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APPLICATION OF SRIVASTAVA-ATTIYA OPERATOR TO THE GENERALIZATION OF MOCANU FUNCTIONS

KHALIDA INAYAT NOOR AND SHUJAAT ALI SHAH*

COMSATS University Islamabad, Pakistan

* *Corresponding author:* shahglike@yahoo.com

Abstract. In this paper we introduce certain subclasses of analytic functions by applying Srivastava-Attiya operator. Our main purpose is to derive inclusion results by using concept of conic domain and subordination techniques. We also deduce some new as well as well-known results from our investigations.

1. Introduction

Let χ denotes the class of analytic functions $f(z)$ in the open unit disk
$\mho = \{z : |z| < 1\}$ such that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Subordination of two functions f and g is denoted by $f \prec g$ and defined as $f(z) = g(w(z))$, where $w(z)$ is schwarz function in $\mho$. Let S, S^* and C denotes the subclasses of χ of univalent functions, starlike functions and convex functions respectively. For $0 \leq \delta < 1$, $S^*(\delta)$ and $C(\delta)$ are the subclasses of S of functions f satisfies;

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + (1 - 2\delta)\, z}{1 - z}, \quad (z \in \mho)\,, \tag{1.2}$$

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$$\frac{(zf'(z))'}{f'(z)} \prec \frac{1+(1-2\delta)z}{1-z}, \quad (z \in \mho), \tag{1.3}$$

respectively. Mocanu [13] introduced the class M_α of $\alpha-$convex functions $f \in S$ satisfies;

$$\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right) \prec \frac{1+z}{1-z}, \tag{1.4}$$

where $\alpha \in [0,1]$, $\frac{f(z)}{z}f'(z) \neq 0$. and $z \in \mho$. We see that $M_0 = S^*$ and $M_1 = C$. This class is vastly studied by several authors. See [4, 15, 17–19]. For $k \in [0,\infty)$, Kanas and Wisniowska [8, 9] introduced the classes $k-UCV$ of k-uniformly convex functions and $k-ST$ of k-starlike functions. The analytic conditions for these classes are given [6–9] as;

$$k-UCV = \left\{f \in \mathbf{S} : Re\left(1+\frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|\right\}, \quad (z \in \mho). \tag{1.5}$$

$$k-ST = \left\{f \in \mathbf{S} : Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)}-1\right|\right\}, \quad (z \in \mho). \tag{1.6}$$

We can rewrite the above relations easily as;

$$Re\,(p(z)) > k\,|p(z)-1|, \tag{1.7}$$

where $p(z) = 1+\frac{zf''(z)}{f'(z)}$ or $p(z) = \frac{zf'(z)}{f(z)}$. It is clear that $p(\mho)$ is conic domain defined as;

$$\Omega_k = \{w \in \mathbb{C} : Re\,(w) > k\,|w-1|\}, \tag{1.8}$$

or

$$\Omega_k = \left\{u+iv : u > k\sqrt{(u-1)^2+v^2}\right\}, \quad (0 \leq k < \infty). \tag{1.9}$$

These conic domains are being studied by several authors. See [2, 6, 14, 16]. Sokol and Nonukawa [23] introduced the class defined as;

$$MN = \left\{f \in S{:}Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf'(z)}{f(z)}-1\right|\right\}, \quad (z \in \mho). \tag{1.10}$$

It is obvius that $MN \subset C$. Recently S. Sivasubramanian et al. [22] extend the Sokol and Nonukawa's work in terms of conic domains. They introduced a new class $k-MN$ of functions $f \in S$ such that

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf'(z)}{f(z)}-1\right|, \quad (z \in \mho). \tag{1.11}$$

In motivation of the work [23], A. Rasheed et al. [21], introduced an interesting class $k-UM_\alpha$ $(0 \leq \alpha \leq 1)$ of functions $f \in S$ such that

$$Re\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{(zf'(z))'}{f'(z)}\right]>k\left|\frac{zf'(z)}{f(z)}-1\right|\ ,\quad (z\in\mho)\,. \tag{1.12}$$

Obviously, we can see $k-UM_1=k-MN$ and $1-UM_1=MN$.

