International Journal of Analysis and Applications
Volume 19, Number 3 (2021), 477-493
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-19-2021-477

BOUNDS ON TOEPLITZ DETERMINANT FOR STARLIKE FUNCTIONS WITH RESPECT TO CONJUGATE POINTS

DAUD MOHAMAD, NUR HAZWANI AQILAH ABDUL WAHID*

Faculty of Computer and Mathematical Sciences,
Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia

**Corresponding author:* hazwaniaqilah@tmsk.uitm.edu.my

ABSTRACT. This paper is concerned with the estimate of the upper bounds of the Toeplitz determinants $|T_2(3)|$ and $|T_3(3)|$ for functions belonging to the subclass of starlike functions with respect to conjugate points. The results presented would extend the results for some existing subclasses in the literature.

1. INTRODUCTION

Let A be the class of functions $f(z)$ which are analytic in an open unit disk $E=\{z:|z|<1\}$ and having the power series expansion

(1.1)
$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

in E. Let S be the class of functions $f(z)\in A$ and univalent in E.

Let P be the class of functions $p(z)$ of the form

Received February 28th, 2021; accepted April 7th, 2021; published May 4th, 2021.
2010 Mathematics Subject Classification. 30C45, 30C50.
Key words and phrases. coefficient bounds; univalent functions; starlike functions; Toeplitz determinants

(1.2) $$p(z)=1+\sum_{n=1}^{\infty}p_n z^n$$

that is analytic in E and satisfying the condition $\operatorname{Re} p(z)>0,\ z\in E.$ Functions in P are called Carathéodory functions. It is well known that if $p(z)\in P,$ then a Schwarz function $\omega(z)$ exists with $\omega(0)=0,\ |\omega(z)|<1,\ z\in E$ such that [1]

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}.$$

For two functions $F(z)$ and $G(z)$ analytic in $E,$ we say that the function $F(z)$ is subordinate to $G(z)$ and we write it as $F(z)\prec G(z)$ if there exists a Schwarz function $\omega(z)$ which is analytic in E with $\omega(0)=0,\ |\omega(z)|<1,$ such that $F(z)=G(\omega(z)).$ Further, if $G(z)$ is univalent in $E,$ then $F(z)\prec G(z)\Leftrightarrow F(0)=G(0)$ and $F(E)=G(E)$ (see Miller and Mocanu [2, 3] for details).

Let S^* denote the class of starlike functions in S. It is known that $f(z)\in S^*$ if and only if

(1.3) $$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0,\ z\in E.$$

El-Ashwah and Thomas [4] defined the following class:

(1.4) $$S_C^*=\left\{f(z)\in \mathrm{A}:\ \operatorname{Re}\left\{\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}\right\}>0,\ z\in E\right\}.$$

Functions in the class S_C^* are called starlike functions with respect to conjugate points.
Halim [5] defined the following class:

(1.5) $$S_C^*(\delta)=\left\{f(z)\in \mathrm{A}:\ \operatorname{Re}\left\{\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}\right\}>\delta,\ 0\le\delta<1,\ z\in E\right\}.$$

In terms of subordination, Dahhar and Janteng [6] generalized the class S_C^* and it is denoted by $S_C^*(A,B).$ This class is defined as follows:

(1.6) $$S_C^*(A,B)=\left\{f(z)\in A:\ \frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}\prec\frac{1+Az}{1+Bz},\ -1\le B<A\le 1,\ z\in E\right\}.$$

Wahid et al. [7] introduced the subclass of tilted starlike functions with respect to conjugate points of order δ, $S_C^*(\alpha,\delta,A,B)$ and it is given by

(1.7) $$S_C^*(\alpha,\delta,A,B)=\left\{f(z)\in A:\ \left[e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right]\frac{1}{t_{\alpha\delta}}\prec\frac{1+Az}{1+Bz},\ z\in E\right\},$$

where $g(z)=\dfrac{f(z)+\overline{f(\bar{z})}}{2}$, $t_{\alpha\delta}=\cos\alpha-\delta>0,\ 0\le\delta<1,\ |\alpha|<\dfrac{\pi}{2}$ and $-1\le B<A\le 1$.

In particular, $S_C^*(0)\equiv S_C^*$, $S_C^*(1,-1)\equiv S_C^*$ and $S_C^*(0,0,1,-1)\equiv S_C^*$.

Toeplitz matrices are one of the well-studied classes of structured matrices. The concept of Toeplitz matrices led to the development of the studies related to Toeplitz determinants, Toeplitz kernel, Toeplitz operators, and q-deformed Toeplitz matrices [8]. In a recent investigation, the Toeplitz determinant has been studied by [9-18], and they succeeded in estimating the coefficient bounds for Toeplitz determinant $|T_q(n)|$, $n,q\ge 1$ for the first few values of n and q over some subclasses of A. The Toeplitz determinant $T_q(n)$, $n,q\ge 1$ of functions $f(z)$ of the form (1.1), is defined by Thomas and Halim [9]

$$T_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_n & \cdots & a_{n+q-2}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n\end{vmatrix},\ a_1=1.$$

However, apart from these works, there was no study of finding estimates for $|T_2(3)|$ and $|T_3(3)|$ for the subclasses introduced by El-Ashwah and Thomas [4], Halim [5], Dahhar and Janteng [6], and Wahid et al. [7]. In fact, as far as we are concerned, no bound for $|T_3(3)|$ was obtained for the class of univalent functions and its subclasses in the existing literature. Therefore, in this paper, we obtain the upper bounds for the Toeplitz determinant for $S_C^*(\alpha,\beta,A,B)$ as defined in (1.7) for the case of $n=3$, $q=2$ and $n=3$, $q=3$ namely

(1.8) $$T_2(3)=\begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}$$

and

(1.9) $$T_3(3)=\begin{vmatrix} a_3 & a_4 & a_5 \\ a_4 & a_3 & a_4 \\ a_5 & a_4 & a_3 \end{vmatrix}.$$

We also give some results for the subclasses introduced by El-Ashwah and Thomas [4], Halim [5], and Dahhar and Janteng [6].

We shall state the following lemmas to prove our main results.

2. PRELIMINARY RESULTS

Lemma 2.1. *[19] For a function $p(z)\in P$ of the form (1.2), the sharp inequality $|p_n|\le 2$ holds for each $n\ge 1$. Equality holds for the function $p(z)=\frac{1+z}{1-z}$.*

Lemma 2.2. *[20] Let $p(z)\in P$ of the form (1.2) and $\mu\in\mathbb{C}$. Then*

$$|p_n-\mu p_k p_{n-k}|\le 2\max\{1,|2\mu-1|\},\ 1\le k\le n-1.$$

If $|2\mu-1|\ge 1$, then the inequality is sharp for the function $p(z)=\frac{1+z}{1-z}$ or its rotations. If $|2\mu-1|<1$, then the inequality is sharp for the function $p(z)=\frac{1+z^n}{1-z^n}$ or its rotations.

