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ON q-MOCANU TYPE FUNCTIONS ASSOCIATED WITH q-RUSCHEWEYH DERIVATIVE OPERATOR

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Abstract. In this paper, we introduce certain subclasses of analytic functions defined by using the q-difference operator. Mainly we give several inclusion results for defined classes. Also, certain applications due to q-Ruscheweyh derivative operator will be discussed.

1. Introduction

Let $\mathbf{A}$ denotes the class of analytic functions $f(z)$ in the open unit disk $E = \{z : |z| < 1\}$ such that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Subordination of two functions f and g is denoted by $f \prec g$ and defined as $f(z) = g(w(z))$, where $w(z)$ is Schwartz function in E (see [10]). Let S, S^* and C denote the subclasses of $\mathbf{A}$ of univalent functions, starlike functions and convex functions respectively. Mocanu [11] introduced the class $M(\alpha)$ of $\alpha-$convex functions $f \in S$ satisfies;

$$\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right) \prec \frac{1+z}{1-z},$$

where $\alpha \in [0,1]$, $\frac{f(z)}{z}f'(z) \neq 0$ and $z \in E$. We see that $M_0 = S^*$ and $M_1 = C$. This class is vastly studied by several authors, see [2, 14].

We recall here some basic definitions and concept details of q-calculus that are used in this paper.

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The q-difference operator, which was introduced by Jackson [7], defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}; \quad q \neq 1, \quad z \neq 0,$$

for $q \in (0,1)$. It is clear that $\lim_{q\to 1^-} D_q f(z) = f'(z)$, where $f'(z)$ is the ordinary derivative of the function. For more properties of D_q; see [3–5, 9, 18].

It can easily be seen that, for $n \in \mathbb{N} = \{1, 2, 3, ..\}$ and $z \in E$,

$$D_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n]_q \, z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots \, .$$

We have the following rules of D_q.

$$D_q \left(af(z) \pm bg(z) \right) = aD_q f(z) \pm bD_q g(z) \, .$$

$$D_q \left(f(z) \, g(z) \right) = f(qz) \, D_q \left(g(z) \right) + g(z) D_q \left(f(z) \right).$$

$$D_q \left(\frac{f(z)}{g(z)} \right) = \frac{D_q \left(f(z) \right) g(z) - f(z) D_q \left(g(z) \right)}{g(qz) g(z)}, \quad g(qz)g(z) \neq 0.$$

$$D_q \left(\log f(z) \right) = \frac{D_q \left(f(z) \right)}{f(z)}.$$

Some properties related with function theory involving q-theory were first introduced by Ismail et al. [6]. Moreover, several authors studied in this matter such as [1, 12, 13, 15].

Now, by making use of the principle of subordination together with q-difference operator, we have the following classes:

Let a function $p \in \mathbf{A}$ with $p(0) = 1$ is in the class $\widetilde{P}_q(\beta)$ if and only if

$$p(z) \prec p_{q,\beta}(z), \quad \textit{where } p_{q,\beta}(z) = \left(\frac{1+z}{1-qz} \right)^{\beta}, \quad (0 < \beta \leq 1) \, . \tag{1.2}$$

It is very easy to see that $p_{q,\beta}(z)$ is convex univalent in E for $0 < \beta \leq 1$. Aslo, $p_{q,\beta}(z)$ is symmetric with respect to the real axis, that is,

$$0 < \Re \left(p_{q,\beta}(z) \right) < \left(\frac{1}{1-q} \right)^{\beta}.$$

Definition 1.1. *Let function $f \in \mathbf{A}$ and $0 \le \alpha \le 1$, $q \in (0,1)$. Then $f \in M_q^{\beta}(\alpha)$ if and only if*

$$J_q(\alpha, f) \in \widetilde{P}_q(\beta),$$

where

$$J_q(\alpha, f) = (1-\alpha)\frac{zD_q f}{f} + \alpha\frac{D_q(zD_q f)}{D_q f}.$$

Moreover, let us denote

$$M_q^{\beta}(0) = S_q^*(\beta), \qquad M_q^{\beta}(1) = C_q(\beta).$$

A function $f \in \mathbf{A}$ is said to be in $S_q^*(\beta)$ and $C_q(\beta)$ if and only if

$$\frac{zD_q f(z)}{f(z)} \prec p_{q,\beta}(z) \text{ and } \frac{D_q(zD_q f(z))}{D_q f(z)} \prec p_{q,\beta}(z),$$

respectively.