We recall a Hurwitz-Lerch Zeta function $\Phi(s,b;z)$ [25] defined by

$$\Phi(s,a;z)=\sum_{n=2}^{\infty}\frac{z^n}{(n+b)^s}, \tag{1.13}$$

($b\ \epsilon\ \mathbb{C}\setminus\mathbb{Z}_0^-$; $s\ \epsilon\ \mathbb{C}$ when $|z|<1$; $Re(s)>1$ when $|z|=1$),

where $\mathbb{C}$ and $\mathbb{Z}_0^-$ denotes the set of complex numbers and the set of negative integers respectively.

Srivastava and Attiya [24] introduced the linear operator $J_{s,b}:\chi\to\chi$ defined in terms of the convolution (or Hadamard product), by

$$J_{s,b}f(z)=G_{s,b}(z)*f(z), \tag{1.14}$$

where

$$G_{s,b}(z)=(1+b)^s\left[\Phi(s,b;z)-b^s\right], \tag{1.15}$$

with $z\ \epsilon\ \mho$, $b\ \epsilon\ \mathbb{C}\setminus\mathbb{Z}_0^-$ and $s\ \epsilon\ \mathbb{C}$. Therefore, using (1.13) to (1.15), we have

$$J_{s,b}f(z)=z+\sum_{n=2}^{\infty}\left(\frac{1+b}{n+b}\right)^s a_nz^n, \tag{1.16}$$

where $z\ \epsilon\ \mho$, $b\ \epsilon\ \mathbb{C}\setminus\mathbb{Z}_0^-$ and $s\ \epsilon\ \mathbb{C}$.

The srivastava-Attiya operator generalizes the integral operators introduced by Alexandar [1], Libera [10], Bernardi [3] and Jung et al. [5].

In 2007, Raducanu and Srivastava [20] introduced and studied the class $S^*_{s,b}(\delta)$ of functions $f\ \epsilon\ \chi$ satisfies $J_{s,b}f(z)\ \epsilon\ S^*(\delta)$.

Now by using concepts of conic domains and Srivastava-Attiya integral operator, we introduce new classes as following.

Definition 1.1. *Let $k\in[0,\infty)$ and $\alpha,\beta\in[0,1]$. Then $f\in k-UM(\alpha,\beta)$ if and only if*

$$Re\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{(zf'(z))'}{f'(z)}\right]>k\left|(1-\beta)\frac{zf'(z)}{f(z)}+\beta\frac{(zf'(z))'}{f'(z)}-1\right|\ ,\quad (z\in\mho)\,.$$

Some of the special cases are given below and we refer to [8, 9, 21–23].

Special cases:

(i) For $\beta=0$, the class $k-UM(\alpha,\beta)$ reduces to the class $k-UM_\alpha$. See [21].

(ii) For $\alpha=1$ and $\beta=0$, the class $k-UM(\alpha,\beta)$ reduces to the class $k-MN$. See [22].

(iii) For $\alpha=1$, $\beta=0$ and $k=1$, the class $k-UM(\alpha,\beta)$ reduces to the class MN. See [23].

(iv) For $\alpha=1$ and $\beta=1$, the class $k-UM(\alpha,\beta)$ reduces to the class $k-UCV$. See [9].

(v) For $\alpha = 0$ and $\beta = 0$, the class $k - UM(\alpha, \beta)$ reduces to the class $k - ST$. See [8].

Definition 1.2. *Let* $\alpha, \beta \in [0,1]$, $k \in [0, \infty)$, $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$ *and* $s \in \mathbb{C}$. *Then* $f \in k - UM_b^s(\alpha, \beta)$ *if and only if* $J_{s,b}f(z) \in k - UM(\alpha, \beta)$.

Clearly, for $s = 0$ the classes $k - UM_b^s(\alpha, \beta)$ and $k - UM(\alpha, \beta)$ coincides.