3. MAIN RESULTS

Theorem 3.1. *If the function $f(z)$ given by (1.1) belongs to the class $S_C^*(\alpha,\delta,A,B)$, then*

$$\begin{aligned}|T_2(3)|\le \frac{T^2}{2304}\Big\{&832+64\left|(1-3\Upsilon)\xi^4+\left(-3\Upsilon+11\Upsilon^2\right)\xi^3+\left(-12\Upsilon^3+2\Upsilon^2\right)\xi^2+4\Upsilon^4\xi\right|\\&+8\left|72\Upsilon^2-144\Upsilon+72\xi^2+(-144\Upsilon+144)\xi\right|+8\left|-128\Upsilon^2+128\Upsilon-72\xi^2+(192\Upsilon-96)\xi\right|\\&+16\left|-16\Upsilon^2+32\Upsilon^3+\left(24\Upsilon-88\Upsilon^2+32\Upsilon^3\right)\xi+(-24-12\Upsilon)\xi^3+\left(-16+92\Upsilon-72\Upsilon^2\right)\xi^2\right|\Big\}\end{aligned}$$

where $\xi=Te^{-i\alpha}$, $T=(A-B)t_{\alpha\delta}$, $t_{\alpha\delta}=\cos\alpha-\delta$ and $\Upsilon=1+B$.

Proof. From (1.7), since $f(z)\in S_C^*(\alpha,\delta,A,B)$, according to subordination relationship, so there exists a Schwarz function $\omega(z)$ such that

(3.1) $$\left\{e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right\}\frac{1}{t_{\alpha\delta}}=\frac{1+A\omega(z)}{1+B\omega(z)},$$

where $g(z)=\dfrac{f(z)+\overline{f(\bar{z})}}{2}$, $t_{\alpha\delta}=\cos\alpha-\delta.$

Define a function

$$h(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+\sum_{n=1}^{\infty}k_nz^n.$$

We have $h(z)\in P$ and

(3.2) $$\omega(z)=\frac{h(z)-1}{h(z)+1}.$$

Using (3.2), from (3.1), we have

(3.3) $$e^{i\alpha}\frac{zf'(z)}{g(z)}=\frac{\left[e^{i\alpha}(1-B)-T\right]+h(z)\left[e^{i\alpha}(1+B)+T\right]}{1-B+h(z)(1+B)}$$

where $T=(A-B)t_{\alpha\delta}$.

Using the series expansion in (3.3), we get

(3.4) $$\begin{aligned}&e^{i\alpha}(1-B)\left(z+2a_2z^2+3a_3z^3+4a_4z^4+\cdots\right)\\&\quad+e^{i\alpha}(1+B)\left(z+2a_2z^2+3a_3z^3+4a_4z^4+\cdots\right)\left(1+k_1z+k_2z^2+k_3z^3+\cdots\right)\\&=\left[e^{i\alpha}(1-B)-T\right]\left(z+a_2z^2+a_3z^3+a_4z^4+\cdots\right)\\&\quad+\left[e^{i\alpha}(1+B)+T\right]\left(z+a_2z^2+a_3z^3+a_4z^4+\cdots\right)\left(1+k_1z+k_2z^2+k_3z^3+\cdots\right).\end{aligned}$$

Equating the coefficients of z^3 and z^4 respectively in the expansion of (3.4) and for simplicity, we take $\xi=Te^{-i\alpha}$ and $\Upsilon=1+B$, give us

(3.5) $$a_3=\frac{2k_2\xi+k_1^2\xi^2-\Upsilon k_1^2\xi}{8}$$

and

(3.6) $$a_4=\frac{8k_3\xi+6k_1k_2\xi^2-8k_1k_2\Upsilon\xi+k_1^3\xi^3-3k_1^3\Upsilon\xi^2+2k_1^3\Upsilon^2\xi}{48}.$$

Squaring (3.5) and (3.6), respectively, we get

(3.7) $$a_3^2=\frac{4k_2^2\xi^2+k_1^4\Upsilon^2\xi^2-4k_1^2k_2\Upsilon\xi^2-2k_1^4\Upsilon\xi^3+4k_1^2k_2\xi^3+k_1^4\xi^4}{64}$$

and

$$\begin{aligned}a_4^2=\frac{\xi^2}{2304}\Big[&64k_3^2+k_1^6\xi^4-3k_1^6\Upsilon\xi^3+2k_1^6\Upsilon^2\xi^2-3k_1^6\Upsilon\xi^4+11k_1^6\Upsilon^2\xi^3-12k_1^6\Upsilon^3\xi^2+4k_1^6\Upsilon^4\xi\\&+96k_1k_2k_3\xi-128\Upsilon k_1k_2k_3+16k_1^3k_3\xi^2-24k_1^3k_3\Upsilon\xi+16\Upsilon^2k_1^3k_3-24k_1^3k_3\Upsilon\xi^2+16k_1^3k_3\Upsilon^2\xi\\&+36k_1^2k_2^2\xi^2-96k_1^2k_2^2\Upsilon\xi+64\Upsilon^2k_1^2k_2^2+12k_1^4k_2\xi^3-34k_1^4k_2\Upsilon\xi^2+36k_1^4k_2\Upsilon^2\xi-16\Upsilon^3k_1^4k_2\\&+6k_1^4k_2\Upsilon\xi^3+36k_1^4k_2\Upsilon^2\xi^2-16k_1^4k_2\Upsilon^3\xi\Big].\end{aligned}$$

(3.8)

From the equations (1.8), (3.7), and (3.8), yield

$$\begin{aligned}|T_2(3)|&=\left|a_4^2-a_3^2\right|\\&=\Big|\frac{\xi^2}{2304}\Big\{\Big[64k_3^2+k_1^6\xi^4-3k_1^6\Upsilon\xi^3+2k_1^6\Upsilon^2\xi^2-3k_1^6\Upsilon\xi^4+11k_1^6\Upsilon^2\xi^3-12k_1^6\Upsilon^3\xi^2+4k_1^6\Upsilon^4\xi\\&+96k_1k_2k_3\xi-128\Upsilon k_1k_2k_3+16k_1^3k_3\xi^2-24k_1^3k_3\Upsilon\xi+16\Upsilon^2k_1^3k_3-24k_1^3k_3\Upsilon\xi^2+16k_1^3k_3\Upsilon^2\xi\\&+36k_1^2k_2^2\xi^2-96k_1^2k_2^2\Upsilon\xi+64\Upsilon^2k_1^2k_2^2+12k_1^4k_2\xi^3-34k_1^4k_2\Upsilon\xi^2+36k_1^4k_2\Upsilon^2\xi-16\Upsilon^3k_1^4k_2\\&+6k_1^4k_2\Upsilon\xi^3+36k_1^4k_2\Upsilon^2\xi^2-16k_1^4k_2\Upsilon^3\xi\Big]\\&-\Big[144k_2^2+36\Upsilon^2k_1^4-144\Upsilon k_1^2k_2-72k_1^4\Upsilon\xi+144k_1^2k_2\xi+36k_1^4\xi^2\Big]\Big\}\Big|\\&=\Big|\frac{\xi^2}{2304}\Big\{64k_3^2-144k_2^2+k_1^6\Big[(1-3\Upsilon)\xi^4+\left(-3\Upsilon+11\Upsilon^2\right)\xi^3+\left(2\Upsilon^2-12\Upsilon^3\right)\xi^2+4\Upsilon^4\xi\Big]\\&+k_1^4\Big[-36\xi^2+72\Upsilon\xi-36\Upsilon^2\Big]+k_1^2k_2\left[-144\xi+144\Upsilon\right]+k_1k_2k_3\left[96\xi-128\Upsilon\right]\\&+k_1^2k_2^2\Big[36\xi^2-96\Upsilon\xi+64\Upsilon^2\Big]+k_1^3k_3\Big[16\Upsilon^2+(16-24\Upsilon)\xi^2+\left(-24\Upsilon+16\Upsilon^2\right)\xi\Big]\\&+k_1^4k_2\Big[(12+6\Upsilon)\xi^3+\left(-34\Upsilon+36\Upsilon^2\right)\xi^2+\left(36\Upsilon^2-16\Upsilon^3\right)\xi-16\Upsilon^3\Big]\Big\}\Big|.\end{aligned}$$