Special cases:

(i) If $q \to 1^-$, then the class $M_q^{\beta}(\alpha)$ reduces to the class $M^{\beta}(\alpha)$.

(ii) If $q \to 1^-$ and $\beta = 1$, then the class $M_q^{\beta}(\alpha)$ reduces to the class $M(\alpha)$ introduced by Mocanu [11].

(iii) If $q \to 1^-$, $\alpha = 0$ and $\beta = 1$, then the class $M_q^{\beta}(\alpha)$ reduces to the well known class S^* of starlike functions.

(iv) If $q \to 1^-$, $\alpha = 1$ and $\beta = 1$, then the class $M_q^{\beta}(\alpha)$ reduces to the well known class C of convex functions.

The authors in [8], introduced an operator $R_q^{\lambda} : \mathbf{A} \to \mathbf{A}$ defined as:

$$\begin{aligned} R_q^{\lambda} f(z) &= F_{\lambda+1,q}(z) * f(z) && (1.3) \\ &= z + \sum_{n=2}^{\infty} \frac{[n+\lambda-1]_q!}{[\lambda]_q!\,[n-1]_q!} a_n z^n, && (1.4) \end{aligned}$$

where $f \in \mathbf{A}$, $F_{\lambda+1,q}(z) = z + \sum_{n=2}^{\infty} \frac{[n+\lambda-1]_q!}{[\lambda]_q![n-1]_q!} z^n$ and $*$ denotes convolution.

This series (1.4) is absolutely convergent in E. For $q \to 1^-$, we have the operator R^{λ}, called Ruscheweyh derivative operator introduced in [16].

In this case

$$\begin{aligned} R^{\lambda} f(z) &= \lim_{q \to 1^-} R_q^{\lambda} f(z) = z + \sum_{n=2}^{\infty} \frac{(n+\lambda-1)!}{\lambda!\,(n-1)!} a_n z^n \\ &= \frac{z}{(1-z)^{\lambda+1}} * f(z). \end{aligned}$$

We note that $R_q^0 f(z) = f(z)$ and $R_q^1 f(z) = zD_q f(z)$. Also

$$R_q^n f(z) = \frac{zD_q^n\left(z^{n-1} f(z)\right)}{[n]_q!}; \; n \in \mathbb{N} = \{1, 2, 3, \ldots\}.$$

The following identity can be easily obtained from (1.4)

$$zD_q\left(R_q^{\lambda}f(z)\right)=\left(1+\frac{[\lambda]_q}{q^{\lambda}}\right)R_q^{\lambda+1}f(z)-\frac{[\lambda]_q}{q^{\lambda}}R_q^{\lambda}f(z). \tag{1.5}$$

Now, we define

Definition 1.2. *Let $f\in\mathbf{A}$ and $n\in\mathbb{N}$, $0\leq\alpha\leq 1$, $q\in(0,1)$ and $\beta\in(0,1]$. Then*

$$f\in M_q^{\beta}(n,\alpha)\ \textit{if and only if}\ R_q^n f(z)\in M_q^{\beta}(\alpha).$$

Moreover, let us denote

$$M_q^{\beta}(n,0)=S_q^*(n,\beta)\ \textit{and}\ M_q^{\beta}(n,1)=C_q(n,\beta).$$

Note that

$$f\in C_q(n,\beta)\Leftrightarrow zD_qf\in S_q^*(n,\beta). \tag{1.6}$$

2. Main Results

We need the following basic result to prove our main results:

Lemma 2.1. *[17] Let β and γ be complex numbers with $\beta\neq 0$ and let $h(z)$ be analytic in E with $h(0)=1$ and $Re\{\beta h(z)+\gamma\}>0$. If $p(z)=1+p_1z+p_2z^2+...$ is analytic in E, then*

$$p(z)+\frac{zD_qp(z)}{\beta p(z)+\gamma}\prec h(z)$$

implies that $p(z)\prec h(z)$.