2. Preliminaries

Lemma 2.1. *[12] Let* $\hbar$ *be an analytic function on* $\overline{\mho}$ *except for at most one pole on* $\partial\mho$ *and univalent on* $\overline{\mho}$, $\wp$ *be an analytic function in* $\mho$ *with* $\wp(0) = \hbar(0)$ *and* $\wp(z) \neq \wp(0)$, $z \in \mho$. *If* $\wp$ *is not subordinate to* $\hbar$, *then there exist points* $z_0 \in \mho$, $\xi_0 \in \partial\mho$ *and* $\varepsilon \geq 1$ *for which*

$$\wp\left(|z| < |z_0|\right) \subset \hbar\left(\mho\right), \quad \wp(z_0) = \hbar(\xi_0), \quad z_0\wp'(z_0) = \varepsilon\xi_0\wp'(\xi_0).$$

Lemma 2.2. *[6] If* $f \in S^*(\alpha)$ *for some* $\alpha \in \left[\frac{1}{2}, 0\right)$, *then*

$$Re\left(\frac{f(z)}{z}\right) > \frac{1}{3 - 2\alpha}.$$

Lemma 2.3. *[6] If* $Re\left(\sqrt{f'(z)}\right) > \alpha$ *for some* $\alpha \in \left[\frac{1}{2}, 0\right]$, *then*

$$Re\left(\frac{f(z)}{z}\right) > \frac{2\alpha^2 + 1}{3}.$$

3. Main Results

Theorem 3.1. *Let* $k \in [0, \infty)$ *and* $\alpha, \beta \in [0,1]$. *Also, let* p *be a function analytic in the unit disk such that* $p(0) = 1$. *If*

$$Re\left[p(z) + \alpha\frac{zp'(z)}{p(z)}\right] - k\left|p(z) - 1 + \beta\frac{zp'(z)}{p(z)}\right| > 0,$$

then

$$p(z) \prec \frac{1 + (1 - 2\gamma)z}{1 - z} := h(z),$$

where $\gamma = \gamma(k, \alpha, \beta)$ *is given by*

$$\gamma(k, \alpha, \beta) = \frac{1}{4}\left[\sqrt{\frac{\left(\alpha - 2k + \beta k\right)^2}{\left(1 + k\right)^2} + \frac{8\left(\alpha + \beta k\right)}{(1 + k)}} - \frac{(\alpha - 2k + \beta k)}{(1 + k)}\right]. \tag{3.1}$$

Proof. We may assume that $\gamma \geq \frac{1}{2}$ since the condition $Re\left(p(z) + \frac{zp'(z)}{p(z)}\right) > 0$ implies at least $Re\left(p(z)\right) > \frac{1}{2}$. (See [11]). Suppose now, on the contrary that $p \not\prec h$. Then, by Lemma 2.1, there exist $z_0 \in \mho$, $\xi_0 \in \partial\mho$ and $m \geq 1$ such that

$$p(z_0) = \gamma + ix,\ z_0 p'(z_0) = my,\ where\ y \leq -\frac{(1-\gamma)^2 + x^2}{2\,(1-\gamma)},\ (x, y \in \mathbb{R}).$$

Using these relations, we have

$$Re\left[p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)}\right] - k\left|p(z_0) - 1 + \beta \frac{z_0 p'(z_0)}{p(z_0)}\right| > 0,$$

or

$$0 < \ Re\left[p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)}\right] - k\left|p(z_0) - 1 + \beta \frac{z_0 p'(z_0)}{p(z_0)}\right|$$

$$= Re\left[\gamma + ix + \alpha \frac{my}{\gamma + ix}\right] - k\left|\gamma + ix - 1 + \beta \frac{my}{\gamma + ix}\right|$$

$$= \gamma + \frac{\alpha m y \gamma}{\gamma^2 + x^2} - k\left|\frac{(\gamma + ix)^2 - (\gamma + ix) + \beta m y}{\gamma + ix}\right|$$

$$\leq \gamma - \frac{\alpha\gamma}{2(1-\gamma)}\left(\frac{(1-\gamma)^2 + x^2}{\gamma^2 + x^2}\right) - k\frac{\sqrt{(X + Yx^2)^2 + Tx^2}}{\gamma^2 + x^2} = R(x),$$

where $X = \frac{(2\gamma+\beta)(1-\gamma)}{2}$, $Y = \frac{2(1-\gamma)+\beta}{2(1-\gamma)}$ and $T = (2\gamma - 1)^2$. The function $R(x)$ is even in regard of x. Now we have to show that $R(x)$ has maximum value at $x = 0$ when $\alpha, \beta \in [0, 1]$ and $\gamma \in \left[\frac{1}{2}, 1\right)$. We can easily check

$$R'(x) = -x\left[\frac{\alpha\gamma\,(2\gamma - 1)}{(1-\gamma)\,(\gamma^2 + x^2)} - k\left\{2Y(X + Yx^2) + T - \frac{2\sqrt{(X + Yx^2)^2 + Tx^2}}{\gamma^2 + x^2}\right\}\right].$$