Further, by suitably arranging the terms yield

$$\begin{aligned}|T_2(3)| = \Bigg| \frac{\xi^2}{2304} \Big\{ & 64k_3^2 - 144k_2^2 + k_1^6\left[(1-3\Upsilon)\xi^4 + \left(-3\Upsilon + 11\Upsilon^2\right)\xi^3 + \left(2\Upsilon^2 - 12\Upsilon^3\right)\xi^2 + 4\Upsilon^4\xi\right] \\ & + k_1^2(-144\xi + 144\Upsilon)\left[k_2 - k_1^2\left(\frac{36\Upsilon^2 + 36\xi^2 - 72\Upsilon\xi}{-144\xi + 144\Upsilon}\right)\right] \\ & + k_1k_2(96\xi - 128\Upsilon)\left[k_3 - k_1k_2\left(\frac{-36\xi^2 + 96\Upsilon\xi - 64\Upsilon^2}{96\xi - 128\Upsilon}\right)\right] \\ & + k_1^3\left(16\Upsilon^2 + 16\xi^2 - 24\Upsilon\xi^2 + 16\Upsilon^2\xi - 24\Upsilon\xi\right) \bullet \\ & \left[k_3 - k_1k_2\left(\frac{(-12-6\Upsilon)\xi^3 + (-36\Upsilon + 34)\Upsilon\xi^2}{16\Upsilon^2 + (16-24\Upsilon)\xi^2 + \left(16\Upsilon^2 - 24\Upsilon\right)\xi} + \frac{(-36+16\Upsilon)\Upsilon^2\xi + 16\Upsilon^3}{16\Upsilon^2 + (16-24\Upsilon)\xi^2 + \left(16\Upsilon^2 - 24\Upsilon\right)\xi}\right)\right]\Big\}\Bigg| \\ = \Bigg| \frac{\xi^2}{2304} \Big\{ & 64k_3^2 - 144k_2^2 + k_1^6\left[(1-3\Upsilon)\xi^4 + \left(-3\Upsilon + 11\Upsilon^2\right)\xi^3 + \left(2\Upsilon^2 - 12\Upsilon^3\right)\xi^2 + 4\Upsilon^4\xi\right] \\ & + k_1^2(-144\xi + 144\Upsilon)\left[k_2 - \mu k_1^2\right] + k_1k_2(96\xi - 128\Upsilon)\left[k_3 - \chi k_1k_2\right] \\ & + k_1^3\left(16\Upsilon^2 + (16-24\Upsilon)\xi^2 + \left(16\Upsilon^2 - 24\Upsilon\right)\xi\right)\left[k_3 - \lambda k_1k_2\right]\Big\}\Bigg|\end{aligned}$$

(3.9)

where

$$\mu = \frac{36\Upsilon^2 + 36\xi^2 - 72\Upsilon\xi}{-144\xi + 144\Upsilon},$$

$$\chi = \frac{-36\xi^2 + 96\Upsilon\xi - 64\Upsilon^2}{96\xi - 128\Upsilon}$$

and

$$\lambda = \frac{-12\xi^3 - 6\Upsilon\xi^3 - 36\Upsilon^2\xi^2 + 34\Upsilon\xi^2 - 36\Upsilon^2\xi + 16\Upsilon^3\xi + 16\Upsilon^3}{16\Upsilon^2 + 16\xi^2 - 24\Upsilon\xi^2 + 16\Upsilon^2\xi - 24\Upsilon\xi}.$$

Consequently, by the triangle inequality, from (3.9), we get

$$\begin{aligned}|T_2(3)| \le \frac{T^2}{2304}\Big\{ & 64|k_3|^2 + 144|k_2|^2 + |k_1|^6\left|(1-3\Upsilon)\xi^4 + \left(-3\Upsilon + 11\Upsilon^2\right)\xi^3 + \left(-12\Upsilon^3 + 2\Upsilon^2\right)\xi^2 + 4\Upsilon^4\xi\right| \\ & + |k_1|^2\left|-144\xi + 144\Upsilon\right|\left|k_2 - \mu k_1^2\right| + |k_1||k_2|\left|96\xi - 128\Upsilon\right|\left|k_3 - \chi k_1k_2\right| \\ & + |k_1|^3\left|16\Upsilon^2 + (16-24\Upsilon)\xi^2 + \left(16\Upsilon^2 - 24\Upsilon\right)\xi\right|\left|k_3 - \lambda k_1k_2\right|\Big\}.\end{aligned}$$

(3.10)

By Lemma 2.2,

(3.11)
$$\left|k_2-\mu k_1^2\right|\le 2\max\left\{1,\left|2\mu-1\right|\right\}$$
$$=2\max\left\{1,\left|\frac{72\Upsilon^2-144\Upsilon+72\xi^2-144\Upsilon\xi+144\xi}{-144\xi+144\Upsilon}\right|\right\},$$

(3.12)
$$\left|k_3-\chi k_1k_2\right|\le 2\max\left\{1,\left|2\chi-1\right|\right\}$$
$$=2\max\left\{1,\left|\frac{-128\Upsilon^2+128\Upsilon-72\xi^2+192\Upsilon\xi-96\xi}{96\xi-128\Upsilon}\right|\right\}$$

and

(3.13)
$$\left|k_3-\lambda k_1k_2\right|\le 2\max\left\{1,\left|2\lambda-1\right|\right\}$$
$$=2\max\left\{1,\left|\frac{-16\Upsilon^2+32\Upsilon^3+\left(24\Upsilon-88\Upsilon^2+32\Upsilon^3\right)\xi+\left(-24-12\Upsilon\right)\xi^3}{16\Upsilon^2+16\xi^2-24\Upsilon\xi^2+16\Upsilon^2\xi-24\Upsilon\xi}\right.\right.$$
$$\left.\left.+\frac{\left(-16+92\Upsilon-72\Upsilon^2\right)\xi^2}{16\Upsilon^2+16\xi^2-24\Upsilon\xi^2+16\Upsilon^2\xi-24\Upsilon\xi}\right|\right\}.$$