Theorem 2.1. *Let $0\leq\alpha\leq 1$, $\beta\in(0,1]$ and $q\in(0,1)$. Then*

$$M_q^{\beta}(\alpha)\subset S_q^*(\beta).$$

Proof. Let $f\in M_q^{\beta}(\alpha)$ and let

$$\frac{zD_qf(z)}{f(z)}=p(z). \tag{2.1}$$

We note that $p(z)$ is analytic in E with $p(0)=1$.

The q-logarithmic differentiation of (2.1) yields

$$\frac{D_q\left(zD_q\left(f(z)\right)\right)}{D_qf(z)}-\frac{D_q\left(f(z)\right)}{f(z)}=\frac{D_qp(z)}{p(z)}.$$

Equivalently

$$\frac{D_q\left(zD_q\left(f(z)\right)\right)}{D_qf(z)}=p(z)+\frac{zD_qp(z)}{p(z)}.$$

Since $f\in M_q^{\beta}(\alpha)$, so we get

$$J_q(\alpha,f)=p(z)+\alpha\frac{zD_qp(z)}{p(z)}\prec p_{q,\beta}(z). \tag{2.2}$$

Since $Re\left\{\frac{1}{\alpha}p_{q,\beta}(z)\right\}>0$ in E, so by (2.2) together with Lemma 2.1, we obtain $p(z)\prec p_{q,\beta}(z)$. Consequently $f\in S_q^*(\beta)$. □

Corollary 2.1. *For $q\to 1^-$, we have $M^\beta(\alpha)\subset S^*(\beta)$. Furthermore, for $\beta=1$, $M(\alpha)\subset S^*$.*

Corollary 2.2. *For $q\to 1^-$, $\alpha=1$ and $\beta=1$, we have well known fundamental result $C\subset S^*$.*

Theorem 2.2. *Let $\alpha>1$, $\beta\in(0,1]$ and $q\in(0,1)$. Then*

$$M_q^\beta(\alpha)\subset C_q(\beta).$$

Proof. Let $f\in M_q^\beta(\alpha)$. Then, by Definition 1.1,

$$(1-\alpha)\frac{zD_qf(z)}{f(z)}+\alpha\frac{D_q(zD_qf(z))}{D_qf(z)}=p_1(z)\in\widetilde{P}_q(\beta).$$

Now,

$$\begin{aligned}\alpha\frac{D_q(zD_qf(z))}{D_qf(z)} &= (1-\alpha)\frac{zD_qf(z)}{f(z)}+\alpha\frac{D_q(zD_qf(z))}{D_qf(z)}+(\alpha-1)\frac{zD_qf(z)}{f(z)}\\ &= (\alpha-1)\frac{zD_qf(z)}{f(z)}+p_1(z).\end{aligned}$$

This implies

$$\begin{aligned}\frac{D_q(zD_qf)}{D_qf} &= \left(\frac{1}{\alpha}-1\right)\frac{zD_qf}{f}+\frac{1}{\alpha}p_1(z)\\ &= \left(\frac{1}{\alpha}-1\right)p_2(z)+\frac{1}{\alpha}p_1(z).\end{aligned}$$

Since $p_1,p_2\in\widetilde{P}_q(\beta)$ and is $\widetilde{P}_q(\beta)$ convex set, so $\frac{D_q(zD_qf)}{D_qf}\in\widetilde{P}_q(\beta)$. Hence, proof is complete. □

Theorem 2.3. *For $0\le\alpha_1<\alpha_2<1$*

$$M_q^\beta(\alpha_2)\subset M_q^\beta(\alpha_1).$$

Proof. For $\alpha_1=0$, this is obvious from Theorem 2.1.

