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$K-g-$FUSION FRAMES IN HILBERT C^*-MODULES

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Abstract. In this paper, we introduce the concepts of $g-$fusion frame and $K-g-$fusion frame in Hilbert C^*-modules and we give some properties. Also, we study the stability problem of $g-$fusion frame. The presented results extend, generalize and improve many existing results in the literature.

1. Introduction and Preliminaries

Frame theory is recently an active research area in mathematics, computer science, and engineering with many exciting applications in a variety of different fields.

A frame is a set of vectors in a Hilbert space that can be used to reconstruct each vector in the space from its inner products with the frame vectors. These inner products are generally called the frame coefficients of the vector. But unlike an orthonormal basis each vector may have infinitely many different representations in terms of its frame coefficients.

Frames for Hilbert spaces were introduced by Duffin and Schaefer [1] in 1952 to study some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [2] for signal processing.

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Hilbert C^*-modules is a generalization of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers.

Many generalizations of the concept of frame have been defined in Hilbert C^*-modules [3, 5, 9–13].

In the following, we recall some definitions and results that will be used to prove our mains results.

Let $\mathcal{A}$ be a unital C^*-algebra, let J be countable index set.

Throughout this paper $\mathcal{H}$ and $\mathcal{K}$ are countably generated Hilbert $\mathcal{A}-$modules and $(\mathcal{K}_j)_{j\in \mathrm{J}}$ is a sequence of closed Hilbert submodules of $\mathcal{K}$.

For each $j \in \mathrm{J}$, $End_{\mathcal{A}}^*(\mathcal{H},\mathcal{K}_j)$ is the collection of all adjointable $\mathcal{A}-$linear maps from $\mathcal{H}$ to $\mathcal{K}_j$, and $End_{\mathcal{A}}^*(\mathcal{H},\mathcal{H})$ is denoted by $End_{\mathcal{A}}^*(\mathcal{H})$.

Definition 1.1. *[6]*

Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle .,.\rangle : \mathcal{H}\times\mathcal{H}\to\mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle f,f\rangle \geq 0$ *for all* $f\in\mathcal{H}$ *and* $\langle f,f\rangle = 0$ *if and only if* $f=0$.

(ii) $\langle af+g,h\rangle = a\langle f,h\rangle + \langle g,h\rangle$ *for all* $a\in\mathcal{A}$ *and* $f,g,h\in\mathcal{H}$.

(iii) $\langle f,g\rangle = \langle g,f\rangle^*$ *for all* $f,g\in\mathcal{H}$.

For $f\in\mathcal{H}$, we define $||f|| = ||\langle f,f\rangle||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in a C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|f| = \langle f,f\rangle^{\frac{1}{2}}$ for $f\in\mathcal{H}$.

Define $l^2((\mathcal{K}_j)_{j\in\mathrm{J}})$ by

$$l^2((\mathcal{K}_j)_{j\in\mathrm{J}}) = \{(f_j)_{j\in\mathrm{J}} : f_j\in\mathcal{K}_j, ||\sum_{j\in\mathrm{J}}\langle f_j,f_j\rangle|| < \infty\}.$$

With $\mathcal{A}-$valued inner product is given by

$$\langle (f_j)_{j\in\mathrm{J}}, (g_j)_{j\in\mathrm{J}}\rangle = \sum_{j\in\mathrm{J}}\langle f_j,g_j\rangle,$$

$l^2((\mathcal{K}_j)_{j\in\mathrm{J}})$ is a Hilbert $\mathcal{A}-$module.

Lemma 1.1. *[7] Let $T\in End_{\mathcal{A}}^*(\mathcal{H},\mathcal{K})$ and $\mathcal{H},\mathcal{K}$ are Hilberts $\mathcal{A}-$modules.The following statemnts are multually equivalent:*

(i) *T is surjective.*

(ii) *T^* is bounded below with respect to the norm, i.e., there is $m>0$ such that $||T^*f||\geq m||f||$ for all $f\in\mathcal{K}$.*

(iii) *T^* is bounded below with respect to the inner product, i.e, there is $m^{'}>0$ such that $\langle T^*f,T^*f\rangle \geq m^{'}\langle f,f\rangle$ for all $f\in\mathcal{K}$.*

Lemma 1.2. *[7] Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$, and $T \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $T^* = T$. The following statements are equivalent:*

(i) *T is surjective.*

(ii) *There are $m, M > 0$ such that $m\|f\| \leq \|Tf\| \leq M\|f\|$, for all $f \in \mathcal{H}$.*

(iii) *There are $m', M' > 0$ such that $m'\langle f, f\rangle \leq \langle Tf, Tf\rangle \leq M'\langle f, f\rangle$ for all $f \in \mathcal{H}$.*

Lemma 1.3. *[4] Let $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert $\mathcal{A}$-modules and $T \in End^*(\mathcal{H}, \mathcal{K})$. Then:*

(i) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and*

$$\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2.$$

(ii) *If T is surjective, then the adjointable map TT^* is invertible and*

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.$$

Lemma 1.4. *[8] Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module. If $T \in End^*_{\mathcal{A}}(\mathcal{H})$, then*

$$\langle Tf, Tf\rangle \leq \|T\|^2\langle f, f\rangle, \qquad \forall f \in \mathcal{H}.$$

Lemma 1.5. *[14] Let $\mathcal{A}$ be a C^*-algebra, E, $\mathcal{H}$, $\mathcal{K}$ be Hilbert $\mathcal{A}$-modules. Let $T \in End^*_{\mathcal{A}}(E, \mathcal{K})$ and $T^{'} \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. If $\overline{\mathcal{R}(T^*)}$ is orthogonally complemented, then the following statements are equivalent:*

(i) *$\mathcal{R}(T^{'}) \subseteq \mathcal{R}(T)$.*

(ii) *$T^{'}(T^{'})^* \leq \lambda TT^*$ for some $\lambda > 0$.*

(iii) *There exists a positive real number $\mu > 0$ such that $\|(T^{'})^*f\| \leq \mu\|T^*f\|$, for all $f \in \mathcal{K}$.*

(iv) *There exists a solution $X \in End^*_{\mathcal{A}}(\mathcal{H}, E)$ of the so-called douglas equation $T^{'} = TX$.*

2. $K-g-$fusion frames in Hilbert C^*-modules

We begin this section with the following lemma.

Lemma 2.1. *Let $(W_j)_{j\in\mathrm{J}}$ be a sequence of orthogonally complemented closed submodules of $\mathcal{H}$ and $T \in End^*_{\mathcal{A}}(\mathcal{H})$ invertible, if $T^*TW_j \subset W_j$ for each $j \in \mathrm{J}$, then $(TW_j)_{j\in\mathrm{J}}$ is a sequence of orthogonally complemented closed submodules and $P_{W_j}T^* = P_{W_j}T^*P_{TW_j}$.*

Proof. Firstly for each $j \in \mathrm{J}$, $T : W_j \to TW_j$ is invertible, so each TW_j is a closed submodule of $\mathcal{H}$. We show that $\mathcal{H} = TW_j \oplus T(W_j^{\perp})$. Since $\mathcal{H} = T\mathcal{H}$, then for each $f \in \mathcal{H}$, there exists $g \in \mathcal{H}$ sutch that $f = Tg$. On the other hand $g = u + v$, for some $u \in W_j$ and $v \in W_j^{\perp}$. Hence $f = Tu + Tv$, where $Tu \in TW_j$ and $Tv \in T(W_j^{\perp})$ plainly $TW_j \cap T(W_j^{\perp}) = (0)$, therefore $\mathcal{H} = TW_j \oplus T(W_j^{\perp})$. Hence for every $g \in W_j$, $h \in W_j^{\perp}$ we have $T^*Tg \in W_j$ and therefore $\langle Tg, Th\rangle = \langle T^*Tg, h\rangle = 0$, so $T(W_j^{\perp}) \subset (TW_j)^{\perp}$ and consequently $T(W_j^{\perp}) = (TW_j)^{\perp}$ witch implies that TW_j is orthogonally complemented.

Let $f \in \mathcal{H}$ we have $f = P_{TW_j}f + g$, for some $g \in (TW_j)^{\perp}$, then $T^*f = T^*P_{TW_j}f + T^*g$. Let $v \in W_j$ then $\langle T^*g, v\rangle = \langle g, Tv\rangle = 0$ then $T^*g \in W_j^{\perp}$ and we have $P_{W_j}T^*f = P_{W_j}T^*P_{TW_j}f + P_{W_j}T^*g$, then $P_{W_j}T^*f = P_{W_j}T^*P_{TW_j}f$ thus implies that for each $j \in \mathrm{J}$ we have $P_{W_j}T^* = P_{W_j}T^*P_{TW_j}$. □

Now we define the notion of $K-g-$fusion frame in Hilbert C^*-modules.

Definition 2.1. *let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a countably generated Hilbert $\mathcal{A}-$module. let $(v_j)_{j\in\mathrm{J}}$ be a family of weights in $\mathcal{A}$,i.e.,each v_j is a positive invertible element frome the center of $\mathcal{A}$. Let $(W_j)_{j\in\mathrm{J}}$ be a collection of orthogonally complemented closed submodules of $\mathcal{H}$. Let $(\mathcal{K}_j)_{j\in\mathrm{J}}$ a sequence of closed submodules of $\mathcal{K}$ and $\Lambda_j \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_j)$ for each $j \in \mathrm{J}$. we say $\Lambda = (W_j, \Lambda_j, v_j)_{j\in\mathrm{J}}$ is $g-$fusion frame for $\mathcal{H}$ with respect to $(\mathcal{K}_j)_{j\in\mathrm{J}}$ if there exist real constants $0 < A \leq B < \infty$ such that*

$$A\langle f, f\rangle \leq \sum_{j\in\mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}f, \Lambda_j P_{W_j}f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in \mathcal{H}.$$

The counstants A and B are called the lower and upper bounds of $g-$fusion frame, respectively. If $A = B$ then Λ is called tight $g-$fusion frame and if $A = B = 1$ then we say Λ is a parseval $g-$fusion frame. if the family Λ satisfies

$$\sum_{j\in\mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}f, \Lambda_j P_{W_j}f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in \mathcal{H}.$$

Then it is called a $g-$fusion bessel sequence for $\mathcal{H}$ with bound B.