By making use of Lemma 2.1 together with (3.11)-(3.13), we find that

(3.14)
$$\left|k_1\right|^2\left|-144\xi+144\Upsilon\right|\left|k_2-\mu k_1^2\right|\le 8\left|-144\xi+144\Upsilon\right|\left|\frac{72\Upsilon^2-144\Upsilon+72\xi^2+\left(-144\Upsilon+144\right)\xi}{-144\xi+144\Upsilon}\right|,$$

(3.15)
$$\left|k_1\right|\left|k_2\right|\left|96\xi-128\Upsilon\right|\left|k_3-\chi k_1k_2\right|\le 8\left|96\xi-128\Upsilon\right|\left|\frac{-128\Upsilon^2+128\Upsilon-72\xi^2+\left(192\Upsilon-96\right)\xi}{96\xi-128\Upsilon}\right|$$

and

$$\left|k_1\right|^3\left|16\Upsilon^2+\left(16-24\Upsilon\right)\xi^2+\left(16\Upsilon^2-24\Upsilon\right)\xi\right|\left|k_3-\lambda k_1k_2\right|$$
$$\le 16\left|16\Upsilon^2+\left(16-24\Upsilon\right)\xi^2+\left(16\Upsilon^2-24\Upsilon\right)\xi\right|\left|\frac{-16\Upsilon^2+32\Upsilon^3+\left(24\Upsilon-88\Upsilon^2+32\Upsilon^3\right)\xi}{16\Upsilon^2+\left(16-24\Upsilon\right)\xi^2+\left(16\Upsilon^2-24\Upsilon\right)\xi}\right.$$
$$\left.+\frac{\left(-24-12\Upsilon\right)\xi^3+\left(-16+92\Upsilon-72\Upsilon^2\right)\xi^2}{16\Upsilon^2+\left(16-24\Upsilon\right)\xi^2+\left(16\Upsilon^2-24\Upsilon\right)\xi}\right|.$$

(3.16)

Again by applying Lemma 2.1 along with (3.14)-(3.16), from (3.10), we obtain

$$\left|T_2(3)\right| \le \frac{T^2}{2304}\Big\{832+64\left|(1-3\Upsilon)\xi^4+\left(-3\Upsilon+11\Upsilon^2\right)\xi^3+\left(-12\Upsilon^3+2\Upsilon^2\right)\xi^2+4\Upsilon^4\xi\right|$$
$$+8\left|72\Upsilon^2-144\Upsilon+72\xi^2+(-144\Upsilon+144)\xi\right|+8\left|-128\Upsilon^2+128\Upsilon-72\xi^2+(192\Upsilon-96)\xi\right|$$
$$+16\left|-16\Upsilon^2+32\Upsilon^3+\left(24\Upsilon-88\Upsilon^2+32\Upsilon^3\right)\xi+(-24-12\Upsilon)\xi^3+\left(-16+92\Upsilon-72\Upsilon^2\right)\xi^2\right|\Big\}.$$

The result is sharp for the function given by $\left\{e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right\}\frac{1}{t_{\alpha\delta}}=\frac{1+z}{1-z}$. This completes the proof of Theorem 3.1.

Remark 3.1. *For $\alpha=0$, $\delta=0$, $A=1$ and $B=-1$, Theorem 3.1 yields $\left|T_2(3)\right|\le 25$. This inequality coincides with the result obtained by Ali et al. [14] for S^*.*

Theorem 3.2. *If the function $f(z)$ given by (1.1) belongs to the class $S_C^*(\alpha,\delta,A,B)$, then*

$$\left|T_3(3)\right| \le \frac{T}{384}\Big\{8\left|-12\Upsilon^3+36\Upsilon^2+\left(22\Upsilon^2-44\Upsilon\right)\xi+(-12\Upsilon+12)\xi^2+2\xi^3\right|$$
$$+16\left|12\Upsilon-6-8\xi\right|+4\left|-48\Upsilon+48\xi\right|+4\left|24\Upsilon-12\xi\right|+192\Big\}\bullet$$
$$\frac{T^2}{9216}\Big\{2304+2048+8\left|-144\Upsilon+144\xi\right|+8\left|-288\Upsilon^2+576\Upsilon-288\xi^2+(576\Upsilon-576)\xi\right|$$
$$+64\left|(24\Upsilon-5)\xi^4+\left(3\Upsilon-88\Upsilon^2\right)\xi^3+\left(35\Upsilon^2+96\Upsilon^3\right)\xi^2+\left(-51\Upsilon^3-32\Upsilon^4\right)\xi+18\Upsilon^4\right|$$
$$+4\left|-144\xi+288\Upsilon-288\right|+8\left|360\xi^2+(-792\Upsilon+576)\xi+448\Upsilon^2-736\Upsilon\right|$$
$$+16\left|(108+96\Upsilon)\xi^3+\left(32-328\Upsilon+576\Upsilon^2\right)\xi^2+\left(48\Upsilon+92\Upsilon^2+32\Upsilon^3\right)\xi-256\Upsilon^3-16\Upsilon^2\right|\Big\}$$

where $\xi=Te^{-i\alpha}$, $T=(A-B)t_{\alpha\delta}$, $t_{\alpha\delta}=\cos\alpha-\delta$ and $\Upsilon=1+B$.

Proof. Upon simplification of (1.9), the determinant $T_3(3)$ can be written as

$$\left|T_3(3)\right|=\left|(a_3-a_5)\left(a_3^2-2a_4^2+a_3a_5\right)\right|$$

and by using the triangle inequality, we get

$$\left|T_3(3)\right|\le\left|a_3-a_5\right|\left|a_3^2-2a_4^2+a_3a_5\right|.$$

Now, equating the coefficient of z^5 in the expansion of (3.4) and for simplicity, we take $\xi=Te^{-i\alpha}$ and $\Upsilon=1+B$, give us

$$a_5=\frac{48k_4\xi+32k_1k_3\xi^2-48k_1k_3\Upsilon\xi+12k_2^2\xi^2-24k_2^2\Upsilon\xi+k_1^4\xi^4-6k_1^4\Upsilon\xi^3}{384}$$
$$+\frac{11k_1^4\Upsilon^2\xi^2-6k_1^4\Upsilon^3\xi+12k_1^2k_2\xi^3-44k_1^2k_2\Upsilon\xi^2+36k_1^2k_2\Upsilon^2\xi}{384}. \tag{3.17}$$

From the equations (3.5) and (3.17), we obtain

$$\left|a_3-a_5\right|=\left|\frac{1}{384}\left\{\left[96k_2\xi+48k_1{}^2\xi^2-48k_1{}^2\Upsilon\xi\right]-\left[48k_4\xi+32k_1k_3\xi^2-48k_1k_3\Upsilon\xi+12k_2{}^2\xi^2-24k_2{}^2\Upsilon\xi+k_1{}^4\xi^4\right.\right.\right.$$
$$\left.\left.\left.-6k_1{}^4\Upsilon\xi^3+11k_1{}^4\Upsilon^2\xi^2-6k_1{}^4\Upsilon^3\xi+12k_1{}^2k_2\xi^3-44k_1{}^2k_2\Upsilon\xi^2+36k_1{}^2k_2\Upsilon^2\xi\right]\right\}\right|$$
$$=\frac{1}{384}\left|k_1{}^4\xi\left[6\Upsilon^3-11\Upsilon^2\xi+6\Upsilon\xi^2-\xi^3\right]+k_1{}^2k_2\xi\left[-36\Upsilon^2+44\Upsilon\xi-12\xi^2\right]\right.$$
$$\left.+k_1{}^2\xi\left[-48\Upsilon+48\xi\right]+k_2{}^2\xi\left[24\Upsilon-12\xi\right]+k_1k_3\xi\left[48\Upsilon-32\xi\right]+96k_2\xi-48k_4\xi\right|.$$

