenomena such as flood analysis, horse racing, human lifespans, maximum rainfalls and river discharges in hydrology. It is considered as one of the extreme value distributions (EVD_s), known as the EVD Type II. The Fr distribution (with scale and shape parameters $a,b>0$) has CDF and PDF given by:

$$G(x;a,b)=e^{-(\frac{a}{x})^b}, \quad x>0, \quad a,b>0 \tag{3.13}$$

and

$$g(x;a,b)=ba^bx^{-b-1}e^{-(\frac{a}{x})^b}, \quad x>0, \quad a,b>0 \tag{3.14}$$

The CDF and PDF of Dagum-Fréchet distribution can be obtained by substituting Equations (3.13) and Equations (3.14) in Equations (2.3) and Equations (2.4) as follows:

$$F(x;\lambda,\delta,\beta,a,b)=\left(1+\lambda\left[\frac{e^{-(\frac{a}{x})^b}}{\left(1-e^{-(\frac{a}{x})^b}\right)}\right]^{-\delta}\right)^{-\beta}, \quad x>0, \quad \lambda,\delta,\beta,a,b>0, \tag{3.15}$$

and

$$f(x;\lambda,\delta,\beta,a,b)=\beta\lambda\delta ba^bx^{-b-1}\frac{\left(e^{-(\frac{a}{x})^b}\right)^{-\delta}}{\left(1-e^{-(\frac{a}{x})^b}\right)^{-\delta+1}}\left(1+\lambda\left[\frac{e^{-(\frac{a}{x})^b}}{\left(1-e^{-(\frac{a}{x})^b}\right)}\right]^{-\delta}\right)^{-\beta-1}, x>0,$$

$\lambda,\delta,\beta,a,b>0.$

4. Mathematical Properties of the Dagum-X Family

This section describes some of mathematical properties of the Dagum-X family of distributions.

4.1. **Survival and Hazard rate functions.** Let the random variable T be the time to failure of the Dagum-X family of distributions. The survival and hazard rate functions of Dagum-X family of distributions are, respectively, given by:

$$S(t;\underline{\theta})=1-\left(1+\lambda\left[\frac{G(t;\underline{\theta})}{\overline{G}(t;\underline{\theta})}\right]^{-\delta}\right)^{-\beta}, \tag{4.1}$$

and

$$h(t;\underline{\theta})=\frac{\beta\lambda\delta g(t)\frac{[G(t)]^{-\delta-1}}{[1-[G(t)]]^{-\delta+1}}\left[1+\lambda(\frac{G(t)}{1-G(t)})^{-\delta}\right]^{-\beta-1}}{1-\left(1+\lambda\left[\frac{G(t;\underline{\theta})}{\overline{G}(t;\underline{\theta})}\right]^{-\delta}\right)^{-\beta}}, \tag{4.2}$$

where $\underline{\theta} = (\beta, \lambda, \delta)^T$ is a vector of parameters of baseline distribution.

4.2. **Moments.** Let X be a random variable follows the Dagum-X family with the density function given in Equation (2.4). The rth moment of X is given by:

$$E(X^r) = \beta\lambda\delta \int_0^\infty x^r g(x) \frac{[G(x)]^{-\delta-1}}{[1-[G(x)]]^{-\delta+1}} \left[1+\lambda\left(\frac{G(x)}{1-G(x)}\right)^{-\delta}\right]^{-\beta-1} dx.$$

Using the expansion (see [23]):

$$(1+x)^{-(n+1)} = \sum_{k=0}^{\infty} \binom{n+k}{k} (-1)^k x^k, \tag{4.3}$$

we have

$$\begin{aligned} E(X^r) =& \beta\lambda\delta \int_0^\infty x^r g(x) \frac{[G(x)]^{-\delta-1}}{[1-[G(x)]]^{-\delta+1}} \sum_{k=0}^{\infty} (-1)^k \lambda^k \binom{\beta+k}{k} \left(\frac{G(x)}{1-G(x)}\right)^{-\delta k} dx, \\ =& \beta\delta \int_0^\infty x^r g(x) \sum_{k=0}^{\infty} (-1)^k \lambda^{k+1} \binom{\beta+k}{k} [G(x)]^{-\delta(k+1)-1} [1-G(x)]^{\delta(k+1)-1} dx, \end{aligned}$$

and using the following expansion (see [24])

$$(1-x)^n = \sum_{k=0}^{\infty} \binom{n}{k} (-1)^k x^k, \tag{4.4}$$

we have

$$\begin{aligned} E(X^r) =& \beta\delta \int_0^\infty x^r g(x) \sum_{k=0}^{\infty} (-1)^k \lambda^{k+1} \binom{\beta+k}{k} [G(x)]^{-\delta(k+1)-1} \sum_{m=0}^{\infty} (-1)^m \binom{\delta(k+1)-1}{m} [G(x)]^m dx, \\ =& \beta\delta \int_0^\infty x^r g(x) \sum_{k,m=0}^{\infty} (-1)^{k+m} \lambda^{k+1} \binom{\beta+k}{k} \binom{\delta(k+1)-1}{m} [G(x)]^J dx \end{aligned}$$

Therefore,

$$E(X^r) = C_1 \int_0^\infty x^r g(x) [G(x)]^J dx. \tag{4.5}$$

where $C_1 = \beta\delta \sum_{k,m=0}^{\infty} (-1)^{k+m} \lambda^{k+1} \binom{\beta+k}{k} \binom{\delta(k+1)-1}{m}$, $\quad J = m - \delta(k+1) - 1$.