Lemma 2.2. *let $\Lambda = (W_j, \Lambda_j, v_j)_{j\in\mathrm{J}}$ be a $g-$fusion bessel sequence for $\mathcal{H}$ with bound B. Then for each sequence $(f_j)_{j\in\mathrm{J}} \in l^2((\mathcal{K}_j)_{j\in\mathrm{J}})$, the series $\sum_{j\in\mathrm{J}} v_jP_{W_j}\Lambda_j^*f_j$ is converge unconditionally.*

Proof. let I be a finite subset of J, then

$$\begin{aligned}
||\sum_{j\in\mathrm{I}} v_jP_{W_j}\Lambda_j^*f_j|| &= \sup_{||g||=1} ||\langle\sum_{j\in\mathrm{I}} v_jP_{W_j}\Lambda_j^*f_j, g\rangle|| \\
&\leq ||\sum_{j\in\mathrm{I}}\langle f_j, f_j\rangle||^{\frac{1}{2}} \sup_{||g||=1} ||\sum_{j\in\mathrm{I}} v_j^2\langle\Lambda_jP_{W_j}g, \Lambda_jP_{W_j}g\rangle||^{\frac{1}{2}} \\
&\leq \sqrt{B}||\sum_{j\in\mathrm{I}}\langle f_j, f_j\rangle||^{\frac{1}{2}}.
\end{aligned}$$

And it follows that $\sum_{j\in\mathrm{J}} v_jP_{W_j}\Lambda_j^*f_j$ is unconditionally convergent in $\mathcal{H}$. □

Now, we can define the synthesis operator by lemma 2.2

Definition 2.2. *let $\Lambda = (W_j, \Lambda_j, v_j)_{j\in\mathrm{J}}$ be a $g-$fusion bessel sequence for $\mathcal{H}$. Then the operator T_Λ : $l^2((\mathcal{K}_j)_{j\in\mathrm{J}})) \to \mathcal{H}$ defined by*

$$T_\Lambda((f_j)_{j\in\mathrm{J}}) = \sum_{j\in\mathrm{J}} v_jP_{W_j}\Lambda_j^*f_j, \qquad \forall (f_j)_{j\in\mathrm{J}} \in l^2((\mathcal{K}_j)_{j\in\mathrm{J}}).$$

Is called synthesis operator. We say the adjoint T_Λ^* of the synthesis the analysis operator and it is defined by $T_\Lambda^* : \mathcal{H} \to l^2((\mathcal{K}_j)_{j\in \mathrm{J}})$ such that

$$T_\Lambda^*(f) = (v_j \Lambda_j P_{W_j}(f))_{j\in \mathrm{J}}, \qquad \forall f \in \mathcal{H}.$$

The operator $\mathcal{S}_\Lambda : \mathcal{H} \to \mathcal{H}$ defined by

$$\mathcal{S}_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{j\in \mathrm{J}} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j}(f), \qquad \forall f \in \mathcal{H}.$$

Is called $g-$fusion frame operator. It can be easily verify that

$$\langle \mathcal{S}_\Lambda f, f\rangle = \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j}(f), \Lambda_j P_{W_j}(f)\rangle, \qquad \forall f \in \mathcal{H}. \tag{2.1}$$

Furthermore, if Λ is a $g-$fusion frame with bounds A and B, then

$$A\langle f, f\rangle \leq \langle \mathcal{S}_\Lambda f, f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in \mathcal{H}.$$

It easy to see that the operator $\mathcal{S}_\Lambda$ is bounded, self-adjoint, positive, now we proof the inversibility of $\mathcal{S}_\Lambda$. Let $f \in \mathcal{H}$ we have

$$||T_\Lambda^*(f)|| = ||(v_j \Lambda_j P_{W_j}(f))_{j\in \mathrm{J}}|| = ||\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j}(f), \Lambda_j P_{W_j}(f)\rangle||^{\frac{1}{2}}.$$

Since Λ is $g-$fusion frame then

$$\sqrt{A}||\langle f, f\rangle||^{\frac{1}{2}} \leq ||T_\Lambda^* f||.$$

Then

$$\sqrt{A}||f|| \leq ||T_\Lambda^* f||.$$

Frome lemma1.1, T_Λ is surjective and by lemma1.3, $T_\Lambda T_\Lambda^* = \mathcal{S}_\Lambda$ is invertible. We now, $AI_\mathcal{H} \leq \mathcal{S}_\Lambda \leq BI_\mathcal{H}$ and this gives $B^{-1}I_\mathcal{H} \leq \mathcal{S}_\Lambda^{-1} \leq A^{-1}I_\mathcal{H}$

Definition 2.3. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a countably generated Hilbert $\mathcal{A}-$module. let $(v_j)_{j\in \mathrm{J}}$ be a family of weights in $\mathcal{A}$,i.e.,each v_j is a positive invertible element frome the center of $\mathcal{A}$, let $(W_j)_{j\in \mathrm{J}}$ be a collection of orthogonally complemented closed submodules of $\mathcal{H}$. Let $(\mathcal{K}_j)_{j\in \mathrm{J}}$ a sequence of closed submodules of $\mathcal{K}$ and $\Lambda_j \in End_\mathcal{A}^*(\mathcal{H}, \mathcal{K}_j)$ for each $j \in \mathrm{J}$ and $K \in End_\mathcal{A}^*(\mathcal{H})$. We say $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ is $K-g-$fusion frame for $\mathcal{H}$ with respect to $(\mathcal{K}_j)_{j\in \mathrm{J}}$ if there exist real constants $0 < A \leq B < \infty$ such that*

$$A\langle K^* f, K^* f\rangle \leq \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \leq B\langle f, f\rangle, \qquad \forall f \in \mathcal{H}. \tag{2.2}$$

The constants A and B are called a lower and upper bounds of $K-g-$fusion frame, respectively. If the left-hand inequality of (2.2) is an equality, we say that Λ is a tight $K-g-$fusion frame. If $K = I_\mathcal{H}$ then Λ is a $g-$fusion frame and if $K = I_\mathcal{H}$ and $\Lambda_j = P_{W_j}$ for any $j \in \mathrm{J}$, then Λ is a fusion frame for $\mathcal{H}$

Example 2.1. *Let H be a Hilbert C^*-module with dimensional 3 and let $\{e_1, e_2, e_3\}$ be standard basis.*

We define the operator K on H by

$$Ke_1 = e_2, \quad Ke_2 = e_3, \quad Ke_3 = e_3;$$

Suppose that $W_j = \mathcal{K}_j = span\{e_j\}$ where $j = 1, 2, 3$. Let

$$\Lambda_j x = \langle x, e_j \rangle e_j,$$

it is clear that $(W_j, \Lambda_j, 1)_{j\in J}$ is a $K - g-$fusion frame for H.

Remark 2.1. *If $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ is $K - g-$fusion frame for $\mathcal{H}$ with bounds A and B, then we have*

$$AKK^* \leq \mathcal{S}_\Lambda \leq BI_{\mathcal{H}} \tag{2.3}$$

From inequalities (2.1) and (2.3), we have

Lemma 2.3. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ be a $g-$fusion bessel sequence for $\mathcal{H}$. Then Λ is $K - g-$fusion frame for $\mathcal{H}$ if and only if there exist a constant $A > 0$ such that $AKK^* \leq \mathcal{S}_\Lambda$, where $\mathcal{S}_\Lambda$ is the frame operator for Λ.*

Theorem 2.1. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$, and $\Lambda = (W_j, P_{W_j}, v_j)_{j\in \mathrm{J}}$ be a $g-$fusion bessel sequence for $\mathcal{H}$ with frame operator $\mathcal{S}_\Lambda$ such that $\overline{\mathcal{R}(\mathcal{S}_\Lambda^{\frac{1}{2}})}$ is orthogonally complemented. Then Λ is $K - g-$fusion frame for $\mathcal{H}$ if and only if $K = \mathcal{S}_\Lambda^{\frac{1}{2}} M$ for some $M \in End^*_{\mathcal{A}}(\mathcal{H})$.*

Proof. Suppose Λ is a $K - g-$fusion frame for $\mathcal{H}$, then there exist $A > 0$ such that $AKK^* \leq \mathcal{S}_\Lambda$, and $\mathcal{S}_\Lambda$ is self-adjoint and positive thus $\mathcal{S}_\Lambda^{\frac{1}{2}}$ is self-adjoint and positive, so we have

$$KK^* \leq \frac{1}{A} \mathcal{S}_\Lambda^{\frac{1}{2}} \mathcal{S}_\Lambda^{\frac{1}{2}}.$$

By lemma1.5, there exists some $M \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $K = \mathcal{S}_\Lambda^{\frac{1}{2}} M$.