(3.18)

Further, by suitably arranging the terms, we get

$$\left|a_3-a_5\right|=\frac{1}{384}\left|k_1{}^2\xi\left[k_2\left(-36\Upsilon^2+44\Upsilon\xi-12\xi^2\right)-k_1{}^2\left(-6\Upsilon^3+11\Upsilon^2\xi-6\Upsilon\xi^2+\xi^3\right)\right]\right.$$
$$\left.+k_1{}^2\xi\left[-48\Upsilon+48\xi\right]+k_2{}^2\xi\left[24\Upsilon-12\xi\right]-8\xi\left[6k_4-k_1k_3\left(6\Upsilon-4\xi\right)\right]+96k_2\xi\right|$$
$$=\frac{|\xi|}{384}\left|k_1{}^2\left(-36\Upsilon^2+44\Upsilon\xi-12\xi^2\right)\left[k_2-k_1{}^2\left(\frac{-6\Upsilon^3+11\Upsilon^2\xi-6\Upsilon\xi^2+\xi^3}{-36\Upsilon^2+44\Upsilon\xi-12\xi^2}\right)\right]\right.$$
$$\left.-48\left[k_4-k_1k_3\left(\frac{6\Upsilon-4\xi}{6}\right)\right]+k_1{}^2\left[-48\Upsilon+48\xi\right]+k_2{}^2\left[24\Upsilon-12\xi\right]+96k_2\right|$$
$$=\frac{|\xi|}{384}\left|k_1{}^2\left(-36\Upsilon^2+44\Upsilon\xi-12\xi^2\right)\left[k_2-\chi k_1{}^2\right]-48\left[k_4-\mu k_1k_3\right]\right.$$
$$\left.+k_1{}^2\left[-48\Upsilon+48\xi\right]+k_2{}^2\left[24\Upsilon-12\xi\right]+96k_2\right|$$

(3.19)

where

$$\chi=\frac{-6\Upsilon^3+11\Upsilon^2\xi-6\Upsilon\xi^2+\xi^3}{-36\Upsilon^2+44\Upsilon\xi-12\xi^2}$$

and

$$\mu=\frac{6\Upsilon-4\xi}{6}.$$

Consequently, by the triangle inequality, from (3.19), we get

$$\left|a_3-a_5\right|\le\frac{T}{384}\left\{\left|k_1\right|^2\left|-36\Upsilon^2+44\Upsilon\xi-12\xi^2\right|\left|k_2-\chi k_1{}^2\right|+48\left|k_4-\mu k_1k_3\right|\right.$$
$$\left.+96\left|k_2\right|+\left|k_1\right|^2\left|-48\Upsilon+48\xi\right|+\left|k_2\right|^2\left|24\Upsilon-12\xi\right|\right\}. \tag{3.20}$$

By making use of Lemma 2.1 and Lemma 2.2, we find that

$$|k_1|^2\left|-36\Upsilon^2+44\Upsilon\xi-12\xi^2\right|\left|k_2-\chi k_1^2\right|\le 8\left|-36\Upsilon^2+44\Upsilon\xi-12\xi^2\right|\bullet \tag{3.21}$$
$$\left|\frac{-12\Upsilon^3+36\Upsilon^2+\left(22\Upsilon^2-44\Upsilon\right)\xi+\left(-12\Upsilon+12\right)\xi^2+2\xi^3}{-36\Upsilon^2+44\Upsilon\xi-12\xi^2}\right|$$

and

$$48\left|k_4-\mu k_1k_3\right|\le 96\left|\frac{12\Upsilon-6-8\xi}{6}\right|. \tag{3.22}$$

Again by applying Lemma 2.1 along with (3.21) and (3.22), from (3.20) yields

$$\begin{aligned}|a_3-a_5|\le\frac{T}{384}\Big\{&8\left|-12\Upsilon^3+36\Upsilon^2+\left(22\Upsilon^2-44\Upsilon\right)\xi+\left(-12\Upsilon+12\right)\xi^2+2\xi^3\right|\\&+16\left|12\Upsilon-6-8\xi\right|+4\left|-48\Upsilon+48\xi\right|+4\left|24\Upsilon-12\xi\right|+192\Big\}.\end{aligned} \tag{3.23}$$

In view of (3.5), (3.7), (3.8), and (3.17), we have

$$\begin{aligned}&\left|a_3^2-2a_4^2+a_3a_5\right|\\
&=\Bigg|\frac{\xi^2}{9216}\left\{576k_2^2+k_1^4\left[144\Upsilon^2+144\xi^2-288\Upsilon\xi\right]+k_1^2k_2\left[-576\Upsilon+576\xi\right]\right\}\\
&+\frac{\xi^2}{9216}\Big\{-512k_3^2+k_1^6\left[-8\xi^4+24\Upsilon\xi^3-16\Upsilon^2\xi^2+24\Upsilon\xi^4-88\Upsilon^2\xi^3+96\Upsilon^3\xi^2-32\Upsilon^4\xi\right]\\
&+k_1k_2k_3\left[-768\xi+1024\Upsilon\right]+k_1^3k_3\left[-128\xi^2+192\Upsilon\xi-128\Upsilon^2+192\Upsilon\xi^2-128\Upsilon^2\xi\right]\\
&+k_1^2k_2^2\left[-288\xi^2+768\Upsilon\xi-512\Upsilon^2\right]+k_1^4k_2\Big[-96\xi^3+272\Upsilon\xi^2-288\Upsilon^2\xi+128\Upsilon^3\\
&-48\Upsilon\xi^3-288\Upsilon^2\xi^2+128\Upsilon^3\xi\Big]\Big\}+\frac{\xi^2}{9216}\Big\{288k_2k_4+k_2^3\left[72\xi-144\Upsilon\right]+k_1k_2k_3\left[192\xi-288\Upsilon\right]\\
&+k_1^6\left[3\xi^4-21\Upsilon\xi^3+51\Upsilon^2\xi^2-51\Upsilon^3\xi+18\Upsilon^4\right]+k_1^3k_3\left[96\xi^2-240\Upsilon\xi+144\Upsilon^2\right]\\
&+k_1^2k_4\left[-144\Upsilon+144\xi\right]+k_1^2k_2^2\left[108\xi^2-372\Upsilon\xi+288\Upsilon^2\right]\\
&+k_1^4k_2\left[42\xi^3-204\Upsilon\xi^2+306\Upsilon^2\xi-144\Upsilon^3\xi\right]\Big\}\Bigg|\\
&=\frac{\left|\xi^2\right|}{9216}\Big|\Big\{576k_2^2-512k_3^2+k_1^2k_4\left[-144\Upsilon+144\xi\right]\\
&+k_1^6\left[\left(24\Upsilon-5\right)\xi^4+\left(3\Upsilon-88\Upsilon^2\right)\xi^3+\left(35\Upsilon^2+96\Upsilon^3\right)\xi^2+\left(-32\Upsilon^4-51\Upsilon^3\right)\xi+18\Upsilon^4\right]\\
&+k_1^2k_2\left[-576\Upsilon+576\xi\right]+k_1^4\left[144\Upsilon^2+144\xi^2-288\Upsilon\xi\right]+288k_2k_4+k_2^3\left[72\xi-144\Upsilon\right]\\
&+k_1k_2k_3\left[-576\xi+736\Upsilon\right]+k_1^2k_2^2\left[-180\xi^2+396\Upsilon\xi-224\Upsilon^2\right]\\
&+k_1^3k_3\left[\left(-32+192\Upsilon\right)\xi^2+\left(-48\Upsilon-128\Upsilon^2\right)\xi+16\Upsilon^2\right]\\
&+k_1^4k_2\left[\left(-54-48\Upsilon\right)\xi^3+\left(68\Upsilon-288\Upsilon^2\right)\xi^2+\left(18\Upsilon^2-16\Upsilon^3\right)\xi+128\Upsilon^3\right]\Big\}\Big|.\end{aligned}$$