4.3. **Quantail function.** Let X be a random variable that has the CDF given in Equation (2.3). The quantile function, $Q(u)$ of X can be derived as follows:

Let

$$u \;= F(x) = \quad \left(1+\lambda\left[\frac{G(x;\underline{\theta})}{\overline{G}(x;\underline{\theta})}\right]^{-\delta}\right)^{-\beta},$$

After simplification, the quantile function is expressed as

$$Q(u) = G^{-1}\left[\frac{\left(\frac{[u]^{-\frac{1}{\beta}}-1}{\lambda}\right)^{\frac{-1}{\delta}}}{1+\left(\frac{[u]^{-\frac{1}{\beta}}-1}{\lambda}\right)^{\frac{-1}{\delta}}}\right], \tag{4.6}$$

where, u is a uniform random number on the interval $(0, 1)$ and $G^{-1}(.)$ is the inverse function of $G(.)$. In particular, $Q(0.5)$ is the median of the family and defined by substituting $u = 0.5$ in Equation (4.6):

$$Q(0.5) = G^{-1}\left[\frac{\left(\frac{[0.5]^{-\frac{1}{\beta}}-1}{\lambda}\right)^{\frac{-1}{\delta}}}{1+\left(\frac{[0.5]^{-\frac{1}{\beta}}-1}{\lambda}\right)^{\frac{-1}{\delta}}}\right].$$

The first and third quartiles can be obtained also by substituting $u = 0.25$ and $u = 0.75$, respectively, in Equation (4.6), as follows:

$$Q(0.25) = G^{-1}\left[\frac{\left(\frac{[0.25]^{-\frac{1}{\beta}}-1}{\lambda}\right)^{\frac{-1}{\delta}}}{1+\left(\frac{[0.25]^{-\frac{1}{\beta}}-1}{\lambda}\right)^{\frac{-1}{\delta}}}\right],$$

and

$$Q(0.75) = G^{-1}\left[\frac{\left(\frac{[0.75]^{-\frac{1}{\beta}}-1}{\lambda}\right)^{\frac{-1}{\delta}}}{1+\left(\frac{[0.75]^{-\frac{1}{\beta}}-1}{\lambda}\right)^{\frac{-1}{\delta}}}\right].$$

4.4. **Rényi Entropys.** The entropy of a random variable X represents a measure of uncertainty variation. Let X be a random variable that has the PDF given in Equation (2.4), then the Rényi entropy of the random variable X is defined as:

$$R_\theta(x) = (1-\theta)^{-1} log\left[\int_0^\infty f(x)^\theta dx\right], \quad \theta > 0 \quad and \quad \theta \neq 1. \tag{4.7}$$

Therefore, by applying Equation (2.4) into Equation (4.7), we have:

$$R_\theta(x) = \frac{1}{1-\theta} log\left[\int_0^\infty (\beta\lambda\delta)^\theta [g(x)]^\theta \left[\frac{[G(x)]^{-\theta(\delta+1)}}{[1-G(x)]^{-\theta(\delta-1)}}\right]\left[1+\lambda\left(\frac{G(x)}{1-G(x)}\right)^{-\delta}\right]^{-\theta(\beta+1)} dx\right].$$

Using the expansions in Equation (4.3) and Equation (4.4), the Rényi entropy of the Dagum-X, $R_\theta(x)$, can be written as:

$$\begin{aligned} R_\theta(x) &= \frac{1}{1-\theta} log \int_0^\infty (\beta\lambda\delta)^\theta [g(x)]^\theta \sum_{k,r=0}^{\infty} \binom{\theta(\beta+1)+k-1}{k}\binom{\theta(\delta-1)+\delta k}{r} \\ &\times \quad (-1)^{r+k}\lambda^k [G(x)]^{-\theta(\delta+1)-\delta k+r} dx \end{aligned}$$

Thus,

$$R_\theta(x) = \frac{1}{1-\theta} log\left[C_1 \int_0^\infty [g(x)]^\theta [G(x)]^{J_1} dx\right] \tag{4.8}$$

where J_1 and C_1 are resbectivaly, defined as follow:

$$\begin{aligned}J_1 =& -\theta(\delta+1)-\delta k+r\\ C_1 =&(\beta\lambda\delta)^{\theta}\sum_{k,r=0}^{\infty}\binom{\theta(\beta+1)+k-1}{k}\binom{\theta(\delta-1)+\delta k}{r}(-1)^{r+k}\lambda^{k}\end{aligned}$$

4.5. **Order Statistics.** Let $x_{1:n}, x_{2:n}, ..., x_{n:n}$ be the order statistics obtained from the Dagum-X with CDF $F(x)$ and PDF $f(x)$, respectively, given in Equation (2.3) and Equation (2.4). The PDF of the ith order statistics can be expressed as:

$$\begin{aligned}f_{i:n}(x) =&\frac{n!\beta\lambda\delta g(x)}{(i-1)!(n-i)!}\frac{[G(x)]^{-\delta-1}}{[1-[G(x)]]^{-\delta+1}}\left[1+\lambda\left(\frac{G(x)}{1-G(x)}\right)^{-\delta}\right]^{-\beta-1}\\ &\times\left[\left[1+\lambda\left(\frac{G(x)}{1-G(x)}\right)^{-\delta}\right]^{-\beta}\right]^{i-1}\left[1-\left[1+\lambda\left(\frac{G(x)}{1-G(x)}\right)^{-\delta}\right]^{-\beta}\right]^{n-i}\end{aligned}$$

Let

$$u=\left[1+\lambda\left(\frac{G(x)}{1-G(x)}\right)^{-\delta}\right]^{-\beta}, \tag{4.9}$$

then

$$\begin{aligned}f_{i:n}(x) =&\frac{n!\beta\lambda\delta g(x)}{(i-1)!(n-i)!}\frac{[G(x)]^{-\delta-1}}{[1-[G(x)]]^{-\delta+1}}u^{(1+\frac{1}{\beta})}[u]^{i-1}[1-u]^{n-i},\\ =&\frac{n!\beta\lambda\delta g(x)}{(i-1)!(n-i)!}\frac{[G(x)]^{-\delta-1}}{[1-[G(x)]]^{-\delta+1}}u^{(i+\frac{1}{\beta})}[1-u]^{n-i},\end{aligned}$$

By applying the expansion in Equation (4.4), we have

$$\begin{aligned}f_{i:n}(x) =& \frac{n!\beta\delta g(x)}{(i-1)!(n-i)!}\sum_{k,m=0}^{\infty}\binom{n-i}{k}\binom{\beta i+\beta k+m}{m}(-1)^{k+m}(\lambda)^{m+1}G(x)^{-(\delta m+\delta+1)}\\ &\times\sum_{l=0}^{\infty}\binom{\delta m+\delta-1}{l}(-1)^{l}G(x)^{l},\end{aligned}$$

$$=\frac{n!\beta\delta g(x)}{(i-1)!(n-i)!}\sum_{k,m,l=0}^{\infty}\binom{n-i}{k}\binom{\beta i+\beta k+m}{m}\binom{\delta m+\delta-1}{l}(-1)^{k+m+l}(\lambda)^{m+1}G(x)^{l-(\delta m+\delta+1)}. \tag{4.10}$$

5. Dagum-Fréchet Distribution and Its Properties

The Fréchet distribution is becoming increasingly a preferred distribution in extending new statistical models. [25] introduced a distribution that generalizes the Fréchet distribution, known as the exponentiated Fréchet distribution and included a detailed analysis of the mathematical properties of this new distribution. [26] introduced and studied three component mixtures of the Fréchet distributions when the shape parameter is known under Bayesian view point. [27] developed a new compound continuous distribution named the Gompertz Fréchet distribution which extends the Frèchet distribution. [28] proposed a new four-parameter Fréchet distribution called the odd Lomax Fréchet distribution. The new model can be expressed as a linear mixture of Fréchet densities.