Suppose that there exists an operator $M \in End^*_{\mathcal{A}}(\mathcal{H})$ so that $K = \mathcal{S}_\Lambda^{\frac{1}{2}} M$. From Lemma1.5 , we know that $AKK^* \leq \mathcal{S}_\Lambda$ for some constant $A > 0$, from Lemma2.3, Λ is a $K - g-$fusion frame. □

Theorem 2.2. *If $U \in End^*_{\mathcal{A}}(\mathcal{H})$ and Λ is a $K-g-$fusion frame for $\mathcal{H}$, and $\mathcal{R}(U) \subset \mathcal{R}(K)$ such that $\overline{\mathcal{R}(K^*)}$ is orthogonaly complemented. Then Λ is $U - g-$fusion frame for $\mathcal{H}$.*

Proof. By Lemma1.5, $\exists \lambda > 0$: $UU^* \leq \lambda KK^*$, then for each $f \in \mathcal{H}$ we have

$$\langle U^* f, U^* f \rangle = \langle UU^* f, f \rangle \leq \langle \lambda KK^* f, f \rangle \leq \lambda \langle K^* f, K^* f \rangle.$$

It follows that,

$$\frac{A}{\lambda} \langle U^* f, U^* f \rangle \leq \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f \rangle, \qquad \forall f \in \mathcal{H}.$$

So Λ is a $U - g-$fusion frame for $\mathcal{H}$. □

Theorem 2.3. *Let $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ and $\Gamma = (V_j, \Gamma_j, u_j)_{j\in \mathrm{J}}$ be two $g-$fusion bessel sequence for $\mathcal{H}$ with bounds B_1 and B_2, respectively. Suppose that T_Λ and T_Γ are their synthesis operators such that $T_\Gamma T_\Lambda^* = K^*$ where $K \in End_{\mathcal{A}}^*(\mathcal{H})$. Then, both Λ and Γ are K and $K^* - g-$fusion frames, respectively.*

Proof. Let $f \in \mathcal{H}$, we have

$$\langle K^*f, K^*f\rangle = \langle T_\Gamma T_\Lambda^* f, T_\Gamma T_\Lambda^* f\rangle \leq ||T_\Gamma||^2 \langle T_\Lambda^* f, T_\Lambda^* f\rangle \leq B_2 \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle.$$

So,

$$B_2^{-1}\langle K^*f, K^*f\rangle \leq \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle.$$

Thus Λ is $K - g-$ fusion frame for $\mathcal{H}$. Similarly, Γ is $K^* - g-$fusion frame for $\mathcal{H}$ with the lower bound B_1^{-1}. □

Theorem 2.4. *Let $K \in End_{\mathcal{A}}^*(\mathcal{H})$ and $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ be a $g-$fusion bessel sequence for $\mathcal{H}$, with synthesis operator T_Λ of Λ. Suppose that $\overline{\mathcal{R}(T_\Lambda^*)}$ and $\overline{\mathcal{R}(K^*)}$ are orthogonaly complemented, the following statements hold:*

1. *If Λ is a tight $K - g-$fusion frame for $\mathcal{H}$, then $\mathcal{R}(K) = \mathcal{R}(T_\Lambda)$.*
2. *$\mathcal{R}(K) = \mathcal{R}(T_\Lambda)$ if and only if there exist two constants $0 < A \leq B < \infty$ such that:*

$$A\langle K^*f, K^*f\rangle \leq \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \leq B\langle K^*f, K^*f\rangle, \qquad \forall f \in \mathcal{H}.$$

Proof. (1) Suppose that Λ is a tight $K - g-$fusion frame for $\mathcal{H}$, then there exixt $A > 0$, such that for each $f \in \mathcal{H}$

$$\begin{aligned} A\langle K^*f, K^*f\rangle &= \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \\ &= \langle (v_j \Lambda_j P_{W_j} f)_{j\in \mathrm{J}}, (v_j \Lambda_j P_{W_j} f)_{j\in \mathrm{J}}\rangle \\ &= \langle T_\Lambda^* f, T_\Lambda^* f\rangle. \end{aligned}$$

Then

$$\langle AKK^*f, f\rangle = \langle T_\Lambda T_\Lambda^* f, f\rangle.$$

So

$$AKK^* = T_\Lambda T_\Lambda^*.$$

Then by lemma1.5 , $\mathcal{R}(T_\Lambda) = \mathcal{R}(K)$

(2) Suppose that $\mathcal{R}(K) = \mathcal{R}(T_\Lambda)$, by lemma1.5 there exist two constants A, $B > 0$ such that

$$AKK^* \leq T_\Lambda T_\Lambda^* \leq BKK^*.$$

Which implies that for each $f \in \mathcal{H}$

$$\langle AKK^*f, f\rangle \leq \langle T_\Lambda T_\Lambda^* f, f\rangle \leq \langle BKK^*f, f\rangle.$$

$$A\langle KK^*f, f\rangle \leq \langle T_\Lambda T_\Lambda^* f, f\rangle \leq B\langle KK^*f, f\rangle.$$

Therefore

$$A\langle K^*f, K^*f\rangle \leq \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \leq B\langle K^*f, K^*f\rangle.$$

Suppose that there exist two constants A, $B > 0$ such that for each $f \in \mathcal{H}$

$$A\langle K^*f, K^*f\rangle \leq \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \leq B\langle K^*f, K^*f\rangle.$$

Then

$$A\langle KK^*f, f\rangle \leq \langle T_\Lambda T_\Lambda^* f, f\rangle \leq B\langle KK^*f, f\rangle.$$

So

$$AKK^* \leq T_\Lambda T_\Lambda^* \leq BKK^*.$$

Since by lemma1.5 , $\mathcal{R}(T_\Lambda) = \mathcal{R}(K)$. □

Theorem 2.5. *Let $U \in End_{\mathcal{A}}^*(\mathcal{H})$ be an invertible operator on $\mathcal{H}$ and $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ be a $K-g-$fusion frame for $\mathcal{H}$ for some $K \in End_{\mathcal{A}}^*(\mathcal{H})$. Suppose that $U^*UW_j \subset W_j$, $\forall j \in \mathrm{J}$. Then $\Gamma = (UW_j, \Lambda_j P_{W_j} U^*, v_j)_{j\in \mathrm{J}}$ is a $UKU^* - g-$fusion frame for $\mathcal{H}$.*

Proof. Since Λ is a $K - g-$fusion frame for $\mathcal{H}$, $\exists\ A, B > 0$ such that

$$A\langle K^*f, K^*f\rangle \leq \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j}(f), \Lambda_j P_{W_j}(f)\rangle \leq B\langle f, f\rangle, \qquad \forall f \in \mathcal{H}.$$

Also, U is an invertible linear operator on $\mathcal{H}$, so for any $j \in \mathrm{J}$, UW_j is closed in $\mathcal{H}$. Now, for each $f \in \mathcal{H}$, using lemma2.1, we obtain

$$\begin{aligned}\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} U^* P_{UW_j}(f), \Lambda_j P_{W_j} U^* P_{UW_j}(f)\rangle &= \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} U^*(f), \Lambda_j P_{W_j} U^*(f)\rangle \\ &\leq B\langle U^*f, U^*f\rangle \\ &\leq B\|U\|^2\langle f, f\rangle.\end{aligned}$$

On the other hand, for each $f \in \mathcal{H}$

$$\begin{aligned}
A\langle (UKU^*)^*f, (UKU^*)^*f\rangle &= A\langle UK^*U^*f, UK^*U^*f\rangle \\
&\leq A||U||^2\langle K^*U^*f, K^*U^*f\rangle \\
&\leq ||U||^2\sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}U^*(f), \Lambda_j P_{W_j}U^*(f)\rangle \\
&\leq ||U||^2\sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}U^*P_{UW_j}(f), \Lambda_j P_{W_j}U^*P_{UW_j}(f)\rangle,
\end{aligned}$$

Then

$$\frac{A}{||U||^2}\langle (UKU^*)^*f, (UKU^*)^*f\rangle \leq \sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}U^*P_{UW_j}(f), \Lambda_j P_{W_j}U^*P_{UW_j}(f)\rangle$$

Therefore, Γ is $UKU^* - g-$fusion frame for $\mathcal{H}$. □

Theorem 2.6. *Let $U \in End_{\mathcal{A}}^*(\mathcal{H})$ be an invertible operator on $\mathcal{H}$ and $\Gamma = (UW_j, \Lambda_j P_{W_j}U^*, v_j)_{j\in \mathrm{J}}$ be a $K - g-$fusion frame for $\mathcal{H}$ for some $K \in End_{\mathcal{A}}^*(\mathcal{H})$. Suppose that $U^*UW_j \subset W_j$, $\forall j \in \mathrm{J}$. Then $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ is a $U^{-1}KU - g-$fusion frame for $\mathcal{H}$.*

Proof. Since $\Gamma = (UW_j, \Lambda_j P_{W_j}, v_j)_{j\in \mathrm{J}}$ is $K - g-$fusion frame for $\mathcal{H}$, for all $f \in \mathcal{H}$, $\exists$ A, $B > 0$ such that

$$A\langle K^*f, K^*f\rangle \leq \sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}U^*P_{UW_j}, \Lambda_j P_{W_j}U^*P_{UW_j}\rangle \leq B\langle f, f\rangle.$$

Let $f \in \mathcal{H}$, we have

$$\begin{aligned}
A\langle (U^{-1}KU)^*f, (U^{-1}KU)^*f\rangle &= A\langle U^*K^*(U^{-1})^*f, U^*K^*(U^{-1})^*f\rangle \\
&\leq A||U^*||^2\langle K^*(U^{-1})^*f, K^*(U^{-1})^*f\rangle \\
&\leq ||U||^2\sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}U^*P_{UW_j}(U^{-1})^*f, \Lambda_j P_{W_j}U^*P_{UW_j}(U^{-1})^*f\rangle \\
&\leq ||U||^2\sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}U^*(U^{-1})^*f, \Lambda_j P_{W_j}U^*(U^{-1})^*f\rangle \\
&= ||U||^2\sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}f, \Lambda_j P_{W_j}f\rangle.
\end{aligned}$$

Then, for each $f \in \mathcal{H}$, we have

$$\frac{A}{||U||^2}\langle (U^{-1}KU)^*f, (U^{-1}KU)^*f\rangle \leq \sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}f, \Lambda_j P_{W_j}f\rangle.$$

Also, for each $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle &= \sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j} U^*(U^{-1})^* f, \Lambda_j P_{W_j} U^*(U^{-1})^* f\rangle \\ &= \sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j} U^* P_{UW_j}(U^{-1})^* f, \Lambda_j P_{W_j} U^* P_{UW_j}(U^{-1})^* f\rangle \\ &\leq B\langle (U^{-1})^* f, (U^{-1})^* f\rangle \\ &\leq B\|U^{-1}\|^2\langle f, f\rangle.\end{aligned}$$