Let $f\in M_q^\beta(\alpha_2)$. Then, by Definition 1.1,

$$(1-\alpha_2)\frac{zD_qf(z)}{f(z)}+\alpha_2\frac{D_q(zD_qf(z))}{D_qf(z)}=q_1(z)\in\widetilde{P}_q(\beta). \tag{2.3}$$

Now, we can easily write

$$J_q(\alpha_1,f(z))=\frac{\alpha_1}{\alpha_2}q_1(z)+\left(1-\frac{\alpha_1}{\alpha_2}\right)q_2(z), \tag{2.4}$$

where we have used (2.3) and $\frac{zD_qf(z)}{f(z)}=q_2(z)\in\widetilde{P}_q(\beta)$. Since $\widetilde{P}_q(\beta)$ is convex set, so (2.4) follows our required result. □

Remark 2.1. *If $\alpha_2 = 1$ and let $f \in M_q^\beta(1) = C_q(\beta)$. Then, from Theorem 2.3, we can write*

$$f \in M_q^\beta(\alpha_1), for\ 0 \leq \alpha_1 < 1,$$

Now, by making use of Theorem 2.1, we obtain $f \in S_q^*(\beta)$. Thus we have, $C_q(\beta) \subset S_q^*(\beta)$.

We develop some applications in terms of q-linear operator, which we call q-Ruscheweyh derivative operator, given by (1.3).

Theorem 2.4. *Let $0 \leq \alpha \leq 1$, $\beta \in (0,1]$, $n \in \mathbb{N}_0$ and $q \in (0,1)$. Then*

$$M_q^\beta(n+1,\alpha) \subset S_q^*(n+1,\beta).$$

Proof. One can easily prove this result by using similar arguments as used in Theorem 2.1 and letting

$$\frac{zD_q f_{n+1,q}(z)}{f_{n+1,q}(z)} = p(z) \ \left(for \ \ f_{n+1,q}(z) = R_q^{n+1} f(z)\right),$$

where $p(z)$ is analytic in E with $p(0) = 1$. □

Theorem 2.5. *Let $0 \leq \alpha \leq 1$, $\beta \in (0,1]$, $n \in \mathbb{N}_0$ and $q \in (0,1)$. Then*

$$S_q^*(n+1,\beta) \subset S_q^*(n,\beta).$$

Proof. Let $f \in S_q^*(n+1,\beta)$ and let $f_{n+1}(z) = R_q^{n+1} f(z)$. Then

$$\frac{zD_q f_{n+1,q}(z)}{f_{n+1,q}(z)} \prec p_{q,\beta}(z),$$

where $p_{q,\beta}(z)$ is given by (1.2).

Now, let

$$\frac{zD_q f_{n,q}(z)}{f_{n,q}(z)} = H(z), \tag{2.5}$$

where $H(z)$ is analytic in E with $H(0) = 1$. Using identity (1.5) and (2.5), we get

$$\frac{zD_q\left(f_{n,q}(z)\right)}{f_{n,q}(z)} = (1+N_q)\frac{f_{n+1,q}(z)}{f_{n,q}(z)} - N_q,$$

equivalently

$$(1+N_q)\frac{f_{n+1,q}(z)}{f_{n,q}(z)} = H(z) + N_q, \quad \left(for\ N_q = \frac{[n]_q}{q^n}\right).$$

The q-logarithmic differentiation yields,

$$\frac{zD_q\left(f_{n+1,q}(z)\right)}{f_{n+1,q}(z)} = p(z) + \frac{zD_q H(z)}{H(z)+N_q}. \tag{2.6}$$

Since $f \in S_q^*(n+1,\beta)$, So (2.6) implies

$$p(z) + \frac{zD_q H(z)}{H(z)+N_q} \prec p_{q,\beta}(z). \tag{2.7}$$

Since $Re\left\{p_{q,\beta}(z)+N_q\right\}>0$ in E, we use Lemma 2.1 along with (2.7), to get
$H(z)\prec p_{q,\beta}(z)$. Consequently, $f\in S_q^*(n,\beta)$. □