The D-Fr distribution is introduced briefly in (3.4) as a special model of the Dagum-X family. The CDF and PDF of the distribution are given in Equations (3.15) and (**??**), respectively. In this section, mathematical properties of the new distribution are presented and the maximum likelihood estimation is employed to estimate the parameters of the new distribution. Monte Carlo Simulation by using R program to assess the performance of the maximum likelihood estimation is applied and discussed. Finally, real data sets are analyzed to illustrate the performance of the proposed distribution. The plot of the PDF is presented using different values for the five parameters to study its behaviour, as shown in Figure (1).

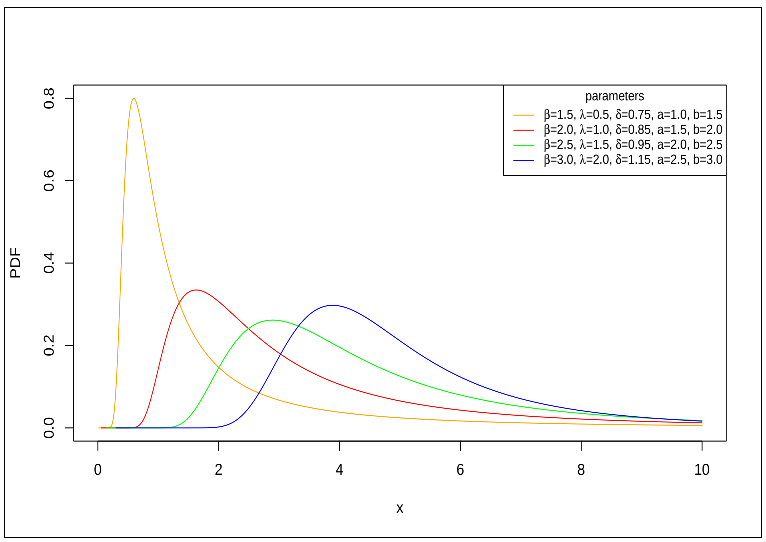

Figure 1. The D-Fr density function when all shape and scale parameters are changing.

Figure (1) displays the density function of the D-Fr for different values of the shape and scale parameters. It is right skewed and has different levels of kurtosis which shows the flexibility of the distribution for modelling skew data.

5.1. **Survival and Hazrd functions.** The survival and hazard rate functions of D-Fr are given by substituting Equation (3.13) and (3.14) in Equation (4.1) and (4.2) respectively as follows:

$$S(x;\underline{\theta}) = 1 - \left(1 + \lambda\left[\frac{e^{-(\frac{a}{x})^b}}{(1 - e^{-(\frac{a}{x})^b})}\right]^{-\delta}\right)^{-\beta}, \tag{5.1}$$

and

$$h(x;\underline{\theta}) = \frac{\beta\lambda\delta b a^b x^{-b-1}\left(\frac{(e^{-(\frac{a}{x})^b})^{-\delta}}{(1-e^{-(\frac{a}{x})^b})^{-\delta+1}}\right)\left(1+\lambda\left[\frac{e^{-(\frac{a}{x})^b}}{(1-e^{-(\frac{a}{x})^b})}\right]^{-\delta}\right)^{-\beta-1}}{1-\left(1+\lambda\left[\frac{e^{-(\frac{a}{x})^b}}{(1-e^{-(\frac{a}{x})^b})}\right]^{-\delta}\right)^{-\beta}}. \quad (5.2)$$

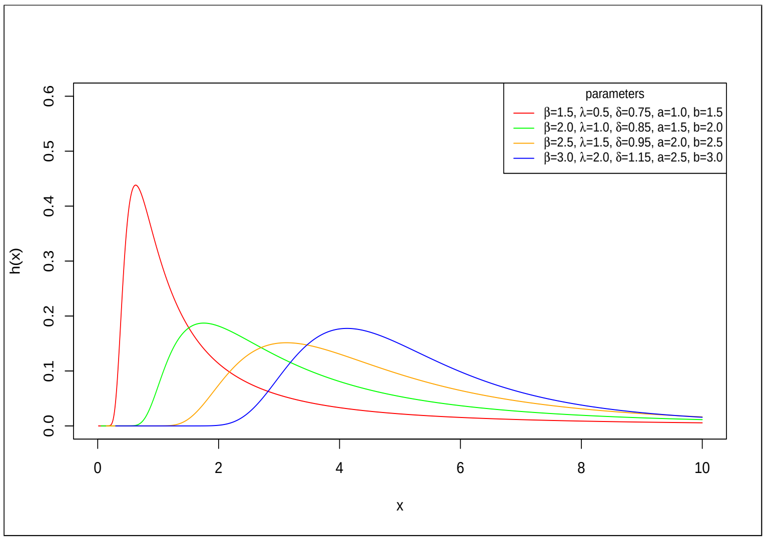

Figure 2. The D-Fr hazard rate function when all shape and scale parameters are changing.

Various curves of the hazard function of Dagum-Fréchet distribution are shown in Figure (2). By assuming different values of the shape and scale parameters, the curves appear to be unimodal and positive skewed with different levels of skewness and kurtosis.