Thus, Λ is a $U^{-1}KU - g-$fusion frame for $\mathcal{H}$. □

Theorem 2.7. *Let $K \in End_{\mathcal{A}}^*(\mathcal{H})$ be an invertible operator on $\mathcal{H}$ and $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ be a $g-$fusion frame for $\mathcal{H}$ with frame bounds A, B and $\mathcal{S}_\Lambda$ be the associated $g-$fusion frame operator. Suppose that for all $j \in \mathrm{J}$, $T^*TW_j \subset W_j$, where $T = K\mathcal{S}_\Lambda^{-1}$. Then $(K\mathcal{S}_\Lambda^{-1}W_j, \Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1}K^*, v_j)_{j\in \mathrm{J}}$ is a $K - g-$fusion frame for $\mathcal{H}$ with the corresponding $g-$fusion frame operator $K\mathcal{S}_\Lambda^{-1}K^*$.*

Proof. We now $T = K\mathcal{S}_\Lambda^{-1}$ is invertible on $\mathcal{H}$ and $T^* = (K\mathcal{S}_\Lambda^{-1})^* = \mathcal{S}_\Lambda^{-1}K^*$. For each $f \in \mathcal{H}$, we have

$$\begin{aligned}\langle K^* f, K^* f\rangle &= \langle \mathcal{S}_\Lambda\mathcal{S}_\Lambda^{-1}K^* f, \mathcal{S}_\Lambda\mathcal{S}_\Lambda^{-1}K^* f\rangle \\ &\leq \|\mathcal{S}_\Lambda\|^2\langle \mathcal{S}_\Lambda^{-1}K^* f, \mathcal{S}_\Lambda^{-1}K^* f\rangle \\ &\leq B^2\langle \mathcal{S}_\Lambda^{-1}K^* f, \mathcal{S}_\Lambda^{-1}K^* f\rangle.\end{aligned}$$

Now for each $f \in \mathcal{H}$, we get

$$\begin{aligned}\sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j} T^* P_{TW_j}(f), \Lambda_j P_{W_j} T^* P_{TW_j}(f)\rangle &= \sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j} T^*(f), \Lambda_j P_{W_j} T^*(f)\rangle \\ &\leq B\langle T^* f, T^* f\rangle \\ &\leq B\|T\|^2\langle f, f\rangle \\ &\leq B\|\mathcal{S}_\Lambda^{-1}\|^2\|K\|^2\langle f, f\rangle \\ &\leq \frac{B}{A^2}\|K\|^2\langle f, f\rangle.\end{aligned}$$

On the other hand, for each $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j} T^* P_{TW_j}(f), \Lambda_j P_{W_j} T^* P_{TW_j}(f)\rangle &= \sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j} T^*(f), \Lambda_j P_{W_j} T^*(f)\rangle \\ &\geq A\langle T^* f, T^* f\rangle \\ &= A\langle \mathcal{S}_\Lambda^{-1}K^* f, \mathcal{S}_\Lambda^{-1}K^* f\rangle \\ &\geq \frac{A}{B^2}\langle K^* f, K^* f\rangle.\end{aligned}$$

Thus $(K\mathcal{S}_\Lambda^{-1}W_j, \Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1}K^*, v_j)_{j\in \mathrm{J}}$ is a $K-g-$fusion frame for $\mathcal{H}$.

For each $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j\in\mathrm{J}} v_j^2 P_{TW_j}(\Lambda_j P_{W_j}T^*)^*(\Lambda_j P_{W_j}T^*)P_{TW_j}f &= \sum_{j\in\mathrm{J}} v_j^2 P_{TW_j}TP_{W_j}\Lambda_j^*(\Lambda_j P_{W_j}T^*)P_{TW_j}f \\ &= \sum_{j\in\mathrm{J}} v_j^2 (P_{W_j}T^*P_{TW_j})^*\Lambda_j^*\Lambda_j(P_{W_j}T^*P_{TWj})f \\ &= \sum_{j\in\mathrm{J}} v_j^2 TP_{W_j}\Lambda_j^*\Lambda_j P_{W_j}T^*f \\ &= T(\sum_{j\in\mathrm{J}} v_j^2 P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}T^*f) \\ &= T\mathcal{S}_\Lambda T^*(f) = K\mathcal{S}_\Lambda^{-1}K^*(f).\end{aligned}$$

This implies that $K\mathcal{S}_\Lambda^{-1}K^*$ is the associated $g-$fusion frame operator. □

Theorem 2.8. *Let $\Lambda = (W_j, \Lambda_j, v_j)_{j\in\mathrm{J}}$ be a $K-g-$fusion frame for $\mathcal{H}$ with bounds A, B and for each $j \in J$, $T_j \in End_{\mathcal{A}}^*(\mathcal{K}_j)$ be invertible operator. Suppose*

$$0 < m = \inf_{j\in\mathrm{J}} \frac{1}{||T_j^{-1}||} \leq \sup_{j\in\mathrm{J}} ||T_j|| = M.$$

If $T \in End_{\mathcal{A}}^*(\mathcal{H})$ is an invertible operator on $\mathcal{H}$ with $KT = TK$ and $T^*TW_j \subset W_j$, $\forall j \in \mathrm{J}$ then $\Gamma = (TW_j, T_j\Lambda_j P_{W_j}T^*, v_j)_{j\in\mathrm{J}}$ is a $K-g-$fusion frame for $\mathcal{H}$.

Proof. Since T and T_j(for each $j \in \mathrm{J}$) are invertible, so

$$\begin{aligned}\langle K^*f, K^*f\rangle &= \langle (T^{-1})^*T^*K^*f, (T^{-1})^*T^*K^*f\rangle \\ &\leq ||(T^{-1})||^2\langle T^*K^*f, T^*K^*f\rangle,\end{aligned}$$

$$\begin{aligned}\langle f, f\rangle &= \langle T_j^{-1}T_jf, T_j^{-1}T_jf\rangle \\ &\leq ||T_j^{-1}||^2\langle T_jf, T_jf\rangle.\end{aligned}$$

For each $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j\in\mathrm{J}} v_j^2\langle T_j\Lambda_j P_{W_j}T^*P_{TW_j}f, T_j\Lambda_j P_{W_j}T^*P_{TW_j}f\rangle &= \sum_{j\in\mathrm{J}} v_j^2\langle T_j\Lambda_j P_{W_j}T^*f, T_j\Lambda_j P_{W_j}T^*f\rangle \\ &\leq \sum_{j\in\mathrm{J}} ||T_j||^2 v_j^2\langle \Lambda_j P_{W_j}T^*f, \Lambda_j P_{W_j}T^*f\rangle \\ &\leq M^2\sum_{j\in\mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}T^*f, \Lambda_j P_{W_j}T^*f\rangle \\ &\leq M^2B\langle T^*f, T^*f\rangle \\ &\leq M^2B||T||^2\langle f, f\rangle.\end{aligned}$$

On the other hand, for each $f \in \mathcal{H}$, we have

$$\begin{aligned}\sum_{j\in \mathrm{J}} v_j^2\langle T_j\Lambda_j P_{W_j}T^*P_{TW_j}f, T_j\Lambda_j P_{W_j}T^*P_{TW_j}f\rangle &= \sum_{j\in \mathrm{J}} v_j^2\langle T_j\Lambda_j P_{W_j}T^*f, T_j\Lambda_j P_{W_j}T^*f\rangle \\ &\geq \sum_{j\in \mathrm{J}} \frac{1}{\|T_j^{-1}\|^2} v_j^2\langle \Lambda_j P_{W_j}T^*f, \Lambda_j P_{W_j}T^*f\rangle \\ &\geq m^2\sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}T^*f, \Lambda_j P_{W_j}T^*f\rangle \\ &\geq m^2A\langle K^*T^*f, K^*T^*f\rangle \\ &\geq \frac{m^2A}{\|(T^{-1})\|^2}\langle K^*f, K^*f\rangle.\end{aligned}$$

Thus, Γ is a $K-g-$fusion frame for $\mathcal{H}$. □

In this theorem we give a necessary and sufficient condition for a quotient operator to be bounded.

Theorem 2.9. *Let $K \in End_{\mathcal{A}}^*(\mathcal{H})$, and $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ be a $K-g-$fusion frame for $\mathcal{H}$ with frame operator $\mathcal{S}_\Lambda$ and frame bounds A and B. Let $U \in End_{\mathcal{A}}^*(\mathcal{H})$ be an invertible operator on $\mathcal{H}$, and $U^*UW_j \subset W_j, \forall j \in \mathrm{J}$. Then the following statements are equivalent:*

1. *$\Gamma = (UW_j, \Lambda_j P_{W_j}U^*, v_j)_{j\in \mathrm{J}}$ is a $UK-g-$fusion frame.*
2. *The quotient operator $[(UK)^*/\mathcal{S}_\Lambda^{\frac{1}{2}}U^*]$ is bounded.*
3. *The quotient operator $[(UK)^*/(U\mathcal{S}_\Lambda U^*)^{\frac{1}{2}}]$ is bounded.*

Proof. (1) $\implies$ (2) Since Γ is $K-g-$fusion frame then there exist A, $B > 0$ such that for each $f \in \mathcal{H}$

$$A\langle (UK)^*f, (UK)^*\rangle \leq \sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}U^*P_{UW_j}f, \Lambda_j P_{W_j}U^*P_{UW_j}f\rangle \leq B\langle f, f\rangle$$

For each $f \in \mathcal{H}$ we have

$$\begin{aligned}\sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}U^*P_{UW_j}f, \Lambda_j P_{W_j}U^*P_{UW_j}f\rangle &= \sum_{j\in \mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}U^*f, \Lambda_j P_{W_j}U^*f\rangle \\ &= \langle \mathcal{S}_\Lambda U^*f, U^*f\rangle \\ &= \langle \mathcal{S}_\Lambda^{\frac{1}{2}}U^*f, \mathcal{S}_\Lambda^{\frac{1}{2}}U^*f\rangle.\end{aligned}$$

Then

$$A\langle (UK)^*f, (UK)^*f\rangle \leq \langle \mathcal{S}_\Lambda^{\frac{1}{2}}(U^*f), \mathcal{S}_\Lambda^{\frac{1}{2}}(U^*f)\rangle.$$

We define the operator: $T : \mathcal{R}(\mathcal{S}_\Lambda^{\frac{1}{2}}U^*) \to \mathcal{R}((UK)^*)$ by

$$T(\mathcal{S}_\Lambda^{\frac{1}{2}}U^*f) = (UK)^*f, \qquad \forall f \in \mathcal{H}.$$

T is linear operator and $Ker(\mathcal{S}_\Lambda^{\frac{1}{2}}U^*) \subset Ker((UK)^*)$. Thus T is well-defined quotient operator. Therefore for each $f \in \mathcal{H}$

$$\begin{aligned} ||T(\mathcal{S}_\Lambda^{\frac{1}{2}}U^*f)|| &= ||(UK)^*f|| \\ &\leq \frac{1}{\sqrt{A}}||\langle \mathcal{S}_\Lambda^{\frac{1}{2}}U^*f, \mathcal{S}_\Lambda^{\frac{1}{2}}U^*f\rangle||^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{A}}||\mathcal{S}_\Lambda^{\frac{1}{2}}U^*f||. \end{aligned}$$

So T is bounded.

