A COMMUTATIVE AND COMPACT DERIVATIONS FOR $W^*$ ALGEBRAS

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ABSTRACT. In this paper, we study the compact derivations on $W^*$ algebras. Let $M$ be $W^*$-algebra, let $LS(M)$ be algebra of all measurable operators with $M$, it is show that the results in the maximum set of orthogonal predictions. We have found that $W^*$ algebra $A$ contains the Center of a $W^*$ algebra $B$ and is either a commutative operation or properly infinite. We have considered derivations from $W^*$ algebra two-sided ideals.

1. INTRODUCTION

Let $M$ be a $W^*$-algebra and let $Z(M)$ be the center of $M$. Fix $a \in M$ and consider the inner derivation $\delta_a$ on $M$ generated by the component $a$, which is $\delta_a(\cdot) := [a, \cdot]$.

The norm closing two sided ideal $f(B)$ generated by the finite projections of a $W^*$ algebra $B$ behaves somewhat similar to the idealized compact operators of $B(H)$ (see [11],[8],[9]). Therefore, it is natural to ask about any sub-algebras $d$ of $B$ that is any derivation from $A$ into $f(B)$ implemented from an element of $y(B)$.
We perform two main difficulties: the presence of the center of \( B \) and the fact that the main characteristic in [8] proof (that is, if \( Q_n \), is a sequence of mutually orthogonal projections and \( T \in B(H) \) hence \( \|Q_n T Q_n\| > \alpha > 0 \) for all \( n \) implies that \( T \) is not compact) failure to generalize to the case in which \( g \) is of Type II\(_\infty\).

Finally, we have considered derivations from \( d \) at the two-sided \( C_{1+\varepsilon}(B, \tau) = B \cap L^{1+\varepsilon}(B, \tau)(1 \leq 1+\varepsilon < \infty) \) to obtain faithful finite normal trace \( \tau \) on \( B \).

2. NOTATIONS

**Lemma (1).** Let \( B \) be a semi-finite algebra, let \( Q_0 \in p(B) \) and \( x_0 \in Q_0 \) be such that \( \omega_{x_0} \), is a faithful trace on \( B_{Q_0} \). Assume there are \( Q_n \in p(B) \), \( F_n \in p(\ell) \) and \( U_n \in B \) for \( n = n_1, n_{i+1}, \ldots, \) such that the projections \( Q_n \) are mutually orthogonal and \( Q_n = U_n U_n^* \), \( Q_n F_n = U_n^* U_n \) for all \( n \) (i.e., \( Q_n \sim Q_{0_n} F_n \)). Let \( x_n = U_n F_n x_0 \). Then \( x_n \to_{jRW} O \).

**Proof.** Assume that \( \sum_{n=0}^\infty Q_n = n \). Let \( \tau \) be a faithful semi-finite normal \( (\text{fin}) \) trace on \( B^+ \) to be agreed on \( B_{Q_0} \) with \( \omega_{x_0} \). Then for all \( B \in B_{Q_0}^+ \) we have

\[
\tau(B) = \tau(U_n U_n^* B U_n^* U_n^*) = \tau(U_n^* U_n U_n^* B U_n^*) = \tau(Q_{0_n} F_n U_n^* B U_n^* F_n Q_{0_n}) = \omega_{x_n}(F_n U_n^* B U_n^* F_n) = \omega_{x_n}(B).
\]

Let \( P \in p(B) \) be any semi-finite projection. Then by [11] there is a central decomposition of the identity \( \sum_{\gamma \in \Gamma} E_\gamma = 1, E_\gamma \in p(\ell), E_\gamma E_{\gamma'} = 0 \) for \( \gamma \neq \gamma' \) such that \( \tau(PE_\gamma) < \infty \) for all \( \gamma \in \Gamma \). Then

\[
\tau(PE_\gamma) = \sum_{n=1}^\infty \tau(Q_n PE_\gamma Q_n) = \sum_{n=1}^\infty \omega_{x_n}(Q_n PE_\gamma Q_n) \leq \sum_{n=1}^\infty \|PE_n x_n\|^2 < \infty.
\]
whence \( \|PE\gamma x_n\| < 0 \) for all \( \gamma \in \Gamma \). Let \( \varepsilon > 0 \) and let \( \Lambda \subset \Gamma \) be a finite index set such that 
\[
\sum_{\gamma \in \Lambda} \|E\gamma x_0\|^2 < \varepsilon.
\]
Then for all \( n \),
\[
\sum_{\gamma \in \Lambda} \|PE\gamma x_n\|^2 = \sum_{\gamma \in \Lambda} \|PE\gamma U_n F_n x_0\|^2
\]
\[
= \sum_{\gamma \in \Lambda} \|PU_n F_n E\gamma x_0\|^2
\]
\[
\leq \sum_{\gamma \in \Lambda} \|E\gamma x_0\|^2 < \varepsilon.
\]
Hence from \( \|Px_n\|^2 \leq \sum_{\gamma \in \Lambda} \|PE\gamma x_n\|^2 + \varepsilon \) where \( \|Px_n\| \to 0 \), to completes the proof.

**Lemma (2).** Let \( T \notin f (P) \), then there is an \( \alpha > 0 \) and \( 0 \neq E \in p(\ell) \) such that for every \( 0 \neq F \in p(\ell) \) with \( F \leq E \) we have \( \|\pi (TF)\| > \alpha \).

**Proof.** Let \( \alpha = \frac{1}{2}\|\pi (T)\| \neq 0 \) and let \( G \) be the sum of a maximal family of mutually orthogonal central projections \( G\gamma \) such that \( \|\pi (TG\gamma)\| \leq \alpha \). Then
\[
\|\pi (TG)\| = \sup_{\gamma} \|\pi (TG\gamma)\| \leq \alpha,
\]
hence \( G \neq 1 \). Let \( E = Z - G \) and let \( 0 \neq F \in p(\ell) \) with \( F \leq E \).
Since \( FG = 0 \), by the maximally of the family we have \( \|\pi (TF)\| > \alpha \).