5.2. **Moments.** The r^{th} moment of the D-Fr distribution is obtained by substituting Fréchet distribution's CDF and PDF in Equations (3.13) and (3.14) into the rth moment of Dagum-X in Equation (4.5). As a result, the rth moment of D-Fr distribution is given as

$$\begin{aligned} E(X^r) &= C_1\int_0^\infty x^r g(x)[G(x)]^J dx, \\ &= C_1\int_0^\infty x^r \left(ba^b x^{-b-1}e^{-(\frac{a}{x})^b}\right)\left[e^{-(\frac{a}{x})^b}\right]^J dx, \\ &= C_2\int_0^\infty x^{r-b-1}\left[e^{-(\frac{a}{x})^b}\right]^{J_2} dx, \end{aligned}$$

where,

$$\begin{aligned} C_1 &= \beta\delta\sum_{k,m=0}^{\infty}(-1)^{k+m}\lambda^{k+1}\binom{\beta+k}{k}\binom{\delta(k+1)-1}{m}, \\ C_2 &= C_1 b a^b, \end{aligned}$$

and

$$J_2 \;=\; J+1 = m-\delta(k+1)$$

Using integration by substitution,

let

$$u = J_2\left[(\frac{a}{x})^{b}\right] \; then \; x = \left[a(\frac{u}{J_2})^{-(\frac{1}{b})}\right] \; and \; dx = (\frac{-a}{bJ_2})(\frac{u}{J_2})^{(\frac{-1}{b}-1)}du.$$

Hence,

$$\begin{aligned}
E(X^r) &= C_2\int_0^\infty x^{r-b-1}\left[e^{-(\frac{a}{x})^b}\right]^{J_2}dx \\
&= C_2\int_\infty^0 \left[a(\frac{u}{J_2})^{-(\frac{1}{b})}\right]^{r-b-1} e^{-u}(\frac{-a}{bJ_2})(\frac{u}{J_2})^{(\frac{-1}{b}-1)}du \\
&= C_2\frac{a^{r-b}}{b}\left(\frac{1}{J_2}\right)^{\frac{-r}{b}+1}\int_0^\infty u^{\frac{-r}{b}}e^{-u}du \\
&= C_2\frac{a^{r-b}}{b}\left(\frac{1}{J_2}\right)^{\frac{-r}{b}+1}\Gamma(-\frac{r}{b}+1) \\
&= \beta\delta b a^b\sum_{k,m=0}^{\infty}(-1)^{k+m}\lambda^{k+1}\binom{\beta+k}{k}\binom{\delta(k+1)-1}{m}\frac{a^{r-b}}{b}\left(\frac{1}{J_2}\right)^{\frac{-r}{b}+1}\Gamma(-\frac{r}{b}+1) \\
&= \beta\delta a^r\sum_{k,m=0}^{\infty}(-1)^{k+m}\lambda^{k+1}\binom{\beta+k}{k}\binom{\delta(k+1)-1}{m}\left(\frac{1}{J_2}\right)^{\frac{-r}{b}+1}\Gamma(-\frac{r}{b}+1)
\end{aligned}$$

Then, the moment of Dagum-Fréche distribution is given as

$$E(X^r) = C_3\left(\frac{1}{J_2}\right)^{\frac{-r}{b}+1}a^r\Gamma(-\frac{r}{b}+1), \quad [1-\frac{r}{b}]>0 \tag{5.3}$$

where,

$$C_3 = \beta\delta\sum_{k,m=0}^{\infty}(-1)^{k+m}\lambda^{k+1}\binom{\beta+k}{k}\binom{\delta(k+1)-1}{m},$$

and

$$J_2 = m-\delta(k+1).$$

5.2.1. *Mean and Variance.* The mean of D-Fr distribution can be obtained by setting ($r=1$) in Equation (5.3), which results in the following form:

$$E(X) = C_3\left(\frac{1}{J_2}\right)^{\frac{-1}{b}+1}a\Gamma(-\frac{1}{b}+1), \tag{5.4}$$

where,

$$C_3 = \beta\delta\sum_{k,m=0}^{\infty}(-1)^{k+m}\lambda^{k+1}\binom{\beta+k}{k}\binom{\delta(k+1)-1}{m},$$

and

$$J_2 = m - \delta(k+1).$$

The 2^{nd} moment $E(X^2)$ can be found by setting $(r = 2)$ in Equation (5.3), then the variance of D-Fr distribution can be obtained as follows:

$$Var(X) = [C_3\left(\frac{1}{J_2}\right)^{\frac{-2}{b}+1} a^2\Gamma(-\frac{2}{b}+1)] - [C_3\left(\frac{1}{J_2}\right)^{\frac{-1}{b}+1} a\Gamma(-\frac{1}{b}+1)]^2 . \tag{5.5}$$

where,

$$C_3 = \beta\delta \sum_{k,m=0}^{\infty} (-1)^{k+m}\lambda^{k+1}\binom{\beta+k}{k}\binom{\delta(k+1)-1}{m},$$

and

$$J_2 = m - \delta(k+1).$$

5.3. **Quantail function.** The quantail function of the D-Fr distribution, $x = F^{-1}(u)$, can be obtained by inverting the CDF in Equation (3.15) as follows:

$$x = Q(u) = \frac{a}{\left[-log\frac{\left[\frac{(u)^{-\frac{1}{\beta}}-1}{\lambda}\right]^{-\frac{1}{\delta}}}{\left[1+\left(\frac{(u)^{-\frac{1}{\beta}}-1}{\lambda}\right)^{-\frac{1}{\delta}}\right]}\right]^{\frac{1}{b}}}. \tag{5.6}$$

where, u is a uniform random number on the interval $(0, 1)$. Therefore, the median of the D-Fr can be found by substituting $u = 0.5$ in Equation (5.6) as follows:

$$Q(0.5) = \frac{a}{\left[-log\frac{\left[\frac{(0.5)^{-\frac{1}{\beta}}-1}{\lambda}\right]^{-\frac{1}{\delta}}}{\left[1+\left(\frac{(0.5)^{-\frac{1}{\beta}}-1}{\lambda}\right)^{-\frac{1}{\delta}}\right]}\right]^{\frac{1}{b}}}. \tag{5.7}$$