(2) $\implies$ (3) Suppose that the quotion operator $[(UK)^*/\mathcal{S}_\Lambda^{\frac{1}{2}}U^*]$ is bounded. Thus for all $f \in \mathcal{H}$, $\exists C > 0$ such that

$$\begin{aligned} ||(UK)^*f|| &\leq C||\mathcal{S}_\Lambda^{\frac{1}{2}}U^*f|| \\ &\leq C||\langle \mathcal{S}_\Lambda^{\frac{1}{2}}U^*f, \mathcal{S}_\Lambda^{\frac{1}{2}}U^*f\rangle||^{\frac{1}{2}} \\ &\leq C||\langle U\mathcal{S}_\Lambda U^*f, f\rangle||^{\frac{1}{2}} \\ &\leq C||\langle (UKU^*)^{\frac{1}{2}}f, (UKU^*)^{\frac{1}{2}}f\rangle||^{\frac{1}{2}} \\ &\leq C||(U\mathcal{S}_\Lambda U^*)^{\frac{1}{2}}f||. \end{aligned}$$

Hence the quotient operator $[(UK)^*/(U\mathcal{S}_\Lambda U^*)^{\frac{1}{2}}]$ is bounded.

(3) $\implies$ (1) For each $f \in \mathcal{H}$, we have

$$\begin{aligned} \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} U^* P_{UW_j} f, \Lambda_j P_{W_j} U^* P_{UW_j} f\rangle &= \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} U^* f, \Lambda_j P_{W_j} U^* f\rangle \\ &\geq A\langle K^*(U^*f), K^*(U^*f)\rangle \\ &= A\langle (UK)^*f, (UK)^*f\rangle. \end{aligned}$$

On the other hand for each $f \in \mathcal{H}$

$$\begin{aligned} \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} U^* P_{UW_j} f, \Lambda_j P_{W_j} U^* P_{UW_j} f\rangle &= \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} U^* f, \Lambda_j P_{W_j} U^* f\rangle \\ &\leq B\langle U^*f, U^*f\rangle \\ &\leq B||U||^2\langle f, f\rangle. \end{aligned}$$

Hence Γ is a $UK-g-$fusion frame for $\mathcal{H}$. □

Studying $g-$fusion frame in Hilbert C^*-modules with different C^*-algebras is interesting and important. In the following, we study this situation.

In the next theorem we take $\mathcal{K}_j \subset \mathcal{H}$ for each $j \in \mathrm{J}$.

Theorem 2.10. *Let $(\mathcal{H}, \mathcal{A}, \langle ., .\rangle_{\mathcal{A}})$ and $(\mathcal{H}, \mathcal{B}, \langle ., .\rangle_{\mathcal{B}})$ be two Hilbert C^*-modules and $\phi : \mathcal{A} \to \mathcal{B}$ be a $*-$homomorphism and θ be a map on $\mathcal{H}$ such that $\langle \theta f, \theta g\rangle_{\mathcal{B}} = \phi(\langle f, g\rangle_{\mathcal{A}})$ for all $f, g \in \mathcal{H}$. Also, suppose that $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ is a $g-$fusion frame for $(\mathcal{H}, \mathcal{A}, \langle ., .\rangle_{\mathcal{A}})$ with $g-$fusion frame operator $\mathcal{S}_{\mathcal{A}}$ and lower and upper $g-$fusion frame bounds A, B respectively. If θ is surjective and $\theta\Lambda_j P_{W_j} = \Lambda_j P_{W_j}\theta$ for each $j \in \mathrm{J}$, then $\Lambda = (W_j, \Lambda_j, \phi(v_j))_{j\in \mathrm{J}}$ is a $g-$fusion frame for $(\mathcal{H}, \mathcal{B}, \langle ., .\rangle_{\mathcal{B}})$ with $g-$fusion frame operator $\mathcal{S}_{\mathcal{B}}$ and lower and upper $g-$fusion frame bounds $\phi(A)$, $\phi(B)$ respectively, and $\langle \mathcal{S}_{\mathcal{B}}\theta f, \theta g\rangle_{\mathcal{B}} = \phi(\langle \mathcal{S}_{\mathcal{A}} f, g\rangle_{\mathcal{A}})$.*

Proof. Let $g \in \mathcal{H}$ then there exists $f \in \mathcal{H}$ such that $\theta f = g$. By the definition of $g-$fusion frame we have

$$A\langle f, f\rangle_{\mathcal{A}} \leq \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle_{\mathcal{A}} \leq B\langle f, f\rangle_{\mathcal{A}}.$$

Then

$$\phi(A\langle f, f\rangle_{\mathcal{A}}) \leq \sum_{j\in \mathrm{J}} \phi(v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle_{\mathcal{A}}) \leq \phi(B\langle f, f\rangle_{\mathcal{A}}).$$

By the definition of $*-$homomorphism we have

$$A\phi(\langle f, f\rangle_{\mathcal{A}}) \leq \sum_{j\in \mathrm{J}} \phi(v_j^2)\phi(\langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle_{\mathcal{A}}) \leq B\phi(\langle f, f\rangle_{\mathcal{A}}).$$

By the relation betwen θ and ϕ we get

$$A\langle \theta f, \theta f\rangle_{\mathcal{B}} \leq \sum_{j\in \mathrm{J}} \phi(v_j)^2 \langle \theta\Lambda_j P_{W_j} f, \theta\Lambda_j P_{W_j} f\rangle_{\mathcal{B}} \leq B\langle \theta f, \theta f\rangle_{\mathcal{B}}.$$

Then

$$A\langle \theta f, \theta f\rangle_{\mathcal{B}} \leq \sum_{j\in \mathrm{J}} \phi(v_j)^2 \langle \Lambda_j P_{W_j} \theta f, \Lambda_j P_{W_j} \theta f\rangle_{\mathcal{B}} \leq B\langle \theta f, \theta f\rangle_{\mathcal{B}}.$$

So, we have

$$A\langle g, g\rangle_{\mathcal{B}} \leq \sum_{j\in \mathrm{J}} \phi(v_j)^2 \langle \Lambda_j P_{W_j} g, \Lambda_j P_{W_j} g\rangle_{\mathcal{B}} \leq B\langle g, g\rangle_{\mathcal{B}}, \qquad \forall g \in \mathcal{H}.$$

On the other hand we have

$$\begin{aligned}
\phi(\langle \mathcal{S}_{\mathcal{A}} f, g\rangle_{\mathcal{A}}) &= \phi(\langle \sum_{j\in \mathrm{J}} v_j^2 P_{W_j}\Lambda_j^*\Lambda_j P_{W_j} f, g\rangle_{\mathcal{A}}) \\
&= \sum_{j\in \mathrm{J}} \phi(v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} g\rangle_{\mathcal{A}}) \\
&= \sum_{j\in \mathrm{J}} \phi(v_j)^2 \langle \theta\Lambda_j P_{W_j} f, \theta\Lambda_j P_{W_j} g\rangle_{\mathcal{B}} \\
&= \sum_{j\in \mathrm{J}} \phi(v_j)^2 \langle \Lambda_j P_{W_j} \theta f, \Lambda_j P_{W_j} \theta g\rangle_{\mathcal{B}} \\
&= \sum_{j\in \mathrm{J}} \phi(v_j)^2 \langle P_{W_j}\Lambda_j^*\Lambda_j P_{W_j} \theta f, \theta g\rangle_{\mathcal{B}} \\
&= \langle \sum_{j\in \mathrm{J}} \phi(v_j)^2 P_{W_j}\Lambda_j^*\Lambda_j P_{W_j} P_{W_j} \theta f, \theta g\rangle_{\mathcal{B}} \\
&= \langle \mathcal{S}_{\mathcal{B}} \theta f, \theta g\rangle_{\mathcal{B}}.
\end{aligned}$$