### 3. RELATIVELY COMPACT DERIVATION

Let \( M \) be a \( W^* \)-algebra and let \( Z(M) \) be the center of \( M \). Fix \( a \in M \) and consider the inner derivation \( \delta_a \) on \( M \) generated by the element \( a \), that is \( \delta_a (\cdot) := [a, \cdot] \). Obviously, \( \delta_a \) there is a linear bounded operator on \( (M, \| \cdot \|_M) \), where \( \| \cdot \|_M \) is a \( C^* \) -norm on \( M \). It is known that there exists \( c \in Z(M) \) such that the following estimate holds: \( \|\delta_a\| \geq \|a - c\|_M \). In view of this result, it is natural to ask whether there exists an element \( y \in M \) with \( \|y\| \leq 1 \) and \( c \in Z(M) \) such that \( [a, y] \geq |a - c| \).

**Definition (3).** A linear subspace \( I \) in the \( W^* \) algebra \( M \) equipped with a norm \( \| \cdot \|_I \) is said to be a symmetric operator ideal if
(i) \( \| S \|_s \geq \| S \| \) for all \( S \in I \).

(ii) \( \| S^* \|_r = \| S \|_s \) for all \( S \in I \).

(iii) \( \| A S B \| \leq \| A \| \| S \| \| B \| \) for all \( S \in I \), \( A, B \in M \).

Observe, that every symmetric operator ideal \( I \) is a two-sided ideal in \( M \), and therefore by [13], it follows from \( 0 \leq S \leq T \) and \( T \in I \) that \( S \in I \) and \( \| S \|_s \leq \| T \|_s \).

**Corollary (4).** Let \( M \) be a \( W^* \)-algebra and let \( I \) be an ideal in \( M \). Let \( \delta : M \to I \) be a derivation. Then there exists an element \( a \in I \), such that \( \delta = \delta_a \).

**Proof.** Since \( \delta \) is a derivation on a \( W^* \)-algebra, it is necessarily inner [8]. Thus, there exists an element \( d \in M \), such that \( \delta(\cdot) = \delta d(\cdot) = [d, \cdot] \). It follows from the hypothesis that \( [d, M] \subseteq I \).

Using [22] (or [20]), we obtain \( [d^*, M] = -[d, M]^* \subseteq I^* = I \) and \( [d_k, M] \subseteq I, k = 1, 2 \), where \( d = d_1 + id_2 \), \( d_k = d_k^* \in M \), for \( k = 1, 2 \). It follows now, that there exist \( c_1, c_2 \in Z(M) \) and \( u_1, u_2 \in U(M) \), such that \( \| [d_k, u_k] \| \geq 1/2 |d_k - c_k| \) for \( k = 1, 2 \). Again applying [20], we obtain \( d_k - c_k \in I \), for \( k = 1, 2 \). Setting \( a := (d_1 - c_1) + i(d_2 - c_2) \), we deduce that \( a \in I \) and \( \delta = [a, \cdot] \).

**Corollary (5).** Let \( M \) be a semi-finite \( W^* \)-algebra and let \( E \) be a symmetric operator space. Fix \( a = a^* \in S(M) \) and consider inner derivation \( \delta = \delta_a \) on the algebra \( LS(M) \) given by \( \delta(x) = [a, x] \), \( x \in LS(M) \). If \( \delta(M) \subseteq E \), then there exists \( d \in E \) satisfying the inequality \( \| d \|_E \leq \| \delta \|_{M \to E} \) and such that \( \delta(x) = [d, x] \).

**Proof.** The existence of \( d \in E \) such that \( \delta(x) = [d, x] \). Now, if \( u \in U(M) \), then
\[
\| \delta(u) \|_E = \| du - ud \|_E \leq \| du \|_E + \| ud \|_E = 2 \| d \|_E .
\]
Hence, if \( x \in M_1 = \{ x \in M : \| x \| \leq 1 \} \), then
\[
x = \sum_{i=1}^4 \alpha_i u_i , \quad \text{where} \quad u_i \in U(M) \quad \text{and} \quad |\alpha_i| \leq 1 \quad \text{for} \quad i = 1, 2, 3, 4 , \quad \text{and so}
\]
\[
\| \delta(x) \|_E \leq \sum_{i=1}^4 \| \delta(\alpha_i u_i) \|_E \leq 8 \| d \|_E , \quad \text{that is} \quad \| \delta \|_{M \to E} \leq 8 \| d \|_E < \infty .
\]
4. **A Commutative Operation on $W^*$ Sub-algebras**

When $A$ a commutative operation is is crucial because it provides the following explicit way to find an operator $T \in B$ implementing the derivation.

For the rest of this section let $A$ be any a commutative operation sub-algebras of $B$ and $\delta : A \to B$ be any derivation. Let $u$ be the unitary group of $A$ and $M$ be a given invariant mean on $u$, i.e., a linear functional on the algebra of bounded complex-valued functions on $u$ such that

(i) For all real $f$, $\inf \{ f(U) \mid U \in u \} \leq Mf \leq \sup \{ f(U) \mid U \in u \}$

(ii) For all $U \in u$, $Mf_U = MS$, where $f_U(V) = f(UV)$ for $V \in u$.

Thus $M$ is bounded and $|Mf| \leq \sup \{ |f(U)| \mid U \in u \}$ for all $f$ (see [8] for the existence and properties of $M$).

For each $\phi \in B$, the map

$$\phi \mapsto M\phi(U^*\delta(U))$$

is linear and bounded and hence defines an element $T \in (B_u)^\prime$. Explicitly,

$$\phi(T) \mapsto M\phi(U^*\delta(U)) \quad \text{for all } \phi \in B_u$$

The same easy computation as in [8] shows that $\delta = aAT$. Notice that for all $A \in B$ the map

$$\phi \mapsto M\phi(U^*BU) = \phi(E(B))$$

defines an element $E(B)$ which clearly belongs to $A \cap B$. Moreover it is easy to see that $E$ is a conditional expectation (i.e., a projection of norm one) from $B$ onto $A \cap B$ (see [6]).