Then $R'(x) = 0$, if and only if, $x = 0$. Since $\alpha, \beta \in [0, 1]$ and $\gamma \in \left[\frac{1}{2}, 1\right)$. So one can see

$$R''(x) = -\left[\frac{\alpha\,(2\gamma - 1)}{\gamma(1-\gamma)} - \frac{k}{2}\left\{(2(1-\gamma) + \beta)\,(2\gamma + \beta) + 2\,(2\gamma - 1)^2\right\}\right] < 0.$$

Thus $R(x)$ has maximum value at $x = 0$, that is

$$R(x) \leq R(0) = \gamma - \frac{\alpha\gamma(1-\gamma)}{2\gamma} - \frac{k(1-\gamma)\,(2\gamma + \beta)}{2\gamma} = 0,$$

for $\gamma = \gamma(k, \alpha, \beta)$ is given by (3.1), which contradicts the assumption. Hence

$$p(z) \prec \frac{1 + (1-2\gamma)z}{1 - z} := h(z),$$

where $\gamma = \gamma(k, \alpha, \beta)$ is given by (3.1). □

Theorem 3.2. *Let $\alpha, \beta \in [0, 1]$, $k \in [0, \infty)$, $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $s \in \mathbb{C}$. Then*

$$k - UM_b^s(\alpha, \beta) \subset S_{s,b}^*(\gamma),$$

where $\gamma = \gamma(k, \alpha, \beta)$ is given by (3.1).

Proof. Let $f \in k - UM_b^s(\alpha, \beta)$. Then, by Definition 1.2, $J_{s,b}f(z) \epsilon k - UM(\alpha, \beta)$, that is

$$Re\left[(1-\alpha)\frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)}+\alpha\frac{\left(z\left(J_{s,b}f(z)\right)'\right)'}{\left(J_{s,b}f(z)\right)'}\right] > k\left|(1-\beta)\frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)}+\beta\frac{\left(z\left(J_{s,b}f(z)\right)'\right)'}{\left(J_{s,b}f(z)\right)'}-1\right|, \quad (z \in \mho).$$

Putting $p(z) = \frac{z(J_{s,b}f(z))'}{J_{s,b}f(z)}$, we have

$$Re\left[p(z)+\alpha\frac{zp'(z)}{p(z)}\right]-k\left|p(z)-1+\beta\frac{zp'(z)}{p(z)}\right| > 0.$$

Our required result follows easily by using Theorem 3.1. □

When $s = 0$, then we have the following new result for class $k - UM(\alpha, \beta)$

Theorem 3.3. *Let $k \in [0, \infty)$ and $\alpha, \beta \in [0, 1]$. Then*

$$k - UM(\alpha, \beta) \subset S^*(\gamma),$$

where $\gamma = \gamma(k, \alpha, \beta)$ is given by (3.1).

The proof is straight forward by putting $p(z) = \frac{zf'(z)}{f(z)}$ and using Theorem 3.1.

When $k = 0$, then we have the following result for a class $0 - UM(\alpha, \beta) = M_\alpha$, introduced by Mocanu [13].

Corollary 3.1. *Let $f \in M_\alpha$. Then $f \in S^*(\gamma)$, where*

$$\gamma(\alpha) = \frac{-\alpha + \sqrt{\alpha^2 + 8\alpha}}{4}. \tag{3.2}$$

When $\beta = 0$, then we have the following result, proved in [21].

Corollary 3.2. *Let $f \in k - UM(\alpha, 0) = k - UM_\alpha$. Then $f \in S^*(\gamma)$, where*

$$\gamma(\alpha, k) = \frac{(2\vartheta - \eta) + \sqrt{(2\vartheta - \eta)^2 + 8\eta}}{4}, \tag{3.3}$$

where $\vartheta = \frac{k}{k+1}$, $\eta = \frac{\alpha+k}{k+1}$.