Theorem 2.6. *Let* $0\le\alpha\le 1$, $\beta\in(0,1]$, $n\in\mathbb{N}_0$ *and* $q\in(0,1)$. *Then*

$$C_q\left(n+1,\beta\right)\subset C_q\left(n,\beta\right).$$

Proof. Let

$$f\in C_q\left(n+1,\beta\right)$$

$$\Leftrightarrow zf'\in S_q^*\left(n+1,\beta\right) \qquad (by\ (1.6))$$

$$\Rightarrow zf'\in S_q^*\left(n,\beta\right) \qquad (by\ Theorem 2.5)$$

$$\Leftrightarrow f\in C_q\left(n,\beta\right). \qquad (by\ (1.6))$$

□

Remark 2.2. *From Theorem 2.4 and Theorem 2.5, we can extend the inclusions as following*

$$M_q^{\beta}(n+1,\alpha)\subset S_q^*\left(n+1,\beta\right)\subset S_q^*\left(n,\beta\right)\subset ...\subset S_q^*\left(\beta\right).$$

$$C_q\left(n+1,\beta\right)\subset C_q\left(n,\beta\right)\subset ...\subset C_q\left(\beta\right).$$

Theorem 2.7. *Let* $f\in\mathbf{A}$. *Then* $f\in M_q^{\beta}(n+1,\alpha)$, $\alpha\neq 0$, *if and only if there exists* $g\in S_q^*\left(n+1,\beta\right)$ *such that*

$$f(z)=\left[\frac{1}{\alpha}\right]_q\left[\int_0^t t^{\frac{1}{\alpha}-1}\left(\frac{g(t)}{t}\right)^{\frac{1}{\alpha}}d_qt\right]^{\alpha}. \tag{2.8}$$

Proof. Let $f\in M_q^{\beta}(n+1,\alpha)$. Then

$$J_q\left(\alpha,f\right)=(1-\alpha)\frac{zD_qf(z)}{f(z)}+\alpha\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}\in\widetilde{P}_q(\beta). \tag{2.9}$$

On some simple calculations of (2.8), we get

$$zD_qf(z)\left(f(z)\right)^{\frac{1}{\alpha}-1}=\left(g(z)\right)^{\frac{1}{\alpha}}. \tag{2.10}$$

The q-logarithmic differentiation of (2.10), gives

$$(1-\alpha)\frac{zD_qf(z)}{f(z)}+\alpha\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}=\frac{zD_qg(z)}{g(z)}. \tag{2.11}$$

From (2.9) and (2.11), we conclude our required result. □

Theorem 2.8. *Let $f \in \mathbf{A}$ and define, for $f \in M_q^{\beta}(n,\alpha)$,*

$$F_{c,q}(z) = \frac{[c+1]_q}{z^c}\int_0^z t^{b-1} f(t) d_q t. \tag{2.12}$$

Then $F_{c,q} \in S_q^*(n,\beta)$.

Proof. Let $f \in M_q^{\beta}(n,\alpha)$. If we set, for $F_{c,q}^n(z) = R_q^n(F_{c,q}(z))$

$$\frac{zD_q\left(F_{c,q}^n(z)\right)}{F_{c,q}^n(z)} = Q(z), \tag{2.13}$$

where $Q(z)$ is analytic in E with $Q(0) = 1$.