The first and third quartiles can also be obtained by substituting $u = 0.25$ and $u = 0.75$ in Equation (5.6), respectively as follows:

$$Q(0.25) = \frac{a}{\left[-log\frac{\left[\frac{(0.25)^{-\frac{1}{\beta}}-1}{\lambda}\right]^{-\frac{1}{\delta}}}{\left[1+\left(\frac{(0.25)^{-\frac{1}{\beta}}-1}{\lambda}\right)^{-\frac{1}{\delta}}\right]}\right]^{\frac{1}{b}}},$$

and

$$Q(0.75)=\frac{a}{\left[-log\frac{\left[\frac{(0.75)^{-\frac{1}{\beta}}-1}{\lambda}\right]^{-\frac{1}{\delta}}}{\left[1+\left(\frac{(0.75)^{-\frac{1}{\beta}}-1}{\lambda}\right)^{-\frac{1}{\delta}}\right]}\right]^{\frac{1}{b}}}.$$

5.4. **Rényi Entropys.** Using the definition of the Rényi entropy in Equation (4.7), and applying the CDF and PDF of Frechet distribution in Equation (3.13) and (3.14), we have:

$$R_\theta(x)=\frac{1}{1-\theta}log\left[C_1\int_0^\infty (ba^bx^{-b-1}e^{-(\frac{a}{x})^b})^\theta(e^{-(\frac{a}{x})^b})^{J_1}dx\right]$$

$$=\frac{1}{1-\theta}log\left[C_2\int_0^\infty (x^{-\theta(b+1)})(e^{-(\frac{a}{x})^b})^\theta(e^{-(\frac{a}{x})^b})^{J_1}dx\right]$$

$$=\frac{1}{1-\theta}log\left[C_2\int_0^\infty x^{-\theta(b+1)}(e^{-(\frac{a}{x})^bJ_2})dx\right]$$

where,

$$C_1=(\beta\lambda\delta)^\theta\sum_{k,r=0}^{\infty}\binom{\theta(\beta+1)+k-1}{k}\binom{\theta(\delta-1)+\delta k}{r}(-1)^{r+k}\lambda^k,$$

$$C_2=C_1(ba^b)^\theta,$$

$$J_1=-\theta(\delta+1)-\delta k+r,$$

$$and$$

$$J_2=\theta+J_1.$$

Using integration by substitution and after simplification, we get

$$R_\theta(x)=\frac{1}{1-\theta}log\left[C_3\int_0^\infty u^{\theta+\frac{\theta}{b}-\frac{1}{b}-1}e^{-u}du\right],$$

$$R_\theta(x)=\frac{1}{(1-\theta)}\quad log\left[C_3\Gamma(\theta+\frac{\theta}{b}-\frac{1}{b})\right],\quad [\theta+\frac{\theta}{b}-\frac{1}{b}]>0 \tag{5.8}$$

where C_3 and J_2 are, respectively, as follows:

$$C_3=(\beta\delta)^\theta b^{\theta-1}a^{1-\theta}\sum_{k,r=0}^{\infty}\binom{\theta(\beta+1)+k-1}{k}\binom{\theta(\delta-1)+\delta k}{r}(-1)^{r+k}\lambda^{k+\theta}\left(\frac{1}{J_2}\right)^{(\theta+\frac{\theta-1}{b})},$$

$$J_2=-\delta(\theta+k)+r.$$

5.5. **Order Statistics.** The order statistics of D-Fr distribution is obtained by substituting the Fr distribution's CDF and PDF in Equation (3.13) and (3.14) in the order statistics of Dagum-X family, in Equation (4.10), as follows:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\beta\delta b a^b x^{(-b-1)} \sum_{k,m,l=0}^{\infty} \binom{n-i}{k}\binom{\beta i+\beta k+m}{m}\binom{\delta m+\delta-1}{l}(-1)^{k+m+l}$$
$$\times (\lambda)^{m+1}[e^{-(\frac{a}{x})^b}]^{l-\delta m-\delta}. \tag{5.9}$$

6. Maximum Likelihood Estimation

In this section, the MLE method will be applied to estimate the unknown parameters of the D-Fr distribution. Assume that $x_1, x_2, ...x_n$ is a random sample of the D-Fr distribution, then the likelihood function for the vector of parameters $\underline{\theta} = (\beta, \lambda, \delta, a, b)^T$ is given by:

$$L(\underline{\theta}) = \prod_{i=1}^{n} \beta\lambda\delta b a^b x^{-b-1} \frac{(e^{-(\frac{a}{x})^b})^{-\delta}}{(1-e^{-(\frac{a}{x})^b})^{-\delta+1}} \left(1+\lambda\left[\frac{e^{-(\frac{a}{x})^b}}{(1-e^{-(\frac{a}{x})^b})}\right]^{-\delta}\right)^{-\beta-1}, \tag{6.1}$$

then the log likelihood function can be written as:

$$l = logL = nlog\beta + nlog\lambda + nlog\delta + nlogb + nbloga - (b+1)\sum_{i=1}^{n} log[x] - \delta\sum_{i=1}^{n} log[e^{-(\frac{a}{x})^b}]$$
$$+(\delta-1)\sum_{i=1}^{n} log[1-e^{-(\frac{a}{x})^b}] - (\beta+1)\sum_{i=1}^{n} log\left[1+\lambda\left(\frac{e^{-(\frac{a}{x})^b}}{1-e^{-(\frac{a}{x})^b}}\right)^{-\delta}\right]. \tag{6.2}$$

The first partial derivatives of the log likelihood function in Equation (6.2) with respect to $\odot$, $\geq$, $\circ$, a and b are respectively given as follows:

$$\frac{\partial l}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} log\left[1+\lambda\left(\frac{e^{-(\frac{a}{x_i})^b}}{1-e^{-(\frac{a}{x_i})^b}}\right)^{-\delta}\right], \tag{6.3}$$

$$\frac{\partial l}{\partial\lambda} = \frac{n}{\lambda} - (\beta+1)\sum_{i=1}^{n} \frac{\left(\frac{e^{-(\frac{a}{x_i})^b}}{1-e^{-(\frac{a}{x_i})^b}}\right)^{-\delta}}{1+\lambda\left(\frac{e^{-(\frac{a}{x_i})^b}}{1-e^{-(\frac{a}{x_i})^b}}\right)^{-\delta}}, \tag{6.4}$$