**Theorem (6).** Let $A$ be a commutative operation $W^*$ sub-algebras of $B$ containing the center $\ell$ of $B$. For every derivation $\delta : A \to f(B)$ there is a $T \in f(B)$ such that $\delta = aAT$.

We have seen that given an invariant mean $M$ on $u$ there is a unique $T \in B$ such that $\delta = aAT$ and $E(T) = 0$. We are going to show that $T \in A(B)$. Reasoning by contradiction assume that $T \notin A(B)$. We proof requires several reductions to the restricted derivation
\( \delta_{E} : A_E \rightarrow f(B) \) for some \( 0 \neq E \in p(\ell) \). To simplify notations we shall assume each time that \( E = 1 \).

Let us start by noticing that if \( Q_i \in p(A) \) for \( i = n,n+1, \) \( Q_n, Q_{n+1} = 0 \) and \( P = Q_n + Q_{n+1} \), then

\[
PTP = \sum_{i=n}^{n+1} Q_i T Q_i + \delta(Q_{n+1}) Q_n + \delta(Q_n) Q_{n+1}
\]

hence

\[
\|\pi(PTP)\| = \left\| \sum_{i=n}^{n+1} \pi(Q_i T Q_i) \right\| + \max_i \pi(Q_i T Q_i)
\]

**Definition (7).** For every \( Q \in p(A) \) define \( [Q] = [Q, \varepsilon] \) to be the central projection. Set

\[
P = \{ P \in p(A) \mid [P] = 1 \}.
\]

Thus \( P \in p \) iff \( \|\pi(PTPG)\| = \|\pi(TG)\| \) for all \( G \in p(\ell) \). We collect several properties of \( [Q] \).

**Corollary (8).** Let \( B \) be a semi-finite \( W^* \) algebra with a trace \( \tau \), let \( A \) be a properly infinite \( W^* \) sub-algebras of \( B \) and let \( 1 \leq 1 + \varepsilon < \infty \). Then for every derivation \( \delta : A \rightarrow C_{1+\varepsilon}(B,\tau) \) there is \( aT \in C_{1+\varepsilon}(B,\tau) \) such that \( \tilde{\delta} = aAT \).

In the notations introduced there, it is easy to see that \( \phi(C_{1+\varepsilon}(B,\tau)) = C_{1+\varepsilon}(\tilde{B},\tilde{\tau}) \), where \( \tau = \tau \oplus \tau_0 \) and \( \tau_0 \) is the usual trace on \( B(H_0) \). We can actually simplify the proof by choosing \( \tilde{A}_n = I \otimes \ell \) since the condition \( \ell \subset A \) is no longer required.

**Corollary (9).** Let \( P = Q_n + Q_{n+1} \). Then there is a largest central projection \( [Q_n, Q_{n+1}] \) such that for every \( G \in p(\ell) \) with \( G \leq [Q_n, Q_{n+1}] \), we have \( \|\pi(Q_i T Q_i G)\| = \|\pi(PTPG)\| \).

**Proof.** Let \( G_i = \{ G \in p(\ell) \mid \|\pi(Q_i T Q_i G)\| = \|\pi(PTPG)\| \} \) and \( \Xi = \{ G + \varepsilon \in p(\ell) \mid G \in G_n \} \). Since \( \|\pi(PTPG)\| = \max_i \|\pi(Q_i T Q_i G)\| \) for all \( G \in p(\ell) \), we see that \( G_n \cup G_{n+1} = p(\ell) \). Notice that \( \Xi \) is hereditary (i.e., \( G - \varepsilon \in \Xi \) and \( F \in p(\ell) \), \( F \leq G + \varepsilon \) imply \( F \in \Xi \)).
Let $[Q_n, Q_{n+1}] = \sup \Xi$. We have only to show that $[Q_n, Q_{n+1}] \in \Xi$. Let $G + \varepsilon = \sum_\gamma (G + \varepsilon)_\gamma$ be the sum of a maximal collection of mutually orthogonal projections $(G + \varepsilon)_\gamma \in \Xi$. Then for every $F \in \Xi$ we have $([Q_n, Q_{n+1}] - (G + \varepsilon))F = 0$ because of the maximal of the collection of $\Xi$. Then $[Q_n, Q_{n+1}] = G + \varepsilon$. Consider now any $G \in p(\ell), \varepsilon \geq 0$, then $G = \sum_\gamma G (G + \varepsilon)_\gamma$ and since $G (G + \varepsilon)_\gamma \leq (G + \varepsilon)_\gamma \in \Xi$, we have $\|\pi \left( Q_n T Q_n G (G + \varepsilon)_\gamma \right)\| = \|\pi \left( P T P G (G + \varepsilon)_\gamma \right)\|$ for all $\gamma$. Since $\pi \left( Q_n T Q_n G \right) (resp. \pi \left( P T P G \right))$ is the direct sum of then $\pi \left( Q_n T Q_n G \right) (resp. \pi \left( P T P G \right))$, then we have

$$
\|\pi \left( Q_n T Q_n G \right)\| = \sup_\gamma \|\pi \left( Q_n T Q_n G (G + \varepsilon)_\gamma \right)\|
$$

$$
= \sup_\gamma \|\pi \left( P T P G (G + \varepsilon)_\gamma \right)\|
$$

$$
= \|\pi \left( P T P G \right)\|
$$

whence $G \in G_n$. Since $\varepsilon \geq 0$ is arbitrary, we have $G + \varepsilon = [Q_n, Q_{n+1}] \in \Xi$ which completes the proof.

**Corollary (10).** (i) If $Q_n Q_{n+1} = 0$ with $Q_i \in p(A)$ then $1 - [Q_n, Q_{n+1}] \leq [Q_n, Q_{n+1}]$.

(ii) If $Q_n \leq Q_{n+1}$ with $Q_i \in p(A)$ then $[Q_n] \leq [Q_{n+1}]$.

(iii) If $\varepsilon \geq 0$ with $Q \in p(A), Q + \varepsilon \in p$ then $[Q] = [Q, \varepsilon]$ and $1 - [Q] \leq [\varepsilon]$

If $\pi(TG) \neq 0$ for all $0 \neq E \in p(\ell)$ then the following hold:

(iv) If $E \in p(\ell)$ then $E = [E]$. 