From (2.12), we can write

$$\frac{D_q\left(z^c F_{c,q}(z)\right)}{[c+1]_q} = z^{c-1} f(z).$$

Using product rule of the q-difference operator, we get

$$zD_q F_{c,q}(z) = \left(1 + \frac{[c]_q}{q^c}\right) f(z) - \frac{[c]_q}{q^c} F_{c,q}(z). \tag{2.14}$$

From (2.13), (2.14) and (1.3), we have

$$Q(z) = \left(1 + \frac{[c]_q}{q^c}\right)\frac{z\left(f_{n,q}(z)\right)}{F_{c,q}^n(z)} - \frac{[c]_q}{q^c},$$

where $F_{c,q}^n(z) = R_q^n(F_{c,q}(z))$ and $f_{n,q}(z) = R_q^n(f(z))$

On q-logarithmic differentiation, we get

$$\frac{zD_q\left(f_{n,q}(z)\right)}{f_{n,q}(z)} = Q(z) + \frac{zD_q Q(z)}{Q(z) + [N]_q}, \quad \left(for\ N_q = \frac{[c]_q}{q^c}\right). \tag{2.15}$$

Since $f \in M_q^{\beta}(n,\alpha) \subset S_q^*(n,\beta)$, so (2.15) implies

$$Q(z) + \frac{zD_q Q(z)}{Q(z) + [c]_q} \prec p_{q,\beta}(z).$$

Now, by applying Lemma 2.1, we conclude $Q(z) \prec p_{q,\beta}(z)$. Consequently, $\frac{zD_q\left(F_{c,q}^n(z)\right)}{F_{c,q}^n(z)} \prec p_{q,\beta}(z)$. Hence $F_{c,q} \in S_q^*(n,\beta)$. □

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References

[1] O. Altintas, N. Mustafa, Coefficient bounds and distortion theorems for the certain analytic functions, Turk. J. Math. 43 (2019), 985–997.

[2] J. Dziok, Classes of functions associated with bounded Mocanu variation, J. Inequal. Appl. (2013), Art. ID 349.

[3] H. Exton, q-Hypergeometric functions and applications, Ellis Horwood Limited, UK, 1983.

[4] G. Gasper, M. Rahman, Basic hypergeometric series, Cambridge University Press, Cambridge, UK, 1990.

[5] H.A. Ghany, q-derivative of basic hypergeomtric series with respect to parameters, Int. J. Math. Anal. 3 (2009), 1617–1632.

[6] M.E.H. Ismail, E. Merkes, D. Styer, A generalization of starlike functions, Complex Var., Theory Appl. 14 (1990), 77–84.

[7] F.H. Jackson, On q-functions and a certain difference operator, Trans. R. Soc. Edin. 46 (1908), 253–281.

[8] S. Kanas, R. Raducanu, Some classes of analytic functions related to Conic domains, Math. Slovaca. 64 (2014), 1183–1196.

[9] V. Koc, P. Cheung, Quantum Calculus, Springer, 2001.

[10] S.S. Miller, P. T. Mocanu, Differential subordinations theory and applications, Marcel Dekker, New York, Basel, 2000.

[11] P.T. Mocanu, Une propriete de convexite generlise dans la theorie de la representation conforme, Math. (Cluj). 11 (1969), 127–133.

[12] M. Naeem, S. Hussain, T. Mahmood, S. Khan, M. Darus, A new subclass of analytic functions defined by using Salagean q-differential operator, Mathematics. 7 (2019), 458.

[13] K.I. Noor, On generalized q-close-to-convexity, Appl. Math. Inf. Sci. 11 (2017), 1383–1388.

[14] K.I. Noor, S. Hussain, On certain analytic functions associated with Ruscheweyh derivatives and bounded Mocanu variation, J. Math. Anal. Appl. 340 (2008), 1145–1152.

[15] K.I. Noor, S. Riaz, Generalized q-starlike functions, Stud. Sci. Math. Hungerica. 54 (2017), 509–522.

[16] S. Ruscheweyh, New criteria for univalent functions. Proc. Amer. Math. Soc. 49 (1975), 109–115.

[17] H. Shamsan, S. Latha, On genralized bounded Mocanu variation related to q-derivative and conic regions, Ann. Pure Appl. Math. 17 (2018), 67–83.

[18] H.E.O. Ucar, Coefficient inequality for q-starlike functions, Appl. Math. Comput. 276 (2016), 122–126.