$$\frac{\partial l}{\partial\delta} = \frac{n}{\delta} - \sum_{i=1}^{n} log[e^{-(\frac{a}{x_i})^b}] - (\beta+1)\sum_{i=1}^{n} -\frac{\left(\frac{e^{-(\frac{a}{x_i})^b}}{1-e^{-(\frac{a}{x_i})^b}}\right)^{-\delta}\lambda log\left(\frac{e^{-(\frac{a}{x_i})^b}}{1-e^{-(\frac{a}{x_i})^b}}\right)}{1+\lambda\left(\frac{e^{-(\frac{a}{x_i})^b}}{1-e^{-(\frac{a}{x_i})^b}}\right)^{-\delta}} + \sum_{i=1}^{n} log[1-e^{-(\frac{a}{x_i})^b}], \tag{6.5}$$

$$\frac{\partial l}{\partial a}=\frac{nb}{a}-(\beta+1)\sum_{i=1}^{n}\frac{\left[\lambda\left(\frac{be^{-2(\frac{a}{x_i})^b}\left(\frac{e^{-(\frac{a}{x_i})^b}}{1-e^{-(\frac{a}{x_i})^b}}\right)^{-\delta-1}\delta(\frac{a}{x_i})^{-1+b}}{x_i(1-e^{-(\frac{a}{x_i})^b})^2}+\frac{be^{-(\frac{a}{x_i})^b}\left(\frac{e^{-(\frac{a}{x_i})^b}}{1-e^{-(\frac{a}{x_i})^b}}\right)^{-\delta-1}\delta(\frac{a}{x_i})^{-1+b}}{x(1-e^{-(\frac{a}{x_i})^b})}\right)\right]}{\left[1+\lambda\left(\frac{e^{-(\frac{a}{x_i})^b}}{1-e^{-(\frac{a}{x_i})^b}}\right)^{-\delta}\right]-\delta\sum_{i=1}^{n}\frac{b[\frac{a}{x_i}]^{-1+b}}{x_i}+(-1+\delta)\sum_{i=1}^{n}\frac{be^{-(\frac{a}{x_i})^b}(\frac{a}{x_i})^{-1+b}}{x_i[1-e^{-(\frac{a}{x_i})^b}]}}, \tag{6.6}$$

$$\begin{aligned}\frac{\partial l}{\partial b}=&\frac{n}{b}+nloga-\sum_{i=1}^{n}log[x_i]\\&-(\beta+1)\sum_{i=1}^{n}\frac{\left[\lambda\left(\frac{e^{-2(\frac{a}{x_i})^b}\left(\frac{e^{-(\frac{a}{x_i})^b}}{1-e^{-(\frac{a}{x_i})^b}}\right)^{-\delta-1}\delta log(\frac{a}{x_i})(\frac{a}{x_i})^b}{(1-e^{-(\frac{a}{x_i})^b})^2}+\frac{e^{-(\frac{a}{x_i})^b}\left(\frac{e^{-(\frac{a}{x_i})^b}}{1-e^{-(\frac{a}{x_i})^b}}\right)^{-\delta-1}\delta log(\frac{a}{x_i})(\frac{a}{x_i})^b}{1-e^{-(\frac{a}{x_i})^b}}\right)\right]}{\left[1+\lambda\left(\frac{e^{-(\frac{a}{x_i})^b}}{1-e^{-(\frac{a}{x_i})^b}}\right)^{-\delta}\right]-\delta\sum_{i=1}^{n}-log(\frac{a}{x_i})(\frac{a}{x_i})^b+(-1+\delta)\sum_{i=1}^{n}\frac{e^{-(\frac{a}{x_i})^b}log(\frac{a}{x_i})(\frac{a}{x_i})^b}{1-e^{-(\frac{a}{x_i})^b}}}.\end{aligned} \tag{6.7}$$

The MLEs $\hat{\beta}, \hat{\lambda}, \hat{\delta}, \hat{a}, \hat{b}$ of $\beta, \lambda, \delta, a, b$ can be obtained by equating the results to zero and solving the system of nonlinear equations numerically.

For interval estimation of the model parameters, inverting Fisher information matrix is required, but finding the expectation of the Fisher information matrix is not easy. Therefore, the 5x5 observed information matrix is used to generate confidence intervals for the model parameters. The observed information matrix is given as follows:

$$I(\hat{\theta})=\begin{bmatrix}-\frac{\partial^2 l}{\partial\beta^2} & -\frac{\partial^2 l}{\partial\beta\partial\lambda} & -\frac{\partial^2 l}{\partial\beta\partial\delta} & -\frac{\partial^2 l}{\partial\beta\partial a} & -\frac{\partial^2 l}{\partial\beta\partial b}\\ -\frac{\partial^2 l}{\partial\lambda\partial\beta} & -\frac{\partial^2 l}{\partial\lambda^2} & -\frac{\partial^2 l}{\partial\lambda\partial\delta} & -\frac{\partial^2 l}{\partial\lambda\partial a} & -\frac{\partial^2 l}{\partial\lambda\partial b}\\ -\frac{\partial^2 l}{\partial\delta\partial\beta} & -\frac{\partial^2 l}{\partial\delta\partial\lambda} & -\frac{\partial^2 l}{\partial\delta^2} & -\frac{\partial^2 l}{\partial\delta\partial a} & -\frac{\partial^2 l}{\partial\delta\partial b}\\ -\frac{\partial^2 l}{\partial a\partial\beta} & -\frac{\partial^2 l}{\partial a\partial\lambda} & -\frac{\partial^2 l}{\partial a\partial\delta} & -\frac{\partial^2 l}{\partial a^2} & -\frac{\partial^2 l}{\partial a\partial b}\\ -\frac{\partial^2 l}{\partial b\partial\beta} & -\frac{\partial^2 l}{\partial b\partial\lambda} & -\frac{\partial^2 l}{\partial b\partial\delta} & -\frac{\partial^2 l}{\partial b\partial a} & -\frac{\partial^2 l}{\partial b^2}\end{bmatrix}$$

The expectation of the observed information matrix can be solved iteratively using R software. Therefore, the multivariate normal distribution $N_5(0, I^{-1})$ can be used to construct 100(1-$\oslash$)% two sided approximate confidence intervals for the model parameters $\odot$, $\geq$, $\circ$, a and b where α is the significant level.

7. Simulation Study

In this Section, simulation studies have been performed using R program to evaluate the theoretical results of the estimation process. The performance of the MLEs of the parameters has been considered. Furthermore, the approximate confidence intervals with confidence level 90% are obtained. The algorithm for the simulation procedure is described below:

Step 1: 5000 random samples of size n=75, 100, 200, 300, 600 and 1000 are generated from the D-Fr distribution. The true parameter values are assumed as (⊙=0.75, ≥=0.2, ∘=0.1, $a = 0.9$ and $b = 0.7$).