(v) If $Q \in p(A)$ then $[Q] \leq c(Q)$, where $c(Q)$ is the central support of $Q$. 

**Proof.** We have to show that for every $G \in p(\ell), G \leq 1 - [Q_n, Q_{n+1}]$ we have $G \in G_{n+1}$. Let $E + \varepsilon$ be the sum $\sum_\gamma E_\gamma$ of a maximal collection of mutually orthogonal projections of $G_{n+1}$ that are majoried by $G$. Then
\[ \|\pi(Q_nTQ_nF)\| = \sup_{\gamma} \|\pi(Q_nTQ_nF_{\gamma})\| \\
= \sup_{\gamma} \|\pi(Q_{n+1} + Q_{n+1})T(Q_{n+1} + Q_{n+1})F_{\gamma}\| \\
= \|\pi(Q_{n+1} + Q_{n+1})T(Q_{n+1} + Q_{n+1})F\| \]

whence \( E + \varepsilon \in G_{n+1} \). By the maximalist of the collection, \( 0 \leq G - (E + \varepsilon) \) does not majority any nonzero projection of \( G_{n+1} \) and since \( p(\ell) = G_n \cup G_{n+1} \), any central projection \( G' \leq G - (E + \varepsilon) \) must be in \( G_n \). By definition of \( \Xi \), this implies that \( G - (E + \varepsilon) \in \Xi \) whence \( G - (E + \varepsilon) \leq [Q_n, Q_{n+1}] \). So,

\( G - (E + \varepsilon) \leq G \leq 1 - [Q_n, Q_{n+1}] \) and hence \( G = E + \varepsilon \in G_{n+1} \) which completes the proof.

(ii) Let \( G \in p(\ell) \) and \( G \leq [Q_n] \). Then \( \|\pi(TG)\| = \|\pi(Q_nTQ_nG)\| \leq \|\pi(Q_{n+1}TQ_{n+1}G)\| \)

\( \leq \|\pi(TG)\| \) whence equality holds and \( [Q_n] \leq [Q_{n+1}] \) by the maximalist of \( [Q_{n+1}] \).

(iii) \([Q, \varepsilon]\) is maximal under the condition: if \( G \in p(\ell) \) and \( G \leq [Q, \varepsilon] \) then

\[ \|\pi(QTQG)\| \leq \|\pi((Q + \varepsilon)T(Q + \varepsilon)G)\| = \|\pi(TG)\| \]

which is the same condition defining \([Q, I - Q] = [Q]\). Thus \([Q] = [Q, \varepsilon]\). Applying this to \( \varepsilon \) we have \([\varepsilon] = [\varepsilon, Q] \) and thus by (i) we have \([\varepsilon] \geq 1 - [Q, \varepsilon] = 1 - [Q] \).

(ii) Let \( E + \varepsilon, E \in p(\ell) \) then \( \|\pi(TEE(E + \varepsilon))\| = \|\pi(T(E + \varepsilon))\| \). This implies that if \( \varepsilon \geq 0 \), then \( E + \varepsilon \leq [E] \) so \( E \leq [E] \) and if \( E + \varepsilon = [E] - E \leq [E] \) then

\[ 0 = \|\pi(TEE(E + \varepsilon))\| = \|\pi(T(E + \varepsilon))\| \) whence \( E = [E] \).

(v) Follows at once from (ii) and (iv).

The condition that \( \|\pi(TE)\| \neq 0 \) for all \( 0 \neq E \in p(\ell) \) is of course meaningless unless \( B \) is properly infinite. Hence, we may assume without loss of generality that:

\( B \) is properly infinite and semi-finite.

There is an \( \alpha > 0 \) such that \( \|\pi(TE)\| > \alpha \) for all \( 0 \neq E \in p(\ell) \).
Lemma (11). Let $P \in p$ and $R_n = X_{PTP}[\alpha, \infty)$, $R_{n+1} = X_{PTP}(-\infty, -\alpha]$, where $X_{PTP}(\ )$ denotes the spectral measure of the self-adjoint operator $PTP$. Then there is an $E_n \in p(\ell)$, with $E_n = I - E$ such that $R_i E_i$ are properly infinite and $c(R_i E_i) = E_j$ for $i = n, n+1$.

Proof. Let $R = R_n + R_{n+1} = X_{PTP}[\infty, \alpha)$ and let $F \neq 0$ be any central projection. If $RF$ were finite, we would have

$$\|\pi(TF)\| = \|\pi(PTPF)\|$$
$$= \|\pi(PTP(1-R)F)\|$$
$$= \|\pi([PTP](1-R)F)\|$$
$$\leq \alpha$$

Thus $RF$ is infinite and nonzero. Hence $R$ is properly infinite and $c(R) = n$. Now let $E_1$ be the maximal central projection majored by $c(R_n)$, such that $R_n F_n$ is properly infinite. Then $c(R_n, E_n) = E_n$ and $R_n(n-E_n)$ is finite, hence $R_{n+1}(n-E_n) = R_{n+1}E_{n+1}$ is properly infinite and $c(R_{n+1}, E_{n+1}) = E_{n+1}$.

End of the Proof of Theorem (6). Take any $0 \neq Q_0 \in p(B)$ such that $B_{\omega_0}$ has a faithful trace $\omega_0$ with $x_0 \in Q_0H$ and assume $\|x_0\| = 1$. Let $P_\gamma \in p$, $\gamma \in \Gamma$ be the not decreasing to zero. We are going to construct inductively a sequence $\gamma_n \in \Gamma$, $F_n \in p(\ell)$, $Q_n \in p(B)$, $U_n$ partial isometrics in $B$, $x_n \in H$ such that