When $\alpha = 1$, $\beta = 0$, then we have the following result, proved in [22].

Corollary 3.3. *Let $f \in k - UM(1, 0) = k - MN$. Then $f \in S^*(\gamma)$, where*

$$\gamma(k) = \frac{1}{4}\left[\sqrt{\left(\frac{1-2k}{1+k}\right)^2 + \frac{8}{(1+k)}} - \left(\frac{1-2k}{1+k}\right)\right]. \tag{3.4}$$

When $\alpha = 1$, $\beta = 0$, and $k = 1$, then we have the following result, proved in [22].

Corollary 3.4. *Let $f \in 1 - UM(1, 0) = MN$. Then $f \in S^*(\gamma)$, where $\gamma \simeq 0.6403$.*

When $\alpha = \beta = 1$, then we have the following result, proved in [6].

Corollary 3.5. *Let $f \in k - UM(1,1) = k - UCV$. Then $f \in S^*(\gamma)$, where*

$$\gamma(k) = \frac{1}{4}\left[\sqrt{\left(\frac{1-k}{1+k}\right)^2 + 8} - \left(\frac{1-k}{1+k}\right)\right]. \tag{3.5}$$

Theorem 3.4. *Let $f \in k - UM_b^s(\alpha, \beta)$. Then*

$$\frac{J_{s,b}f(z)}{z} \prec \frac{1+(1-2\eta)z}{1-z},$$

where $\eta = \frac{1}{3-2\gamma}$ and $\gamma = \gamma(k, \alpha, \beta)$ is given by (3.1).

Proof. Let $f \in k - UM_b^s(\alpha, \beta)$. Then by Theorem 3.2 we have

$$\frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)} \prec \frac{1+(1-2\gamma)z}{1-z},$$

where $\gamma = \gamma(k, \alpha, \beta)$ is given by (3.1). Using Lemma 2.2, we get

$$\frac{J_{s,b}f(z)}{z} \prec \frac{1+(1-2\eta)z}{1-z},$$

where $\eta = \frac{1}{3-2\gamma}$. □

When $s = 0$, then one can prove the following result by using Theorem 3.3 together with Lemma 2.2.

Theorem 3.5. *Let $f \in k - UM(\alpha, \beta)$. Then*

$$\frac{f(z)}{z} \prec \frac{1+(1-2\eta)z}{1-z},$$

where $\eta = \frac{1}{3-2\gamma}$ and $\gamma = \gamma(k, \alpha, \beta)$ is given by (3.1).

When $\alpha = 1$, $\beta = 0$, then we have the following result, proved in [22].

Corollary 3.6. *Let $f \in k - UM(1,0) = k - MN$. Then*

$$\frac{f(z)}{z} \prec \frac{1+(1-2\eta)z}{1-z},$$

where $\eta = \frac{1}{3-2\gamma}$ and $\gamma = \gamma(k)$ is given by (3.4).

When $\alpha = 1$, $\beta = 0$ and $k = 1$, then we have the following result, proved in [22].

Corollary 3.7. *Let $f \in 1 - UM(1,0) = MN$. Then*

$$\frac{f(z)}{z} \prec \frac{1+(1-2\eta)z}{1-z}, \quad \text{where } \eta \simeq 0.58159.$$

When $\alpha = \beta = 1$, then we have the following result, proved in [6].

Corollary 3.8. *Let $f \in k - UM(1,1) = k - UCV$. Then*

$$\frac{f(z)}{z} \prec \frac{1+(1-2\eta)z}{1-z},$$

where $\eta = \frac{1}{3-2\gamma}$ and $\gamma = \gamma(k)$ is given by (3.5).

When $\alpha = \beta = k = 1$, then we have the following result, proved in [6].