□

3. STABILITY OF G-FUSION FRAMES IN HILBERT C^*-MODULES

Frome Theorem 2.7 if $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ is a $g-$fusion frame for $\mathcal{H}$ with associated frame operator $\mathcal{S}_\Lambda$, such that $\mathcal{S}_\Lambda^{-2} W_j \subset W_j$, for all $j \in \mathrm{J}$ then $\tilde{\Lambda} = (\mathcal{S}_\Lambda^{-1} W_j, \Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1}, v_j)_{j\in \mathrm{J}}$ is called the canonical dual $g-$fusion frame of Λ. The frame operator $\mathcal{S}_{\tilde{\Lambda}}$ of $\tilde{\Lambda}$ is described by, for each $f \in \mathcal{H}$

$$\begin{aligned}
\mathcal{S}_{\tilde{\Lambda}}(f) &= \sum_{j\in \mathrm{J}} v_j^2 P_{\mathcal{S}_\Lambda^{-1} W_j} (\Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1})^* (\Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1}) P_{\mathcal{S}_\Lambda^{-1} W_j}(f) \\
&= \sum_{j\in \mathrm{J}} v_j^2 P_{\mathcal{S}_\Lambda^{-1} W_j} \mathcal{S}_\Lambda^{-1} P_{W_j}\Lambda_j^*\Lambda_j (P_{W_j}\mathcal{S}_\Lambda^{-1} P_{\mathcal{S}_\Lambda^{-1} W_j})(f) \\
&= \sum_{j\in \mathrm{J}} v_j^2 (P_{W_j}\mathcal{S}_\Lambda^{-1} P_{\mathcal{S}_\Lambda^{-1} W_j})^* \Lambda_j^*\Lambda_j (P_{W_j}\mathcal{S}_\Lambda^{-1} P_{\mathcal{S}_\Lambda^{-1} W_j})(f) \\
&= \sum_{j\in \mathrm{J}} v_j^2 (P_{W_j}\mathcal{S}_\Lambda^{-1})^* \Lambda_j^*\Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1}(f) \\
&= \sum_{j\in \mathrm{J}} v_j^2 \mathcal{S}_\Lambda^{-1} P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1}(f) \\
&= \mathcal{S}_\Lambda^{-1} \sum_{j\in \mathrm{J}} v_j^2 P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1}(f) \\
&= \mathcal{S}_\Lambda^{-1}(\mathcal{S}_\Lambda(\mathcal{S}_\Lambda^{-1} f)) = \mathcal{S}_\Lambda^{-1}(f).
\end{aligned}$$

Theorem 3.1. *Let $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ and $\Gamma = (V_j, \Gamma_j, v_j)_{j\in \mathrm{J}}$ be two $g-$fusion frames for $\mathcal{H}$ with lower frame bounds A and C, respectively. Suppose that $\mathcal{S}_\Lambda^{-2} W_j \subset W_j$ and $\mathcal{S}_\Gamma^{-2} V_j \subset V_j$, $\forall j \in \mathrm{J}$. If there exist real* *constant $D > 0$ such that for all $f \in \mathcal{H}$*

$$||\sum_{j\in\mathrm{J}} v_j^2\langle\Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle - \sum_{j\in\mathrm{J}} v_j^2\langle\Gamma_j P_{V_j} f, \Gamma_j P_{V_j} f\rangle|| \leq D||\langle f, f\rangle||.$$

Then for all $f \in \mathcal{H}$

$$||\sum_{j\in\mathrm{J}} v_j^2\langle\tilde{\Lambda}_j P_{\tilde{W}_j}(f), \tilde{\Lambda}_j P_{\tilde{W}_j}(f)\rangle - \sum_{j\in\mathrm{J}} v_j^2\langle\tilde{\Gamma}_j P_{\tilde{V}_j}(f), \tilde{\Gamma}_j P_{\tilde{V}_j}(f)\rangle|| \leq \frac{D}{AC}||\langle f, f\rangle||.$$

Such that $\tilde{W}_j = \mathcal{S}_\Lambda^{-1} W_j$, $\tilde{\Lambda}_j = \Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1}$, $\tilde{V}_j = \mathcal{S}_\Gamma^{-1} V_j$, $\tilde{\Gamma}_j = \Gamma_j P_{V_j}\mathcal{S}_\Gamma^{-1}$.

Proof. We have for every $f \in \mathcal{H}$

$$\begin{aligned}
||\mathcal{S}_\Lambda - \mathcal{S}_\Gamma|| &= \sup_{||f||=1} ||\langle(\mathcal{S}_\Lambda - \mathcal{S}_\Gamma)f, f\rangle|| \\
&= \sup_{||f||=1} ||\langle\mathcal{S}_\Lambda f, f\rangle - \langle\mathcal{S}_\Gamma f, f\rangle|| \\
&= \sup_{||f||=1} ||\sum_{j\in\mathrm{J}} v_j^2\langle\Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle - \sum_{j\in\mathrm{J}} v_j^2\langle\Gamma_j P_{V_j} f, \Gamma_j P_{V_j} f\rangle|| \leq D.
\end{aligned}$$

Therefore,

$$||\mathcal{S}_\Lambda^{-1} - \mathcal{S}_\Gamma^{-1}|| \leq ||\mathcal{S}_\Lambda^{-1}||||\mathcal{S}_\Lambda - \mathcal{S}_\Gamma||||\mathcal{S}_\Gamma^{-1}|| \leq \frac{D}{AC}.$$

And for all $f \in \mathcal{H}$

$$\begin{aligned}
\sum_{j\in\mathrm{J}} v_j^2\langle\Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1} P_{\mathcal{S}_\Lambda^{-1}W_j} f, \Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1} P_{\mathcal{S}_\Lambda^{-1}W_j} f\rangle &= \sum_{j\in\mathrm{J}} v_j^2\langle\Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1} f, \Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1} f\rangle \\
&= \sum_{j\in\mathrm{J}} v_j^2\langle P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1} f, \mathcal{S}_\Lambda^{-1} f\rangle \\
&= \langle\sum_{j\in\mathrm{J}} v_j^2 P_{W_j}\Lambda_j^*\Lambda_j P_{W_j}(\mathcal{S}_\Lambda^{-1} f), \mathcal{S}_\Lambda^{-1} f\rangle \\
&= \langle\mathcal{S}_\Lambda(\mathcal{S}_\Lambda^{-1}), \mathcal{S}^{-1} f\rangle \\
&= \langle f, \mathcal{S}_\Lambda^{-1} f\rangle.
\end{aligned}$$

Similarly we have for all $f \in \mathcal{H}$

$$\sum_{j\in\mathrm{J}} v_j^2\langle\Gamma_j P_{V_j}\mathcal{S}_\Gamma^{-1} P_{\mathcal{S}_\Gamma^{-1}V_j} f, \Gamma_j P_{V_j}\mathcal{S}_\Gamma^{-1} P_{\mathcal{S}_\Gamma^{-1}V_j} f\rangle = \langle f, \mathcal{S}_\Gamma^{-1} f\rangle.$$

Then

$$||\sum_{j\in\mathrm{J}} v_j^2\langle\Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1} P_{\mathcal{S}_\Lambda^{-1}W_j} f, \Lambda_j P_{W_j}\mathcal{S}_\Lambda^{-1} P_{\mathcal{S}_\Lambda^{-1}W_j} f\rangle - \sum_{j\in\mathrm{J}}\langle\Gamma_j P_{V_j}\mathcal{S}_\Gamma^{-1} P_{\mathcal{S}_\Gamma^{-1}V_j} f, \Gamma_j P_{V_j}\mathcal{S}_\Gamma^{-1} P_{\mathcal{S}_\Gamma^{-1}V_j} f\rangle||$$

$$= ||\langle f, \mathcal{S}_{\Lambda}^{-1} f\rangle - \langle f, \mathcal{S}_{\Gamma}^{-1} f\rangle||$$
$$= ||\langle f, (\mathcal{S}_{\Lambda}^{-1} - \mathcal{S}_{\Gamma}^{-1}) f\rangle||$$
$$\leq ||\mathcal{S}_{\Lambda}^{-1} - \mathcal{S}_{\Gamma}^{-1}|| ||f||^2$$
$$\leq \frac{D}{AC} ||\langle f, f\rangle||.$$

□

Now we give a characterazation of $g-$fusion frames for Hilbert $\mathcal{A}-$modules.

Theorem 3.2. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}-$module over C^*-algebra. Then $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ is a $g-$fusion frame for $\mathcal{H}$ if and only if there exist two constants $0 < A \leq B < \infty$ such that for all $f \in \mathcal{H}$*

$$A||\langle f, f\rangle|| \leq ||\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle|| \leq B||\langle f, f\rangle||.$$

Proof. Suppose Λ is $g-$fusion frame for $\mathcal{H}$, since there is $\langle f, f\rangle \geq 0$ then for all $f \in \mathcal{H}$,

$$A||\langle f, f\rangle|| \leq ||\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle|| \leq B||\langle f, f\rangle||$$

Conversely for each $f \in \mathcal{H}$ we have

$$||\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle|| = ||\sum_{j\in \mathrm{J}} \langle v_j \Lambda_j P_{W_j} f, v_j \Lambda_j P_{W_j} f\rangle||$$
$$= ||\langle (v_j \Lambda_j P_{W_j} f)_{j\in \mathrm{J}}, (v_j \Lambda_j P_{W_j} f)_{j\in \mathrm{J}}\rangle||$$
$$= ||(v_j \Lambda_j P_{W_j} f)_{j\in \mathrm{J}}||^2.$$

We define the operator $L : \mathcal{H} \to l^2((\mathcal{K}_j)_{j\in \mathrm{J}})$ by $L(f) = (v_j \Lambda_j P_{W_j} f)_{j\in \mathrm{J}}$, then

$$||L(f)||^2 = ||(v_j \Lambda_j P_{W_j} f)_{j\in \mathrm{J}}||^2 \leq B||f||^2.$$

L is $\mathcal{A}-$linear bounded operator, then there exist $C > 0$ sutch that

$$\langle L(f), L(f)\rangle \leq C\langle f, f\rangle, \qquad \forall f \in \mathcal{H}.$$

So

$$\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \leq C\langle f, f\rangle, \qquad \forall f \in \mathcal{H}.$$

Therefore $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ is $g-$fusion bessel sequence for $\mathcal{H}$. Now we cant define the $g-$fusion frame operator $\mathcal{S}_\Lambda$ on $\mathcal{H}$. So

$$\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle = \langle \mathcal{S}_\Lambda f, f\rangle, \qquad \forall f \in \mathcal{H}.$$

Since $\mathcal{S}_\Lambda$ is self-adjoint and positive, then

$$\langle \mathcal{S}_\Lambda^{\frac{1}{2}} f, \mathcal{S}_\Lambda^{\frac{1}{2}} f\rangle = \langle \mathcal{S}_\Lambda f, f\rangle, \qquad \forall f \in \mathcal{H}.$$