(a) $U_nU_n^* = Q_n$, $U_n^*U_n = Q_0F_n$, i.e., $Q_n \sim Q_0F_n$

(b) $x_n = U_nF_n x_0 \in Q_nH$

(c) $Q_n Q_m = 0$ for $n \neq m$

(d) $\gamma_n > \gamma_m$ (hence $P_{\gamma_n} < P_{\gamma_m}$) for $n > m$

(e) $Q_n \leq p_{\gamma_n}$

(f) $\|p_{\gamma_n} x_n\| < \gamma_n$

(g) $|Tx_n, x_n| \geq \alpha_n$. 
The induction can be started with an arbitrary $P_x$; assume we have the construction for $n-1$. Let us apply Lemma(11) to $P = P_x$ and obtain $E_i \in p(\ell), R_i \in p(B)$ for $i = n, n+1$ as defined there. Then

$$1 = \|x_x\|^2 = \|E_n x_0\|^2 + \|E_{n+1} x_0\|^2$$

Let $F_n$ be (any of) the projection $E_n$ or $E_{n+1}$ for which $\|E_n x_0\|^2 \geq \frac{1}{2}$ and let $i$ be the corresponding index. Then $R_i F_n$ is properly infinite and has central support $F_n$. Now $Q_0$ is finite having a finite faithful trace $\omega_{x_0}$, hence so is $Q_j \sim F_j Q_0 \leq Q_0$ for $1 \leq j \leq n-1$ and $(\sum_{j=1}^{n-1} Q_j) F_n$. Let $S_n = \inf \{ R_i F_n, \left(1 - \sum_{j=1}^{n-1} Q_j\right) F_n\}$. By the parallelogram law (see [2]) applied to $F_n$ we have that

$$R_i F_n - S_n \sim \left(\sum_{j=1}^{n-1} Q_j\right) F_n - \inf \left\{ \left(\sum_{j=1}^{n-1} Q_j\right) F_n, \left(1 - R_i\right) F_n\right\}$$

whence $R_i F_n - S_n$ is finite and hence $S_n$ is properly infinite and $c(S_n) = F_n$. Since $Q_0 F_n$ is finite and $c(Q_0 F_n) \leq F_n$ we have $Q_0 F_n \prec S_n$, i.e., there is a partial isometry $U_n \in B$ and a $Q_n \in p(B), Q_n \leq S_n$ such that (a) holds. Let $x_n$ be defined by (b) and choose $\gamma_{n+1} \succ \gamma_n$ so that (d) and (f) hold. Since $Q_n \leq R_i \leq P_x$ we have (e), since $Q_n \leq \left(1 - \sum_{j=1}^{n-1} Q_j\right) F_n$ we have (c). Finally $x_n = R_i x_n = P_x x_n$ hence (g) follows from

$$\left\|(T x_n, x_n)\right\| = \left\|(P_{\gamma_n} T P_{\gamma_n} x_n, x_n)\right\|$$

$$= \left\|(P_{\gamma_n} T P_{\gamma_n} R_i x_n, R_i x_n)\right\|$$

$$\geq \alpha \left\| (R_i x_n, R_i x_n) \right\|$$

$$= \alpha \|x_n\|^2$$

$$= \alpha \|F_n x_0\|^2$$

$$\geq \frac{1}{2} \alpha.$$

Let now $y_n = x_n - P_{\gamma_{n+1}} x_n$. $B$ is semi-finite, hence we can apply Lemma (1) to obtain that $x_n \to B_{BEW} 0$. Since $\left\|P_{\gamma_{n+1}} x_n\right\| \to 0$ we thus have $y_n \to B_{BEW} 0$ and $y_n \in P_n H$, where
\[ P_n = P_{\gamma_n} - P_{\gamma_n+1} \in p(d) \] and are mutually orthogonal by (d). Clearly for \( n \) large enough, 
\[ |(Ty_n, y_n)| = |\omega_{\gamma_n}(T)| > \frac{1}{4}\alpha. \] Since \( \omega_{\gamma_n}(T) = M\omega_{\gamma_n}(U^*\delta(U)) \), by the properties of the invariant mean mentioned, we have that 
\[ \sup \{|\omega_{\gamma_n}(U^*\delta(U))| |U \in u\} > \frac{1}{4}\alpha. \] Thus we can find for every \( n \), a unitary \( V_n \in u \) such that 
\[ \left| \left(V_n^*\delta(V_n)^* y_n, y_n\right) \right| > \frac{1}{4}\alpha. \] Let \( A = \sum_{n=1}^{\infty} V_n P_n \), then \( A \in d \) and 
\[ A^*\delta(A) y_n, y_n = \left| \left(P_n A^*\delta(A) P_n y_n, y_n\right) \right| 
= \left| \left(P_n (A^*AT - ATA) P_n y_n, y_n\right) \right| 
= \left| \left(P_n V_n^*\delta(V_n) P_n y_n, y_n\right) \right| 
= \left| \left(V_n^*\delta(V_n) y_n, y_n\right) \right| 
= \frac{1}{4}\alpha \] for all \( n \). Therefore \( \|\delta(A) y_n\| \to 0 \). But because of (\Pi), we have \( \delta(A) \notin f(B) \), which completes the proof.

5. The Property of Infinite W* Sub-Algebra

**Lemma (12).** Let \( 0 < b \in Z(M) \), \( s(b) = 1; e^a_z(0, \infty) \) be a properly infinite projection and 
\( c\left(e^a_z(0, \infty)\right) = 1. \) Let projection \( q \in P(M) \) be finite or properly infinite, \( c(q) = 1 \) and 
\( q \ll e^a_z(0, \infty). \) Let \( \mathbb{R} \ni \mu_n \downarrow 0. \) For every \( n \in \mathbb{N} \) we denote by \( z_n \) such a projection that \( 1 - z_n \) is the largest central projection, for which \( (1 - z_n) q \geq (1 - z_n) e^a_z(\mu_n b, +\infty) \) holds. We have \( z_n \uparrow_n 1 \) and for 
\[ d := \left\lfloor \mu_1 z_1 + \sum_{n=1}^{\infty} \mu_{n+1}(z_{n+1} - z_n) \right\rfloor b \] the following relations hold: \( q \ll e^a_z(d, +\infty), \) \( 0 < d \leq \mu b \) and \( s(d) = 1. \) Moreover, if all projections \( e^a_z(\mu_n b, +\infty), n \geq 1 \) are finite then \( e^a_z(d, +\infty) \) is a finite projection as well.