By suitably arranging the terms, we get

$$\left|a_3^2-2a_4^2+a_3a_5\right|$$

$$=\frac{\left|\xi^2\right|}{9216}\left|\left\{576k_2^2-512k_3^2+k_1^2k_4\left[-144\Upsilon+144\xi\right]\right.\right.$$

$$+k_1^6\left[\left(24\Upsilon-5\right)\xi^4+\left(3\Upsilon-88\Upsilon^2\right)\xi^3+\left(35\Upsilon^2+96\Upsilon^3\right)\xi^2+\left(-32\Upsilon^4-51\Upsilon^3\right)\xi+18\Upsilon^4\right]$$

$$+k_1^2\left(-576\Upsilon+576\xi\right)\left[k_2-k_1^2\left(\frac{-144\Upsilon^2-144\xi^2+288\Upsilon\xi}{-576\Upsilon+576\xi}\right)\right]+288k_2\left[k_4-k_2^2\left(\frac{-72\xi+144\Upsilon}{288}\right)\right]$$

$$+k_1k_2\left(-576\xi+736\Upsilon\right)\left[k_3-k_1k_2\left(\frac{180\xi^2-396\Upsilon\xi+224\Upsilon^2}{-576\xi+736\Upsilon}\right)\right]$$

$$+k_1^3\left[\left(-32+192\Upsilon\right)\xi^2+\left(-48\Upsilon-128\Upsilon^2\right)\xi+16\Upsilon^2\right]\bullet$$

$$\left.\left.\left[k_3-k_1k_2\left(\frac{\left(54+48\Upsilon\right)\xi^3+\left(-68\Upsilon+288\Upsilon^2\right)\xi^2+\left(-18\Upsilon^2+16\Upsilon^3\right)\xi-128\Upsilon^3}{\left(-32+192\Upsilon\right)\xi^2+\left(-48\Upsilon-128\Upsilon^2\right)\xi+16\Upsilon^2}\right)\right]\right\}\right|$$

and further yields

$$\left|a_3^2-2a_4^2+a_3a_5\right|$$

$$=\frac{\left|\xi^2\right|}{9216}\left|\left\{576k_2^2-512k_3^2+k_1^2k_4\left[-144\Upsilon+144\xi\right]\right.\right.$$

$$+k_1^6\left[\left(24\Upsilon-5\right)\xi^4+\left(3\Upsilon-88\Upsilon^2\right)\xi^3+\left(35\Upsilon^2+96\Upsilon^3\right)\xi^2+\left(-32\Upsilon^4-51\Upsilon^3\right)\xi+18\Upsilon^4\right]$$

$$+k_1^2\left(-576\Upsilon+576\xi\right)\left[k_2-\gamma k_1^2\right]+288k_2\left[k_4-\eta k_2^2\right]+k_1k_2\left(-576\xi+736\Upsilon\right)\left[k_3-\nu k_1k_2\right]$$

$$\left.\left.+k_1^3\left[\left(-32+192\Upsilon\right)\xi^2+\left(-48\Upsilon-128\Upsilon^2\right)\xi+16\Upsilon^2\right]\left[k_3-\lambda k_1k_2\right]\right\}\right|$$

(3.24)

where

$$\gamma=\frac{-144\Upsilon^2-144\xi^2+288\Upsilon\xi}{-576\Upsilon+576\xi},$$

$$\eta=\frac{-72\xi+144\Upsilon}{288},$$

$$\nu=\frac{180\xi^2-396\Upsilon\xi+224\Upsilon^2}{-576\xi+736\Upsilon}$$

and

$$\lambda=\frac{(54+48\Upsilon)\xi^3+(-68\Upsilon+288\Upsilon^2)\xi^2+(-18\Upsilon^2+16\Upsilon^3)\xi-128\Upsilon^3}{(-32+192\Upsilon)\xi^2+(-48\Upsilon-128\Upsilon^2)\xi+16\Upsilon^2}.$$

Consequently, by the triangle inequality, from (3.24), we obtain

$$\begin{aligned}&\left|a_3{}^2-2a_4{}^2+a_3a_5\right|\\&\le\frac{T^2}{9216}\Big\{576|k_2|^2+512|k_3|^2+|k_1|^2|k_4||-144\Upsilon+144\xi|\\&+|k_1|^6\left|(24\Upsilon-5)\xi^4+(3\Upsilon-88\Upsilon^2)\xi^3+(35\Upsilon^2+96\Upsilon^3)\xi^2+(-32\Upsilon^4-51\Upsilon^3)\xi+18\Upsilon^4\right|\\&+|k_1|^2|-576\Upsilon+576\xi|\left|k_2-\gamma k_1{}^2\right|+288|k_2|\left|k_4-\eta k_2{}^2\right|+|k_1||k_2||-576\xi+736\Upsilon||k_3-\nu k_1k_2|\\&+|k_1|^3\left|(-32+192\Upsilon)\xi^2+(-48\Upsilon-128\Upsilon^2)\xi+16\Upsilon^2\right||k_3-\lambda k_1k_2|\Big\}.\end{aligned}$$

(3.25)

By Lemma 2.2,

(3.26)
$$\begin{aligned}\left|k_2-\gamma k_1{}^2\right|&\le 2\max\{1,|2\gamma-1|\}\\&=2\max\left\{1,\left|\frac{-288\Upsilon^2+576\Upsilon-288\xi^2+(576\Upsilon-576)\xi}{-576\Upsilon+576\xi}\right|\right\},\end{aligned}$$

(3.27)
$$\begin{aligned}\left|k_4-\eta k_2{}^2\right|&\le 2\max\{1,|2\eta-1|\}\\&=2\max\left\{1,\frac{|-144\xi+288\Upsilon-288|}{288}\right\},\end{aligned}$$