That implies

$$A||\langle f,f\rangle|| \leq ||\langle \mathcal{S}_{\Lambda}^{\frac{1}{2}}f, \mathcal{S}_{\Lambda}^{\frac{1}{2}}f\rangle|| \leq B||\langle f,f\rangle||, \qquad \forall f\in\mathcal{H}.$$

Frome lemma1.2 there exist two canstants $A^{'}, B^{'} > 0$ such that

$$A^{'}\langle f,f\rangle \leq \langle \mathcal{S}_{\Lambda}^{\frac{1}{2}}f, \mathcal{S}_{\Lambda}^{\frac{1}{2}}f\rangle \leq B^{'}\langle f,f\rangle, \qquad \forall f\in\mathcal{H}.$$

So

$$A^{'}\langle f,f\rangle \leq \sum_{j\in\mathrm{J}} v_j^2\langle \Lambda_j P_{W_j}f, \Lambda_j P_{W_j}f\rangle \leq B^{'}\langle f,f\rangle, \qquad \forall f\in\mathcal{H}.$$

Hence Λ is a $g-$fusion frame for $\mathcal{H}$. □

Theorem 3.3. *Let $\Lambda = (W_j,\Lambda_j,v_j)_{j\in\mathrm{J}}$ be a $g-$fusion frame for $\mathcal{H}$ with frame bounds A and B. If $\Gamma = (W_j,\Gamma_j,v_j)_{j\in\mathrm{J}}$ is $g-$fusion bessel sequence with bound $M < A$, then $(W_j,\Lambda_j+\Gamma_j,v_j)_{j\in\mathrm{J}}$ is $g-$fusion frame for $\mathcal{H}$.*

Proof. Let $f\in\mathcal{H}$, we have

$$||\sum_{j\in\mathrm{J}} v_j^2\langle(\Lambda_j+\Gamma_j)P_{W_j}f,(\Lambda_j+\Gamma_j)P_{W_j}f\rangle||^{\frac{1}{2}} = ||(v_j(\Lambda_j+\Gamma_j)P_{W_j}f)_{j\in\mathrm{J}}||.$$

$$\begin{aligned}
&\leq ||(v_j\Lambda_jP_{W_j}f)_{j\in\mathrm{J}}|| + ||(v_j\Gamma_jP_{W_j}f)_{j\in\mathrm{J}}||\\
&\leq ||\sum_{j\in\mathrm{J}} v_j^2\langle\Lambda_jP_{W_j}f,\Lambda_jP_{W_j}f\rangle||^{\frac{1}{2}} + ||\sum_{j\in\mathrm{J}} v_j^2\langle\Gamma_jP_{W_j}f,\Gamma_jP_{W_j}f\rangle||^{\frac{1}{2}}\\
&\leq \sqrt{B}||f|| + \sqrt{M}||f||\\
&\leq (\sqrt{B}+\sqrt{M})||f||.
\end{aligned}$$

On the other hand, for each $f\in\mathcal{H}$

$$||\sum_{j\in\mathrm{J}} v_j^2\langle(\Lambda_j+\Gamma_j)P_{W_j}f,(\Lambda_j+\Gamma_j)P_{W_j}f\rangle||^{\frac{1}{2}} = ||(v_j(\Lambda_j+\Gamma_j)P_{W_j}f)_{j\in\mathrm{J}}||.$$

$$\begin{aligned}
&\geq ||(v_j\Lambda_jP_{W_j}f)_{j\in\mathrm{J}}|| - ||(v_j\Gamma_jP_{W_j}f)_{j\in\mathrm{J}}||\\
&\geq ||\sum_{j\in\mathrm{J}} v_j^2\langle\Lambda_jP_{W_j}f,\Lambda_jP_{W_j}f\rangle||^{\frac{1}{2}} - ||\sum_{j\in\mathrm{J}} v_j^2\langle\Gamma_jP_{W_j}f,\Gamma_jP_{W_j}f\rangle||^{\frac{1}{2}}\\
&\geq \sqrt{A}||f|| - \sqrt{M}||f||\\
&\geq (\sqrt{A}-\sqrt{M})||f||.
\end{aligned}$$

So,

$$(\sqrt{A}-\sqrt{M})^2||f||^2 \leq ||\sum_{j\in\mathrm{J}} v_j^2\langle(\Lambda_j+\Gamma_j)P_{W_j}f,(\Lambda_j+\Gamma_j)P_{W_j}f\rangle|| \leq (\sqrt{B}+\sqrt{M})^2||f||^2, \qquad \forall f\in\mathcal{H}.$$

Hence $(W_j,\Lambda_j+\Gamma_j,v_j)_{j\in\mathrm{J}}$ is $g-$fusion frame for $\mathcal{H}$. □

Theorem 3.4. *Let $(W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ be a $g-$fusion frame for $\mathcal{H}$ with frame bounds A and B. And $\Gamma_j \in End^{*}_{\mathcal{A}}(\mathcal{H}, \mathcal{K}_j)$, $\forall j \in \mathrm{J}$. Then the following statements are equivalent:*

(1) $(W_j, \Gamma_j, v_j)_{j\in \mathrm{J}}$ *is $g-$fusion frame for $\mathcal{H}$.*

(2) *There exist a canstant M such that $\forall f \in \mathcal{H}$ we have:*

$$||\sum_{j\in \mathrm{J}} v_j^2 \langle (\Lambda_j - \Gamma_j)P_{W_j} f, (\Lambda_j - \Gamma_j)P_{W_j} f\rangle|| \leq M \min(||\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle||, ||\sum_{j\in \mathrm{J}} v_j^2 \langle \Gamma_j P_{W_j} f, \Gamma_j P_{W_j} f\rangle||).$$

Proof. (1) $\Longrightarrow$ (2) Let $(W_j, \Gamma_j, v_j)_{j\in \mathrm{J}}$ be a $g-$fusion frame for $\mathcal{H}$, with frame bounds C and D, then for any $f \in \mathcal{H}$, we have

$$\begin{aligned}
||\sum_{j\in \mathrm{J}} v_j^2 \langle (\Lambda_j - \Gamma_j)P_{W_j} f, (\Lambda_j - \Gamma_j)P_{W_j} f\rangle||^{\frac{1}{2}} &= ||(v_j(\Lambda_j - \Gamma_j)P_{W_j} f)_{j\in \mathrm{J}}|| \\
&\leq ||(v_j \Lambda_j P_{W_j} f)_{j\in \mathrm{J}}|| + ||(v_j \Gamma_j P_{W_j} f)_{j\in \mathrm{J}}|| \\
&\leq ||\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle||^{\frac{1}{2}} + ||\sum_{j\in \mathrm{J}} v_j^2 \langle \Gamma_j P_{W_j} f, \Gamma_j P_{W_j} f\rangle||^{\frac{1}{2}} \\
&\leq ||\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle||^{\frac{1}{2}} + \sqrt{D}||f|| \\
&\leq ||\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle||^{\frac{1}{2}} + \frac{\sqrt{D}}{\sqrt{A}}||\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle||^{\frac{1}{2}} \\
&\leq (1 + \sqrt{\frac{D}{A}})||\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle||^{\frac{1}{2}}.
\end{aligned}$$

Similary, for each $f \in \mathcal{H}$, we can obtain

$$||\sum_{j\in \mathrm{J}} v_j^2 \langle (\Lambda_j - \Gamma_j)P_{W_j} f, (\Lambda_j - \Gamma_j)P_{W_j} f\rangle||^{\frac{1}{2}} \leq (1 + \sqrt{\frac{B}{C}})||\sum_{j\in \mathrm{J}} v_j^2 \langle \Gamma_j P_{W_j} f, \Gamma_j P_{W_j} f\rangle||^{\frac{1}{2}}.$$

We put $M = \min((1 + \sqrt{\frac{B}{C}})^2, (1 + \sqrt{\frac{D}{A}})^2)$.

(2) $\Longrightarrow$ (1) We have for each $f \in \mathcal{H}$

$$\begin{aligned}
\sqrt{A}||f|| &\leq ||\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle||^{\frac{1}{2}} \\
&\leq ||(v_j \Lambda_j P_{W_j} f)_{j\in \mathrm{J}}|| \\
&\leq ||(v_j(\Lambda_j - \Gamma_j)P_{W_j} f)_{j\in \mathrm{J}}|| + ||(v_j \Gamma_j P_{W_j} f)_{j\in \mathrm{J}}|| \\
&\leq \sqrt{M}||\sum_{j\in \mathrm{J}} v_j^2 \langle \Gamma_j P_{W_j} f, \Gamma_j P_{W_j} f\rangle||^{\frac{1}{2}} + ||\sum_{j\in \mathrm{J}} v_j^2 \langle \Gamma_j P_{W_j} f, \Gamma_j P_{W_j} f\rangle||^{\frac{1}{2}} \\
&\leq (\sqrt{M} + 1)||\sum_{j\in \mathrm{J}} v_j^2 \langle \Gamma_j P_{W_j} f, \Gamma_j P_{W_j} f\rangle||^{\frac{1}{2}}.
\end{aligned}$$

And we have $\forall f\in\mathcal{H}$

$$\begin{aligned}||\sum_{j\in \mathrm{J}} v_j^2\langle \Gamma_j P_{W_j}f, \Gamma_j P_{W_j}f\rangle||^{\frac{1}{2}} &= ||(v_j\Gamma_j P_{W_j}f)_{j\in\mathrm{J}}||\\ &\leq ||(v_j\Lambda_j P_{W_j}f)_{j\in\mathrm{J}}|| + ||(v_j(\Lambda_j-\Gamma_j)f)_{j\in\mathrm{J}}||\\ &\leq ||\sum_{j\in\mathrm{J}} v_j^2\langle\Lambda_j P_{W_j}f, \Lambda_j P_{W_j}f\rangle||^{\frac{1}{2}} + \sqrt{M}||\sum_{j\in\mathrm{J}} v_j^2\langle\Lambda_j P_{W_j}f, \Lambda_j P_{W_j}f\rangle||^{\frac{1}{2}}\\ &\leq (\sqrt{M}+1)||\sum_{j\in\mathrm{J}} v_j^2\langle\Lambda_j P_{W_j}f, \Lambda_j P_{W_j}f\rangle||^{\frac{1}{2}}\\ &\leq (\sqrt{M}+1)\sqrt{B}||f||.\end{aligned}$$

So

$$\frac{A}{(1+\sqrt{M})^2}||f||^2 \leq ||\sum_{j\in\mathrm{J}} v_j^2\langle \Gamma_j P_{W_j}f, \Gamma_j P_{W_j}f\rangle|| \leq (\sqrt{M}+1)^2B||f||^2, \qquad \forall f\in\mathcal{H}.$$

Hence $(W_j, \Gamma_j, v_j)_{j\in\mathrm{J}}$ is $g-$fusion frame for $\mathcal{H}$. □

4. Perturbation of K-g-fusion frames

Perturbation of frames has been discussed by Casazza and Christensen. In this section, we present a perturbation of $K-g-$fusion frames. first we give a characterazation of $K-g-$fusion frame for Hilbert $\mathcal{A}-$modules.