**Proof.** Since \( e^a_z(\mu_{n+1} b, +\infty) \geq e^a_z(\mu_n b, +\infty) \) we have
\(e_z^a(1-z_{n+1})q \geq (1-z_{n+1})e_z^a(\mu_{n+1}b, +\infty) \geq (1-z_{n+1})e_z^a(\mu_nb, +\infty)\). Hence, \(z_{n+1} \geq z_n\) for every \(n \in \mathbb{N}\).

In addition, \(e_z^a(\mu_nb, +\infty) \uparrow_n e_z^a(0, +\infty)\) and \(e_z^a(0, +\infty)\) is properly infinite projection. Hence, in the case when \(q\) is finite projection, it follows that \(z_n \uparrow_n 1\). Let us consider the case when \(q\) is a properly infinite projection with \(c(q) = 1\) and such that \(q \prec \prec e_z^a(0, \infty)\). In this case, with \(p = q, q = e_z^a(0, +\infty), q_n = e_z^a(\mu_nb, +\infty)\) and deduce \(\sqrt[n]{\sum_{\infty} z_n} \geq c(q) = 1\).

All other statements follow from the form of element \(d\). Since, \(z_id = \mu_iz_ib, (z_{n+1} - z_n) = \mu_{n+1}(z_{n+1} - z_n)b\) and \(z_nq \prec \prec z_ne_z^a(\mu_nb, +\infty)\) for every \(n \in \mathbb{N}\). Observe also that \(s(d) = s(b)(z_i + \sum_{\infty} (z_{n+1} - z_n)) = 1\).

Finally, let all projections \(e_z^a(\mu_nb, +\infty), n \geq 1\) be finite. Since \(dz_i = \mu_ib(d(z_{n+1} - z_n) = \mu_{n+1}b(z_{n+1} - z_n)\), we have

\[e_z^a(d, +\infty)z_i = e_z^a(\mu_nb, +\infty)z_i,\]

\[e_z^a(d, +\infty)(z_{n+1} - z_n) = e_z^a(\mu_nb, +\infty)(z_{n+1} - z_n)\]

for every \(n \in \mathbb{N}\). There projections standing on the right-hand sides are finite. Hence, \(e_z^a(d, +\infty)\) is finite projection as a sum of the left-hand sides [22].

We shall use a following well-known implication

\[p \prec q \Rightarrow zp \prec zq, \quad \forall z \in P(Z(M))\), \(0 < z \leq c(p) \vee c(q)\).

We supply here a straightforward argument. Let \(z' \in z \in Z(M)\) be such that \(0 < z' \leq c(pz) \vee c(qz) z(c(p) \vee c(q))\). Then \(z' \leq c(p) \vee c(q)\) and therefore

\[z'(zp) = z'p \prec z'q = z'(zq).\] This means \(zp \prec zq\).

As in [6] we can use Theorem (6) to extend the result to the properly infinite case.

**Theorem (13).** Let \(A\) be a properly infinite \(W^*\) sub-algebra of \(B\) containing the center \(\ell\) of \(B\). For every derivation \(\delta : A \to f(B)\) there is \(aT \in f(B)\) such that \(\delta = aA T\).

Before we start the proof let us recall that if \(A\) is properly infinite there is an infinite countable decomposition of the identity into mutually orthogonal projections of \(A\), all
equivalent in $A$ to $I$, and thus a fortify equivalent in $B$ to $1$ [8]. Therefore there is a spatial isomorphism

$$\phi : B \to \tilde{B} = B \otimes B(H_0)$$

with $H_0 = l^{n+1}(\mathbb{Z})$ and

$$\phi(A) = \tilde{A} = A \otimes B(H_0)$$

[5]. Recall also that the elements $B$ of $\tilde{B}$ (or $\tilde{A}$) are represented by bounded matrices $[B_{ij}], i, j \in \mathbb{Z}$ with entries in $B$ (or $A$) by the formula

$$(I \otimes E_j)T(z \otimes E_d) = T_{jk} \otimes E_d$$

where $E_j$ is the canonical matrix unit of $B(H_0)$. In particular if $\ell$, $\varphi$ are the maximal a commutative operation subalgebras of $B(H_0)$ of Laurent (resp. diagonal) matrices, then $B \in B \otimes \ell$ (resp. $B \in B \otimes \varphi$) iff $[B_{ij}]$ is a Laurent matrix with entries in $B$, i.e., $B_{ij} = B_{i-j}$, where $B_{ik}$ denotes the entry along the $k$th diagonal (resp. $B_{ij} = \delta_{ij} B_{nk}$) for all $i, j \in \mathbb{Z}$.

**Proof.** Let $\tilde{\delta} = \phi \circ \delta \circ \phi^{-1}$ then

$$\tilde{\delta} : \tilde{A} \to \phi(f(B)) = f(\tilde{B})$$

is a relative compact derivation. Let us define the following $\mathbb{W}^*$ algebras:

$$\tilde{\ell} = \tilde{B} \cap \tilde{B}, \quad \tilde{A}_n = \ell \otimes \ell, \quad A_n = \phi^{-1}(A_n), \quad \tilde{A}_{n+1} = A \otimes \ell, \quad \text{and} \quad \tilde{A}_{n+2} = A_n \otimes \varphi.$$ First, let us notice that

$$\tilde{A}_n \cap f(\tilde{B}) = (\ell \otimes \ell) \cap (B \otimes B(H_0)) \cap f(\tilde{B})$$

$$= (B \otimes \ell) \cap f(\tilde{B})$$

$$= \{0\}$$

by [22]. Therefore

$$A_n' \cap f(B) = \phi^{-1}(\tilde{A}_n') \cap f(B) = \phi^{-1}(\tilde{A}_n' \cap f(\tilde{B})) = \{0\}$$

because $\phi$ is spatial Now

$$\tilde{\ell} = (B \otimes B(H_0)) \cap (B' \otimes I)$$

$$= \ell \otimes I \subset \tilde{A}_n \subset \tilde{A}.$$
Thus we can apply Theorem(6) to the derivation \( \tilde{\mathcal{D}} \) restricted to the commutative operation sub-algebra \( \tilde{A}_n \) of \( \tilde{B} \) and we obtain a \( T_n \in f(\tilde{B}) \) such that \( \tilde{\mathcal{D}} = \tilde{\mathcal{D}} - aAT_n \) vanishes on \( \tilde{A}_n \).