(3.28)
$$\begin{aligned}|k_3-\nu k_1k_2|&\le 2\max\{1,|2\nu-1|\}\\&=2\max\left\{1,\left|\frac{360\xi^2+(-792\Upsilon+576)\xi+448\Upsilon^2-736\Upsilon}{-576\xi+736\Upsilon}\right|\right\}\end{aligned}$$

and

$$\begin{aligned}|k_3-\lambda k_1k_2|&\le 2\max\{1,|2\lambda-1|\}\\&=2\max\left\{1,\left|\frac{(108+96\Upsilon)\xi^3+(-328\Upsilon+576\Upsilon^2+32)\xi^2+(92\Upsilon^2+32\Upsilon^3+48\Upsilon)\xi-256\Upsilon^3-16\Upsilon^2}{(-32+192\Upsilon)\xi^2+(-48\Upsilon-128\Upsilon^2)\xi+16\Upsilon^2}\right|\right\}.\end{aligned}$$

(3.29)

Hence, applying Lemma 2.1 together with (3.26)-(3.29), we find that

(3.30) $$|k_1|^2|-576\Upsilon+576\xi||k_2-\gamma k_1^2|\le 8|-576\Upsilon+576\xi|\left|\frac{-288\Upsilon^2+576\Upsilon-288\xi^2+(576\Upsilon-576)\xi}{-576\Upsilon+576\xi}\right|,$$

(3.31) $$288|k_2||k_4-\eta k_2^2|\le\frac{1152|-144\xi+288\Upsilon-288|}{288},$$

(3.32) $$|k_1||k_2||-576\xi+736\Upsilon||k_3-\nu k_1k_2|\le 8|-576\xi+736\Upsilon|\left|\frac{360\xi^2+(-792\Upsilon+576)\xi+448\Upsilon^2-736\Upsilon}{-576\xi+736\Upsilon}\right|$$

and

$$|k_1|^3\left|(-32+192\Upsilon)\xi^2+(-48\Upsilon-128\Upsilon^2)\xi+16\Upsilon^2\right||k_3-\lambda k_1k_2|$$
$$\le 16\left|(-32+192\Upsilon)\xi^2+(-48\Upsilon-128\Upsilon^2)\xi+16\Upsilon^2\right|\bullet$$
$$\left|\frac{(108+96\Upsilon)\xi^3+(-328\Upsilon+576\Upsilon^2+32)\xi^2+(92\Upsilon^2+32\Upsilon^3+48\Upsilon)\xi-256\Upsilon^3-16\Upsilon^2}{(-32+192\Upsilon)\xi^2+(-48\Upsilon-128\Upsilon^2)\xi+16\Upsilon^2}\right|.$$

(3.33)

Again by applying Lemma 2.1 along with (3.30)-(3.33), from (3.25) yields

$$\begin{aligned}&|a_3^2-2a_4^2+a_3a_5|\\
&\le\frac{T^2}{9216}\{2304+2048+8|-144\Upsilon+144\xi|+8|-288\Upsilon^2+576\Upsilon-288\xi^2+(576\Upsilon-576)\xi|\\
&+64|(24\Upsilon-5)\xi^4+(3\Upsilon-88\Upsilon^2)\xi^3+(35\Upsilon^2+96\Upsilon^3)\xi^2+(-51\Upsilon^3-32\Upsilon^4)\xi+18\Upsilon^4|\\
&+4|-144\xi+288\Upsilon-288|+8|360\xi^2+(-792\Upsilon+576)\xi+448\Upsilon^2-736\Upsilon|\\
&+16|(108+96\Upsilon)\xi^3+(32-328\Upsilon+576\Upsilon^2)\xi^2+(48\Upsilon+92\Upsilon^2+32\Upsilon^3)\xi-256\Upsilon^3-16\Upsilon^2|\}.\end{aligned}$$

(3.34)

Finally, from (3.23) and (3.34), we obtain

$$\begin{aligned}|T_3(3)|\le&\frac{T}{384}\{8|-12\Upsilon^3+36\Upsilon^2+(22\Upsilon^2-44\Upsilon)\xi+(-12\Upsilon+12)\xi^2+2\xi^3|\\
&+16|12\Upsilon-6-8\xi|+4|-48\Upsilon+48\xi|+4|24\Upsilon-12\xi|+192\}\bullet\\
&\frac{T^2}{9216}\{2304+2048+8|-144\Upsilon+144\xi|+8|-288\Upsilon^2+576\Upsilon-288\xi^2+(576\Upsilon-576)\xi|\\
&+64|(24\Upsilon-5)\xi^4+(3\Upsilon-88\Upsilon^2)\xi^3+(35\Upsilon^2+96\Upsilon^3)\xi^2+(-51\Upsilon^3-32\Upsilon^4)\xi+18\Upsilon^4|\\
&+4|-144\xi+288\Upsilon-288|+8|360\xi^2+(-792\Upsilon+576)\xi+448\Upsilon^2-736\Upsilon|\\
&+16|(108+96\Upsilon)\xi^3+(32-328\Upsilon+576\Upsilon^2)\xi^2+(48\Upsilon+92\Upsilon^2+32\Upsilon^3)\xi-256\Upsilon^3-16\Upsilon^2|\}.\end{aligned}$$

This completes the proof of Theorem 3.2.

By putting the specific values for the parameters α, δ, A and B in Theorem 3.1 and Theorem 3.2, we obtain the coefficient bounds for the Toeplitz determinants for the subclasses introduced by El-Ashwah and Thomas [4], Halim [5], and Dahhar and Janteng [6], respectively as follows.

Corollary 3.1. *For* $f \in S_C^*(0,0,1,-1)$, *we obtain* $|T_2(3)| \le 25$ *and* $|T_3(3)| \le 240$.

Corollary 3.2. *For* $f \in S_C^*(0,\delta,1,-1)$, *we obtain*

$$|T_2(3)| \le \frac{4(1-\delta)^2}{2304}\Big\{832 + 64\left|16(1-\delta)^4\right| + 8\left|288(1-\delta)^2 + 288(1-\delta)\right| \\ +8\left|-288(1-\delta)^2 - 192(1-\delta)\right| + 16\left|-192(1-\delta)^3 - 64(1-\delta)^2\right|\Big\}$$

and

$$|T_3(3)| \le \frac{2(1-\delta)}{384}\{8\left|48(1-\delta)^2 + 16(1-\delta)^3\right| + 16\left|-6-16(1-\delta)\right| + 4\left|96(1-\delta)\right| + 4\left|-24(1-\delta)\right| + 192\}\bullet \\ \frac{4(1-\delta)^2}{9216}\Big\{2304 + 2048 + 8\left|288(1-\delta)\right| + 8\left|-1152(1-\delta)^2 - 1152(1-\delta)\right| \\ +64\left|-80(1-\delta)^4\right| + 4\left|-288(1-\delta) - 288\right| + 8\left|1440(1-\delta)^2 + 1152(1-\delta)\right| \\ +16\left|864(1-\delta)^3 + 128(1-\delta)^2\right|\Big\}.$$