Step 2: The parameters of the distribution are estimated using the MLE method for each sample.

Step 3: The R function (nlminb) is used to solve the five nonlinear likelihood for ⊙, ≥, ∘, a and b.

Step 4: For each simulation, the average biases (ABs) and the mean sqare errors (MSEs) are calculated by: $Bias(\hat{y}) = \sum_{i=1}^{5000} \frac{1}{5000}(\hat{y} - y)$, $\quad MSE(\hat{y}) = \sum_{i=1}^{5000} \frac{1}{5000}(\hat{y} - y)^2$.

Table 1. MLEs, ABs, MSE and 90% confidence limits of the parameters when n= 75, 100, 200, 300, 600 and 1000.

Sample	Parameter	Estimate	Bias	MSE	Lower Limit	Upper Limit	Length
	β	0.8054288	0.053428772	0.121979438	0.23019589	1.3706617	1.1404658
	λ	0.2755135	0.075513528	0.254675089	0.54778984	1.0988169	0.5510270
n=75	δ	0.1209819	0.020981874	0.002012837	0.05554949	0.1864143	0.1308648
	a	0.8423633	-0.057636675	0.153099671	0.20379481	1.4809318	1.2771370
	b	0.7091435	0.009143525	0.009290203	0.55082434	0.8674627	0.3166384
	β	0.8005119	0.050511859	0.084791214	0.33123212	1.2757916	0.9445595
	λ	0.2197650	0.019764989	0.057678640	0.17516100	0.6146910	0.4398520
n=100	δ	0.1173703	0.017370314	0.001444903	0.06158238	0.1731582	0.1115759
	a	0.8595773	-0.040422730	0.100288315	0.34132407	1.3778305	1.0365064
	b	0.6978595	-0.008140522	0.004583588	0.58620669	0.8095123	0.2233056
	β	0.7746440	0.024643973	0.0211146825	0.53835754	1.0109304	0.4725729
	λ	0.1953267	-0.004673258	0.0038030341	0.09386590	0.2967876	0.2029217
n=200	δ	0.1090029	0.009002877	0.0007865224	0.06517773	0.1528280	0.0876503
	a	0.8856043	-0.014395702	0.0190316874	0.65922062	1.1119880	0.4527674
	b	0.6917052	-0.007294776	0.0010332734	0.64046296	0.7429475	0.1024845
	β	0.7622825	0.012282484	0.0073377378	0.62240298	0.9021620	0.27975901
	λ	0.1955416	-0.004458362	0.0008766267	0.14724566	0.2438376	0.09659195
n=300	δ	0.1051773	0.005177325	0.0001848952	0.08443121	0.1259234	0.04149223
	a	0.8932748	-0.006725190	0.0063372406	0.76239317	1.0241565	0.26176328
	b	0.6936197	-0.006380267	0.0004032663	0.66220213	0.7250373	0.06283520
	β	0.7520984	0.002098386	1.419876e-03	0.69002082	0.8141759	0.12415513
	λ	0.1987551	-0.001244911	1.054027e-04	0.18194023	0.2155699	0.03362971
n=600	δ	0.1014451	0.001445127	5.141023e-05	0.08985726	0.1130330	0.02317574
	a	0.8981831	-0.001816934	8.178124e-04	0.85109266	0.9452735	0.09418082
	b	0.6982087	-0.001791341	6.182002e-05	0.68557661	0.7108407	0.02526411
	β	0.7501594	0.0001594468	6.794310e-05	0.73656144	0.7637575	0.027196020
	λ	0.1998049	-0.0001951371	7.570256e-06	0.19527647	0.2043333	0.009056788
n=1000	δ	0.1001842	0.0001841564	2.657683e-06	0.09751148	0.1028568	0.005345358
	a	0.8996841	-0.0003158548	7.301607e-05	0.88559462	0.9137737	0.028179044
	b	0.6997525	-0.0002474826	4.987021e-06	0.69609049	0.7034145	0.007324047

From Table (1), It can be observed that, as the sample size increases, the MLEs approach to the initial values of the parameters. For each sample size n, the MLEs are evaluated using two accuracy measures which are ABs and MSE. As the sample size increases, the ABs and MSEs of the estimated parameters decrease. This indicates that the maximum likelihood estimation method provides consistent estimators for the parameters and approaches the population parameters' values as the sample size increases. It is also noted that the lengths of the confidence intervals for the estimated parameters decrease as the sample size increases.

8. Application

For more illustration, this section compares the efficiency of the goodness-of-fit for the D-Fr distribution with some selected distributions in literature. In particular, two real data sets are used to compare the proposed model with four other distributions, namely, beta-Fréchet(BF) by [29], Gamma-Extended-Fréchet(GEF) by [30], Exponentiated-Exponential-Fréchet(EEF) by [31] and Fréchet(F) distributions by [32] which is also sudied by [33].

The first data set in Table (2) that is used in comparison is provided by Cordeiro and Silva [34]. The data represent the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England.

The second data in Table (3) represents breaking stress of carbon fibers of 50 mm length (GPa) and have been previously used by [35].

Table 2. Strength of 1.5 cm glass fibres data (data set 1).

0.55	0.74	0.77	0.81	0.84	1.24	0.93	1.04	1.11	1.13
1.30	1.25	1.27	1.28	1.29	1.48	1.36	1.39	1.42	1.48
1.51	1.49	1.49	1.50	1.50	1.55	1.52	1.53	1.54	1.55
1.61	1.58	1.59	1.60	1.61	1.63	1.61	1.61	1.62	1.62
1.67	1.64	1.66	1.66	1.66	1.70	1.68	1.68	1.69	1.70
1.78	1.73	1.76	1.76	1.77	1.89	1.81	1.82	1.84	1.84
2.00	2.01	2.24							

Table 3. Breaking stress of carbon fibers of 50 mm length data (data set 2).