Now

\[
\tilde{A}_{n+1} \subset B \otimes \ell \subset \ell' \otimes \ell = \tilde{A}_n'.
\]

Therefore, for all \( A_n \in \tilde{A}_n \) and \( A_{n+1} \in \tilde{A}_{n+1} \) we have

\[
\tilde{\mathcal{D}}_n(A_nA_{n+1}) = A_n \tilde{\mathcal{D}}_n(A_{n+1}) = \tilde{\mathcal{D}}_n(A_{n+1}A_n) = \tilde{\mathcal{D}}_n(A_{n+1})A_n
\]
i.e., \( \tilde{\mathcal{D}}_n(A_{n+1}) \) and \( A_n \) commute and hence

\[
\tilde{\mathcal{D}}_n(\tilde{A}_{n+1}) \subset \tilde{A}_n' \cap f(\tilde{B}) = \{0\}
\]

Thus \( \tilde{\mathcal{D}}_n \) also vanishes on \( \tilde{A}_{n+1} \). Now \( \tilde{A}_n \) is a commutative operation and hence so are \( A_n \) and \( \tilde{A}_{n+2} \). Moreover,

\[
\tilde{\ell} \subset \tilde{A}_n \subset \tilde{A} \subset \tilde{B}
\]

Implies

\[
\ell = \phi^{-1}(\tilde{\ell}) \subset A_n \subset A \subset B
\]
and hence

\[
\tilde{\ell} = \ell \otimes I \subset A_n \otimes I \subset \tilde{A}_{n+2} \subset \tilde{A} \subset \tilde{B}
\]

Thus we can apply again Theorem(6) to the relative compact derivation \( \tilde{\mathcal{D}}_n \) restricted to \( \tilde{A}_{n+2} \).

Let \( T_{n+1} \in f(\tilde{B}) \) be such that \( \tilde{\mathcal{D}}_n \) agrees with ad \( T_{n+1} \) on \( \tilde{A}_{n+2} \). Since

\[
A_n \otimes I \subset A \otimes I \subset A \otimes \ell = \tilde{A}_{n+1}
\]

and \( \tilde{\mathcal{D}}_n \) vanishes on \( \tilde{A}_{n+1} \), we see that ad \( T_{n+1} \) vanishes on \( A_n \otimes I \), i.e.,

\[
T_{n+1} \in (A_n \otimes I)' \cap f(\tilde{B}) = (A_n' \otimes B(H_0)) \cap f(\tilde{B})
\]

Then for all \( i, j \in \mathbb{Z}, (T_{n+1})_{ij} \in A_n' \) and

\[
(T_2)_{ij} \otimes E_{nn} = (I \otimes E_{mi})T_{n+1}(I \otimes E_{jm}) \in f(\tilde{B})
\]
whence by Lemma(12)(a) \( (T_{n+1})_g \in f(B) \). But we saw that \( d'_f \cap f(B) = \{0\} \), hence \( (T_{n+1})_g = 0 \) for all \( i, j \in \mathbb{J} \), so \( T_{n+1} = 0 \). Therefore \( \delta_n \) vanishes also on \( \tilde{A}_{n+1} \) and hence on \( I \otimes \phi \). Now \( \ell \) and \( \phi \) generate \( B(H_0) \), whence \( \tilde{A}_{n+1} = A \otimes \ell \) and \( I \otimes \phi \) generate \( \tilde{A} \). Thus by the \( \sigma \)-weak continuity of \( \delta_n \) (see [6]) we see that

\[
\delta_n = \delta - aAT_n = 0, \text{ i.e., } \delta = aAT_n. \text{ Clearly } \delta = ad \phi^{-1}(T_n) \text{ and } \phi^{-1}(T_n) \in A(B).
\]

Let us assume in this part that \( B \) is semi-finite and let \( \tau \) be a fsn trace on it. Beside the closed ideal \( f(B) \) we can also consider the (non closed) two-sided norm-ideals

\( C_{1+\varepsilon}(B,\tau) \) for \( 1 \leq 1+\varepsilon < \infty \) defined by

\[
C_{1+\varepsilon}(B,\tau) = \left\{ B \in B \mid \tau\left|B|^{1+\varepsilon}\right| < \infty \right\}
\]

\[
\|B\|_{1+\varepsilon} = \tau\left|B|^{1+\varepsilon}\right|^{\frac{1}{1+\varepsilon}} \text{ for } B \in C_{1+\varepsilon}(B,\tau).
\]

Obviously,

\[
C_{1+\varepsilon}(B,\tau) = B \cap L^{1+\varepsilon}(B,\tau),
\]

where the latter is the non commutative \( L^{1+\varepsilon} \)-space of \( B \) relative to \( \tau \) (see [14]).

Recall the following facts about \( L^{1+\varepsilon}(M) \) spaces in the case of a general \( W^* \) algebra \( M \) and \( 1 \leq 1+\varepsilon < \infty \) (\( L^\varepsilon(M) \) is identified with \( M \)): \( L^{1+\varepsilon}(M) \) is a Banach space, its dual is isomorphic to \( L^{\frac{\varepsilon}{1+\varepsilon}}(M) \) (with \( \frac{\varepsilon}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon} = 1 \)), and the duality is established by the functional \( \text{tr} \) on \( L^\varepsilon(M) \),

where if \( A \in L^{1+\varepsilon}(M) \), \( B \in L^{\varepsilon}(M) \) we have \( AB, BA \in L^\varepsilon(M) \) and

\[
\text{tr}(AB) = \text{tr}(BA), \quad \|\text{tr}(AB)\| \leq \|A\|_{1+\varepsilon} \|B\|_{\varepsilon},
\]

\[
\|A\|_{1+\varepsilon} = \left(\text{tr}|A|^{1+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} = \max \left\{ |\text{tr}AB| \mid B \in L^{\varepsilon}(M), \|B\|_{\varepsilon} \leq 1 \right\}
\]