Theorem 4.1. *Let $K\in End_{\mathcal{A}}^*(\mathcal{H})$. Suppose that the operator $T:\mathcal{H}\to l^2((\mathcal{K}_j)_{j\in\mathrm{J}})$ is given by $T(f) = (v_j\Lambda_j P_{W_j}f)_{j\in\mathrm{J}}$, $\forall f\in\mathcal{H}$, and $\overline{\mathcal{R}(T)}$ is orthogonally complemented. Then $\Lambda = (W_j, \Lambda_j, v_j)_{j\in\mathrm{J}}$ is $K-g-$fusion frame for $\mathcal{H}$ if and only if there exist constants $0 < A \leq B < \infty$ such that*

$$A||K^*f||^2 \leq ||\sum_{j\in\mathrm{J}} v_j^2\langle\Lambda_j P_{W_j}f, \Lambda_j P_{W_j}f\rangle|| \leq B||f||^2, \qquad \forall f\in\mathcal{H}. \tag{4.1}$$

Proof. If Λ is $K-g-$fusion frame for $\mathcal{H}$, then the equation (4.1) is satisfies. Conversly, we have for each $(f_j)_{j\in\mathrm{J}}\in l^2((\mathcal{K}_j)_{j\in\mathrm{J}})$ and any finite $\mathrm{I}\subset\mathrm{J}$

$$\begin{aligned}||\sum_{j\in\mathrm{I}} v_j P_{W_j}\Lambda_j^*(f_j)|| &= \sup_{||g||=1} ||\langle\sum_{j\in\mathrm{I}} v_j P_{W_j}\Lambda_j^* f_j, g\rangle||\\ &= \sup_{||g||=1} ||\sum_{j\in\mathrm{I}}\langle f_j, v_j\Lambda_j P_{W_j}g\rangle||\\ &\leq \sup_{||g||=1} ||\sum_{j\in\mathrm{I}}\langle f_j, f_j\rangle||^{\frac{1}{2}}||\sum_{j\in\mathrm{I}} v_j^2\langle\Lambda_j P_{W_j}g\rangle||^{\frac{1}{2}}\\ &\leq \sqrt{B}||\sum_{j\in\mathrm{I}}\langle f_j, f_j\rangle||^{\frac{1}{2}}.\end{aligned}$$

Then $\sum_{j\in\mathrm{J}} v_j P_{W_j}\Lambda_j^* f_j$ converge unconditionally in $\mathcal{H}$, and we have $\forall f\in\mathcal{H}$, $\forall (f_j)_{j\in\mathrm{J}}\in l^2((\mathcal{K}_j)_{j\in\mathrm{J}})$

$$\langle Tf, (f_j)_{j\in\mathrm{J}}\rangle = \langle (v_j\Lambda_j P_{W_j}f)_{j\in\mathrm{J}}, (f_j)_{j\in\mathrm{J}}\rangle = \sum_{j\in\mathrm{J}}\langle v_j\Lambda_j P_{W_j}f, f_j\rangle = \langle f, \sum_{j\in\mathrm{J}} v_j P_{W_j}\Lambda_j^* f_j\rangle$$

So T is adjointable and $T^*((f_j)_{j\in \mathrm{J}}) = \sum_{j\in \mathrm{J}} v_j P_{W_j} \Lambda_j^* f_j$ also frome (4.1) we have

$$||K^*f||^2 \leq \frac{1}{A}||Tf||^2, \qquad \forall f \in \mathcal{H}.$$

By lemma1.5, there exist $\nu > 0$ such that $KK^* \leq \nu T^*T$, then

$$\langle KK^*f, f\rangle \leq \nu\langle Tf, Tf\rangle, \qquad \forall f \in \mathcal{H}.$$

Therefore

$$\frac{1}{\nu}\langle K^*f, K^*f\rangle \leq \sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle, \qquad \forall f \in \mathcal{H}.$$

And we have for each $f \in \mathcal{H}$, $\langle Tf, Tf\rangle \leq ||T||^2\langle f, f\rangle$, then

$$\sum_{j\in \mathrm{J}} v_j^2 \langle \Lambda_j P_{W_j} f, \Lambda_j P_{W_j} f\rangle \leq ||T||^2\langle f, f\rangle, \qquad \forall f \in \mathcal{H}.$$

And the proof is completed. □

Theorem 4.2. *Let $\Lambda = (W_j, \Lambda_j, v_j)_{j\in \mathrm{J}}$ be a $K - g-$fusion frame for $\mathcal{H}$ with frame bounds A, B and let $\Gamma_j \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K}_j)$, for all $j \in \mathrm{J}$. Suppose that $T : \mathcal{H} \to l^2((\mathcal{K}_j)_{j\in \mathrm{J}})$ define by $T(f) = (u_j\Gamma_j P_{V_j} f)_{j\in \mathrm{J}}$, $\forall f \in \mathcal{H}$. And $\overline{\mathcal{R}(T)}$ is orthogonally complemented, such that for each $f \in \mathcal{H}$*

$$||((v_j\Lambda_j P_{W_j} - u_j\Gamma_j P_{V_j})f)_{j\in \mathrm{J}}|| \leq \lambda_1||(v_j\Lambda_j P_{W_j} f)_{j\in \mathrm{J}}|| + \lambda_2||(u_j\Gamma_j P_{V_j} f)_{j\in \mathrm{J}}|| + \epsilon||K^*f||.$$

where $0 < \lambda_1, \lambda_2 < 1$ and $\epsilon > 0$ such that $\epsilon < (1 - \lambda_1)\sqrt{A}$.
Then $\Gamma = (V_j, \Gamma_j, u_j)_{j\in \mathrm{J}}$ is a $K - g-$fusion frame for $\mathcal{H}$.

Proof. We have for each $f \in \mathcal{H}$

$$\begin{aligned}
||\sum_{j\in \mathrm{J}} u_j^2\langle \Gamma_j P_{V_j} f, \Gamma_j P_{V_j} f\rangle||^{\frac{1}{2}} &= ||(u_j\Gamma_j P_{V_j} f)_j|| \\
&= ||(u_j\Gamma_j P_{V_j} f)_j + (v_j\Lambda_j P_{W_j} f)_j - (v_j\Lambda_j P_{W_j} f)_j|| \\
&\leq ||((u_j\Gamma_j P_{V_j} - v_j\Lambda_j P_{W_j})f)_j|| + ||(v_j\Lambda_j P_{W_j} f)_j|| \\
&\leq (\lambda_1 + 1)||(v_j\Lambda_j P_{W_j} f)_j|| + \lambda_2||(u_j\Gamma_j P_{V_j} f)_j|| + \epsilon||K^*f||.
\end{aligned}$$

So

$$(1 - \lambda_2)||(u_j\Gamma_j P_{V_j} f)_j|| \leq (\lambda_1 + 1)\sqrt{B}||f|| + \epsilon||K^*f||.$$

Then

$$\begin{aligned}
||(u_j\Gamma_j P_{V_j} f)_j|| &\leq \frac{(\lambda_1 + 1)\sqrt{B}||f|| + \epsilon||K^*f||}{1 - \lambda_2} \\
&\leq (\frac{(\lambda_1 + 1)\sqrt{B} + \epsilon||K||}{1 - \lambda_2})||f||.
\end{aligned}$$

Hence

$$||\sum_{j\in J}u_j^2\langle\Gamma_jP_{V_j}f,\Gamma_jP_{V_j}f\rangle||\leq(\frac{(\lambda_1+1)\sqrt{B}+\epsilon||K^*||}{1-\lambda_2})^2||f||^2.$$

On the other hand for each $f\in\mathcal{H}$

$$\begin{aligned}||\sum_{j\in J}u_j^2\langle\Gamma_jP_{V_j}f,\Gamma_jP_{V_j}f\rangle||^{\frac{1}{2}}&=||(u_j\Gamma_jP_{V_j}f)_j||\\&=||((u_j\Gamma_jP_{V_j}-v_j\Lambda_jP_{W_j})f)_j+(v_j\Lambda_jP_{W_j}f)_j||\\&\geq||(v_j\Lambda_jP_{W_j}f)_j||-||((u_j\Gamma_jP_{V_j}-v_j\Lambda_jP_{W_j})f)_j||\\&\geq(1-\lambda_1)||(v_j\Lambda_jP_{W_j}f)_j||-\lambda_2||(u_j\Gamma_jP_{V_j}f)_j||-\epsilon||K^*f||.\end{aligned}$$

Hence

$$||\sum_{j\in J}u_j^2\langle\Gamma_jP_{V_j}f,\Gamma_jP_{V_j}f\rangle||\geq(\frac{(1-\lambda_1)\sqrt{A}-\epsilon}{1+\lambda_2})^2||K^*f||^2.$$

□

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