Corollary 3.9. *Let* $f \in 1 - UM(1,1) = 1 - UCV$. *Then*

$$Re\left(\frac{f(z)}{z}\right) > 0.6289.$$

Theorem 3.6. *Let* $\alpha, \beta \in [0,1]$, $k \in [0,\infty)$, $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$ *and* $s \in \mathbb{C}$. *If*

$$Re\left[\sqrt{\left(J_{s,b}f(z)\right)'} + \alpha\frac{z\left(J_{s,b}f(z)\right)''}{2\left(J_{s,b}f(z)\right)'}\right] > k\left|\left(J_{s,b}f(z)\right)' + \beta\frac{z\left(J_{s,b}f(z)\right)''}{2\left(J_{s,b}f(z)\right)'} - 1\right|,$$

then

$$\sqrt{\left(J_{s,b}f(z)\right)'} \prec \frac{1+(1-2\gamma)z}{1-z} \Rightarrow \frac{J_{s,b}f(z)}{z} \prec \frac{1+(1-2\eta)z}{1-z}$$

where $\eta = \frac{2\gamma^2+1}{3}$ *and* $\gamma = \gamma(k,\alpha,\beta)$ *is given by (3.1).*

Proof. If we put $p(z) = \sqrt{\left(J_{s,b}f(z)\right)'}$, then

$$\frac{zp'(z)}{p(z)} = \frac{z\left(J_{s,b}f(z)\right)''}{2\left(J_{s,b}f(z)\right)'}.$$

The proof follows easily by using Theorem 3.1 along with Lemma 2.3. □

We can deduce the following result from Theorem 3.6 by choosing $s = 0$.

Theorem 3.7. *Let* $k \in [0,\infty)$ *and* $\alpha, \beta \in [0,1]$. *If*

$$Re\left[\sqrt{f'(z)} + \alpha\frac{zf''(z)}{2f'(z)}\right] > k\left|\sqrt{f'(z)} + \beta\frac{zf''(z)}{2f'(z)} - 1\right|,$$

then

$$\sqrt{f'(z)} \prec \frac{1+(1-2\gamma)z}{1-z} \Rightarrow \frac{f(z)}{z} \prec \frac{1+(1-2\eta)z}{1-z}$$

where $\eta = \frac{2\gamma^2+1}{3}$ *and* $\gamma = \gamma(k,\alpha,\beta)$ *is given by (3.1).*

When $\alpha = 1$, $\beta = 0$, then we have the following result, proved in [22].

Corollary 3.10. *If*

$$Re\left[\sqrt{f'(z)} + \frac{zf''(z)}{2f'(z)}\right] > k\left|\sqrt{f'(z)} - 1\right|,$$

then

$$Re\left(\sqrt{f'(z)}\right) > \gamma \Rightarrow Re\left(\frac{f(z)}{z}\right) > \eta,$$

where $\eta = \frac{2\gamma^2+1}{3}$ *and* $\gamma = \gamma(k)$ *is given by (3.4).*

When $k = 1$, then we have the following result, proved in [22].

Corollary 3.11. *If*

$$Re\left[\sqrt{f'(z)}+\frac{zf''(z)}{2f'(z)}\right]>\left|\sqrt{f'(z)}-1\right|,$$

then

$$Re\left(\sqrt{f'(z)}\right)>\gamma\simeq 0.64\Rightarrow Re\left(\frac{f(z)}{z}\right)>\eta\simeq 0.60.$$

For $k=0$, we have the following result, refer to [22].

Corollary 3.12. *If*

$$Re\left[\sqrt{f'(z)}+\frac{zf''(z)}{2f'(z)}\right]>0,$$

then

$$Re\left(\sqrt{f'(z)}\right)>\gamma\simeq 0.64\Rightarrow Re\left(\frac{f(z)}{z}\right)>\eta\simeq 0.60.$$

Theorem 3.8. *Let $\alpha,\beta\,\epsilon\,[0,1]$, $k\,\epsilon\,[0,\infty)$, $b\,\epsilon\,\mathbb{C}\setminus\mathbb{Z}_0^-$ and $s\,\epsilon\,\mathbb{C}$. If*

$$Re\left[\frac{J_{s,b}f(z)}{z}+\alpha\left(\frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)}-1\right)\right]>k\left|\frac{J_{s,b}f(z)}{z}+\beta\left(\frac{z\left(J_{s,b}f(z)\right)'}{J_{s,b}f(z)}-1\right)-1\right|,$$

then

$$\frac{J_{s,b}f(z)}{z}\prec\frac{1+(1-2\gamma)z}{1-z},$$

where $\gamma=\gamma(k,\alpha,\beta)$ is given by (3.1).