(see [14]). Of course, if \( M = B \) we can identify \( L^{1+\varepsilon}(M) \) with \( L^{1+\varepsilon}(B,\tau) \) and \( \text{tr} \) with \( \tau \). The following inequality will be used here only in the semi-finite case and in the context of \( C_{1+\varepsilon} \)-ideals, but since the same proof holds for \( L^{1+\varepsilon} \)-spaces, we shall consider the general case.
Corollary (14). Let \( M \) be a \( \mathcal{W}^* \) algebra, \( \varepsilon \geq 0, A \in L^{1+\varepsilon}(M) \) and
\( Q_n, Q_{n+1} \in p(M), Q_nQ_{n+1} = 0, Q_n + Q_{n+1} = 1 \). Then
\[
\|A\|_{1+\varepsilon} \geq \|Q_nAQ_n\|_{1+\varepsilon} + \|Q_{n+1}AQ_{n+1}\|_{1+\varepsilon}
\]

**Proof.** Let us first note that
\[
\sum_{i=n}^{n+1} Q_iA(Q_i)^* = \sum_{i=n}^{n+1} |Q_iA(Q_i)|_{1+\varepsilon}
\]
And
\[
\sum_{i=n}^{n+1} Q_iA(Q_i)^* \leq \sum_{i=n}^{n+1} |Q_iA(Q_i)|_{1+\varepsilon}
\]

Consider first \( 1 + \varepsilon = n \) and take the polar decomposition's
\[
Q_iA(Q_i) = U_i |Q_iA(Q_i)|, \quad i = n, n+1.
\]
Then \( U_i^* \) and \( U_i^*U_i \) are majorized by \( Q_i \) and hence \( U_i \) commutes with \( Q_i \). Therefore
\[
B = (U_n + U_{n+1})^* \text{ commutes with } Q_i \text{ and } \|B\| = 1.
\]
Then
\[
\|A\| \geq |\text{tr}AB|
\]
\[
= \left| \text{tr} \left( \sum_{i=n}^{n+1} Q_iBAQ_i \right) \right|
\]
\[
= \text{tr} \left( \sum_{i=n}^{n+1} Q_iA(Q_i) \right)
\]
\[
= \sum_{i=n}^{n+1} \|Q_iA(Q_i)\|_n.
\]

Consider now \( \varepsilon > 0 \). Let \( B \in L^{\varepsilon,1+\varepsilon}(M) \) be such that \( \|B\|_{1+\varepsilon} \leq 1 \) and
\[
\sum_{i=n}^{n+1} Q_iA(Q_i) \leq \text{tr} \left( \sum_{i=n}^{n+1} Q_iA(Q_i) \right) B.
\]
Take the polar decomposition's \( A = U|A| \) and \( B = V|B| \), then \( VU \) are in \( M \) and \(|A|, |B|\) are in \( L^{1+\varepsilon}(M), L^{1+\varepsilon}(M) \), respectively. Let
Then by standard arguments, it is easy to see that $f$ is analytic on $0 < \Re{z} < n$ and continuous and bounded on $0 \leq \Re{z} \leq n$. Then by the three-line theorem (see [4]) we have

$$f\left(\frac{1}{1+\varepsilon}\right) \leq \text{Max}_{t \in \mathbb{R}} f(it)^{\varepsilon_{\mathcal{Q}}} \text{Max}_{t \in \mathbb{R}} f(1+it)^{\varepsilon_{\mathcal{Q}}}$$

Now $f\left(\frac{1}{1+\varepsilon}\right) = \left\|\sum_{i=n}^{n+1} Q_i A Q_i\right\|_{1+\varepsilon}$ and by Holder’s inequality

$$|f(it)| = \text{tr}\left(\sum_{j=n}^{n+1} Q_j U |A|^{1+\varepsilon} |Q_j V| |B|^{\left(\frac{\varepsilon}{1+\varepsilon}\right)} |B|^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)}\right)$$

$$\leq \left\|\sum_{j=n}^{n+1} Q_j U |A|^{1+\varepsilon} |Q_j V| |B|^{\left(\frac{\varepsilon}{1+\varepsilon}\right)} |B|^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)}\right\|$$

$$\leq \left(\text{Max}_{j} \left\|Q_j U |A|^{1+\varepsilon} |Q_j V| |B|^{\left(\frac{\varepsilon}{1+\varepsilon}\right)} |B|^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)}\right\| \left\|V|B|^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)} |B|^{\left(\frac{\varepsilon}{1+\varepsilon}\right)}\right\|\right)$$

$$\leq n.$$

Again by Holder’s inequality applied twice and by the result already obtained in the $\varepsilon = 0$ case,

$$|f(1+it)| = \text{tr}\left(\sum_{j=n}^{n+1} Q_j U |A|^{1+\varepsilon} |A|^{1+\varepsilon} |Q_j V| |B|^{\left(\frac{\varepsilon}{1+\varepsilon}\right)} |B|^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)}\right)$$

$$\leq \left\|\sum_{j=n}^{n+1} Q_j U |A|^{1+\varepsilon} |A|^{1+\varepsilon} |Q_j V| |B|^{\left(\frac{\varepsilon}{1+\varepsilon}\right)} |B|^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)}\right\|$$

$$\leq \left\|U |A|^{1+\varepsilon} |A|^{1+\varepsilon}\right\|$$

$$\leq \left\|U |A|^{1+\varepsilon}\right\| \left\| |A|^{1+\varepsilon}\right\|$$

$$\leq \left\|A\right\|^{1+\varepsilon}$$

Thus $f\left(\frac{1}{1+\varepsilon}\right) \leq \left\|A\right\|^{1+\varepsilon}$ whence by the second equality in this proof,

$$\left\|A\right\|^{1+\varepsilon} \geq \sum_{i=n}^{n+1} Q_i A Q_i \right\|^{1+\varepsilon} = \sum_{i=n}^{n+1} \left\|Q_i A Q_i\right\|^{1+\varepsilon}$$

Data Availability

No data were used to support this study.
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