0.39	0.85	1.08	1.25	1.47	1.57	1.61	1.61	1.69	1.80	1.84
1.87	1.89	2.03	2.03	2.05	2.12	2.35	2.41	2.43	2.48	2.50
2.53	2.55	2.55	2.56	2.59	2.67	2.73	2.74	2.79	2.81	2.82
2.85	2.87	2.88	2.93	2.95	2.96	2.97	3.09	3.11	3.11	3.15
3.15	3.19	3.22	3.22	3.27	3.28	3.31	3.31	3.33	3.39	3.39
3.56	3.60	3.65	3.68	3.70	3.75	4.20	4.38	4.42	4.70	4.90

Certain criteria are used in order to compare between the distributions. The distribution with best fit is the one that has the lowest value of the information criteria (AIC, AICc, BIC and HQIC) that are defined as

$$AIC = -2l(\underline{\hat{\theta}}) + 2p,$$
$$BIC = -2l(\underline{\hat{\theta}}) + plog(n),$$
$$AICc = AIC + \frac{2p(p+1)}{n-p-1},$$
$$HQIC = -2l(\underline{\hat{\theta}}) + 2plog(log(n)).$$

where $l(\underline{\hat{\theta}})$ is denoted by the log likelihood function evaluated at the maximum likelihood estimates, p is the number of parameters in the model and n is the sample size.

Table 4. The log likelihood, AIC, AICc, BIC and HQIC for the data set 1

Distribution	$\hat{l}$	AIC	AICc	BIC	HQIC
D-Fr	12.7337	35.46739	36.52003	46.18307	39.68192
BF	30.22177	68.44353	69.13319	77.01607	71.81516
GEF	30.66209	69.32417	70.01383	77.89671	72.69579
F	46.85336	97.70672	97.90672	101.993	99.39253
EEF	47.63954	103.2791	103.9687	111.8516	106.6507

Table 5. The log likelihood, AIC, AICc, BIC and HQIC for the data set 2

Distribution	$\hat{l}$	AIC	AICc	BIC	HQIC
D-Fr	85.27445	180.5489	181.5489	191.4972	184.8751
BF	100.276	208.552	209.2077	217.3106	212.0129
EEF	100.4079	208.8158	209.4716	217.5744	212.2768
GEF	101.1724	210.3448	211.0006	219.1035	213.8058
F	121.195	246.39	246.5805	250.7693	248.1205

Table 6. The MLEs for the data sets 1 and 2

Data	Distribution	β	λ	δ	a	b
Data 1	D-Fr	0.2231043	15.0981157	15.6678754	1.0476744	0.9305450
	BF	0.6071596	1.9876964	...	17.9057801	41.5883328
	GEF	0.7120245	1.7786321	...	35.0796598	13.7818068
	F	...	...	...	1.263794	2.887747
	EEF	33.4764197	4.7270858	...	27.9266876	0.3634021
Data 2	D-Fr	0.2502454	0.7341762	22.7850195	1.1852932	0.3223076
	BF	0.3798751	3.1899014	...	20.5420435	40.0668395
	GEF	0.5395719	3.5837019	...	23.1357231	8.5334421
	F	...	...	...	2.034154	1.649719
	EEF	30.5722159	2.2994984	...	34.5713778	0.4487714

Tables (4) and (5), demonstrate that the D-Fr model has the lowest value of the information criteria which implies that the proposed model provides a better fit than the other comparative models.

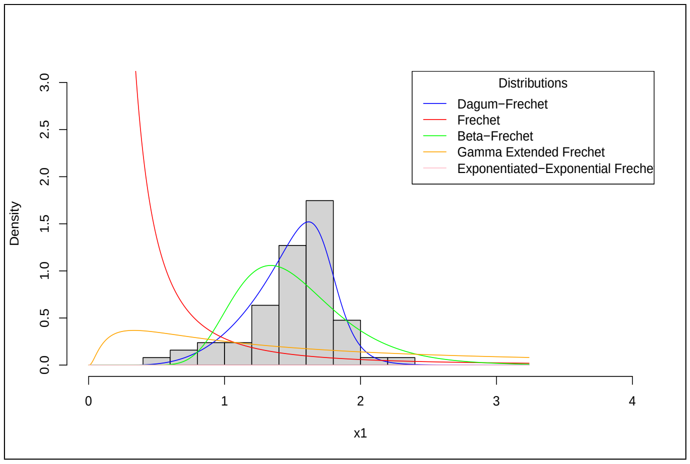

Figure 3. Fitted density curves to the first real data.

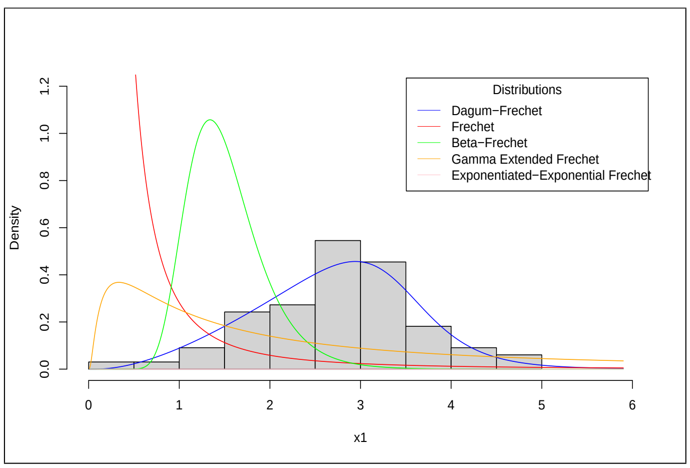

Figure 4. Fitted density curves to the second real data.

Plots of the fitted densities of the five distributions are shown in Figures (3) and (4). The plots illustrate that the D-Fr distribution provides a better fit to the data than other distributions.

9. Conclusion

The development of generalizing families of distributions have attracted the attention of both theoretical and applied statisticians. In this paper, a new family of distributions, called the Dagum-X family of distribution is introduced. The mathematical properties of Dagum-X family of distributions are discussed. A sub model called Dagum-Frechet distribution is presented with some of its properties. The maximum likelihood estimation method was employed for estimating the model parameters and investigated through a simulation study. The simulation study indicates that the maximum likelihood estimation method provides consistent estimators for the parameters. The performance of the Dagum-Frechet distribution was compared to that of beta Frechet, Gamma-Extended-Frechet, Exponentiated-Exponential-Frechet, and Frechet distributions using two real-life data sets for demonstration purposes. The proposed distribution has better fit than other competing distributions. It is concluded that the Dagum-Fréchet distribution is a competitive model for modeling real-life data in different areas.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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