Corollary 3.3. *For* $f \in S_C^*(0,0,A,B)$, *we obtain*

$$|T_2(3)| \le \frac{(A-B)^2}{2304}\Big\{832 + 64\Big|(1-3\Upsilon)(A-B)^4 + \left(-3\Upsilon + 11\Upsilon^2\right)(A-B)^3 \\ +\left(-12\Upsilon^3 + 2\Upsilon^2\right)(A-B)^2 + 4\Upsilon^4(A-B)\Big| \\ +8\left|72\Upsilon^2 - 144\Upsilon + 72(A-B)^2 + (-144\Upsilon + 144)(A-B)\right| \\ +8\left|-128\Upsilon^2 + 128\Upsilon - 72(A-B)^2 + (192\Upsilon - 96)(A-B)\right| \\ +16\Big|-16\Upsilon^2 + 32\Upsilon^3 + \left(24\Upsilon - 88\Upsilon^2 + 32\Upsilon^3\right)(A-B) \\ +(-24-12\Upsilon)(A-B)^3 + \left(-16 + 92\Upsilon - 72\Upsilon^2\right)(A-B)^2\Big|\Big\}$$

and

$$\left|T_3(3)\right| \le \frac{(A-B)}{384}\Big\{8\left|-12\Upsilon^3+36\Upsilon^2+\left(22\Upsilon^2-44\Upsilon\right)(A-B)+(-12\Upsilon+12)(A-B)^2+2(A-B)^3\right|$$

$$+16\left|12\Upsilon-6-8(A-B)\right|+4\left|-48\Upsilon+48(A-B)\right|+4\left|24\Upsilon-12(A-B)\right|+192\Big\}\bullet$$

$$\frac{(A-B)^2}{9216}\Big\{2304+2048+8\left|-144\Upsilon+144(A-B)\right|$$

$$+8\left|-288\Upsilon^2+576\Upsilon-288(A-B)^2+(576\Upsilon-576)(A-B)\right|$$

$$+64\left|(24\Upsilon-5)(A-B)^4+\left(3\Upsilon-88\Upsilon^2\right)(A-B)^3+\left(35\Upsilon^2+96\Upsilon^3\right)(A-B)^2\right.$$

$$\left.+\left(-51\Upsilon^3-32\Upsilon^4\right)(A-B)+18\Upsilon^4\right|+4\left|-144(A-B)+288\Upsilon-288\right|$$

$$+8\left|360(A-B)^2+(-792\Upsilon+576)(A-B)+448\Upsilon^2-736\Upsilon\right|+16\left|(108+96\Upsilon)(A-B)^3\right.$$

$$\left.+\left(32-328\Upsilon+576\Upsilon^2\right)(A-B)^2+\left(48\Upsilon+92\Upsilon^2+32\Upsilon^3\right)(A-B)-256\Upsilon^3-16\Upsilon^2\right|\Big\}.$$

It is observed that the result of $\left|T_2(3)\right|$ for S^* and S_C^* are shown to be equivalent.

4. CONCLUSION

In this paper, we have obtained the coefficient bounds for $\left|T_2(3)\right|$ and $\left|T_3(3)\right|$ for the subclass of tilted starlike functions with respect to conjugate points of order δ, $S_C^*(\alpha,\delta,A,B)$. The results obtained can be reduced to the results for some existing subclasses in the literature by considering specific values for the parameters α, δ, A and B.

Acknowledgements: The authors wish to thank the anonymous referees for their careful reading.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

[1] I. Graham, Geometric function theory in one and higher dimensions, CRC Press, New York, 2003.

[2] S. S. Miller, P. T. Mocanu, Second order differential inequalities in the complex plane, J. Math. Anal. Appl. 65(2) (1978), 289-305.

[3] S. S. Miller, P. T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J. 28(2) (1981), 157-172.

[4] R. M. El-Ashwah, D. K. Thomas, Some subclasses of close-to-convex functions, J. Ramanujan Math. Soc. 2(1) (1987), 85-100.

[5] S. Halim, Functions starlike with respect to other points, Int. J. Math. Math. Sci. 14(3) (1991), 451-456.

[6] S. A. F. M. Dahhar, A. Janteng, A subclass of starlike functions with respect to conjugate points, Int. Math. Forum, 4(28) (2009), 1373-1377.

[7] N. H. A. A. Wahid, D. Mohamad, S. Cik Soh, On a subclass of tilted starlike functions with respect to conjugate points, Menemui Mat. (Discover. Math.) 37(1) (2015), 1-6.

[8] K. Ye, L. H. Lim, Every matrix is a product of Toeplitz matrices, Found. Comput. Math. 16(3) (2016), 577-598.

[9] D. K. Thomas and S. A. Halim, Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions, Bull. Malaysian Math. Sci. Soc. 40(4) (2016), 1781-1790.

[10] V. Radhika, S. Sivasubramanian, G. Murugusundaramoorthy, J. M. Jahangiri, Toeplitz matrices whose elements are the coefficients of functions with bounded boundary rotation, J. Complex Anal. 2016 (2016), Art. ID 4960704.

[11] S. Sivasubramanian, M. Govindaraj, G. Murugusundaramoorthy, Toeplitz matrices whose elements are the coefficients of analytic functions belonging to certain conic domains, Int. J. Pure Appl. Math. 109(10) (2016), 39-49.

[12] C. Ramachandran, D. Kavitha, Toeplitz determinant for some subclasses of analytic functions, Glob. J. Pure Appl. Math. 13(2) (2017), 785-793.

[13] N. Magesh, Ş. Altınkaya, S. Yalçın, Construction of Toeplitz matrices whose elements are the coefficients of univalent functions associated with q-derivative operator, ArXiv:1708.03600 [Math]. (2017).

[14] M. F. Ali, D. K. Thomas, A. Vasudevarao, Toeplitz determinants whose elements are the coefficients of analytic and univalent functions, Bull. Aust. Math. Soc. 97(2) (2018), 253-264.

[15] V. Radhika, J. M. Jahangiri, S. Sivasubramanian, G. Murugusundaramoorthy, Toeplitz matrices whose elements are coefficients of Bazilevič functions, Open Math. 16(1) (2018), 1161-1169.

[16] H. M. Srivastava, Q. Z. Ahmad, N. Khan, B. Khan, Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain, Mathematics, 7(2) (2019), 181.

[17] H. Y. Zhang, R. Srivastava, H. Tang, Third-order Hankel and Toeplitz determinants for starlike functions connected with the sine function, Mathematics, 7(5) (2019), 404.

[18] S. N. Al-Khafaji, A. Al-Fayadh, A. H. Hussain, S. A. Abbas, Toeplitz Determinant whose Its Entries are the Coefficients for Class of Non-Bazilevic Functions, J. Phys.: Conf. Ser. 1660 (2020), 012091.

[19] P. L. Duren, Univalent Functions vol. 259, Springer, New York-Berlin–Heidelberg–Tokyo, 1983.

[20] I. Efraimidis, A generalization of Livingston's coefficient inequalities for functions with positive real part, J. Math. Anal. Appl. 435(1) (2016), 369-379.