The proof follows easily by substituting $p(z)=\frac{J_{s,b}f(z)}{z}$ in Theorem 3.1.

For $s=0$, we can easily deduce the following result.

Theorem 3.9. *Let $k\in[0,\infty)$ and $\alpha,\beta\in[0,1]$. If*

$$Re\left[\frac{f(z)}{z}+\alpha\left(\frac{zf'(z)}{f(z)}-1\right)\right]>k\left|\frac{f(z)}{z}+\beta\left(\frac{zf'(z)}{f(z)}-1\right)-1\right|,$$

then

$$\frac{f(z)}{z}\prec\frac{1+(1-2\gamma)z}{1-z},$$

where $\gamma=\gamma(k,\alpha,\beta)$ is given by (3.1).

When $\alpha=1$, $\beta=0$, then we have the following result, proved in [22].

Corollary 3.13. *If*

$$Re\left[\frac{zf'(z)}{f(z)}+\frac{f(z)}{z}-1\right]>k\left|\frac{f(z)}{z}-1\right|\Rightarrow\frac{f(z)}{z}\prec\frac{1+(1-2\gamma)z}{1-z}$$

where $\gamma=\gamma(k)$ is given by (3.4).

When $\alpha=1$, $\beta=0$ and $k=1$, then we have the following result, proved in [22].

Corollary 3.14. *If*

$$Re\left[\frac{zf'(z)}{f(z)}+\frac{f(z)}{z}-1\right]>\left|\frac{f(z)}{z}-1\right|\Rightarrow Re\left(\frac{f(z)}{z}\right)>\gamma\simeq 0.64.$$

When $\alpha=1$, $\beta=0$ and $k=0$, then we have following result.

Corollary 3.15. *If*

$$Re\left[\frac{zf'(z)}{f(z)}+\frac{f(z)}{z}-1\right]>0\Rightarrow Re\left(\frac{f(z)}{z}\right)>\frac{1}{2}.$$

If we substitute $p(z)=(J_{s,b}f(z))'$ in Theorem 3.1, then we have the following result.

Theorem 3.10. *Let $\alpha,\beta\ \epsilon\ [0,1]$, $k\ \epsilon\ [0,\infty)$, $b\ \epsilon\ \mathbb{C}\setminus\mathbb{Z}_0^-$ and $s\ \epsilon\ \mathbb{C}$. If*

$$Re\left[(J_{s,b}f(z))'+\alpha\frac{z\,(J_{s,b}f(z))''}{(J_{s,b}f(z))'}\right]>k\left|(J_{s,b}f(z))'+\beta\frac{z\,(J_{s,b}f(z))''}{(J_{s,b}f(z))'}-1\right|,$$

then

$$Re\left((J_{s,b}f(z))'\right)>\gamma$$

where $\gamma=\gamma(k,\alpha,\beta)$ is given by (3.1).

For $s=0$, we have the following result.

Theorem 3.11. *Let $k\in[0,\infty)$ and $\alpha,\beta\in[0,1]$. If*

$$Re\left[f'(z)+\alpha\frac{zf''(z)}{f'(z)}\right]>k\left|f'(z)+\beta\frac{zf''(z)}{f'(z)}-1\right|,$$

then

$$Re\left(f'(z)\right)>\gamma$$

where $\gamma=\gamma(k,\alpha,\beta)$ is given by (3.1).

When $\alpha=1$, $\beta=0$, then we have the following result, proved in [22].

Corollary 3.16. *If*

$$Re\left[f'(z)+\frac{zf''(z)}{f'(z)}\right]>k\left|f'(z)-1\right|\Rightarrow Re\left(f'(z)\right)>\gamma\simeq 0.64.$$

When $\alpha=1$, $\beta=k=0$, then we have the following result.

Corollary 3.17. *If $Re\left[f'(z)+\frac{zf''(z)}{f'(z)}\right]>0$, then $Re\left(f'(z)\right)>\frac{1}{2}$.*

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