APPLICATION OF SRIVASTAVA-ATTIYA OPERATOR TO THE GENERALIZATION OF MOCANU FUNCTIONS

KHALIDA INAYAT NOOR AND SHUJAAT ALI SHAH*

COMSATS University Islamabad, Pakistan

*Corresponding author: shahglike@yahoo.com

ABSTRACT. In this paper we introduce certain subclasses of analytic functions by applying Srivastava-Attiya operator. Our main purpose is to derive inclusion results by using concept of conic domain and subordination techniques. We also deduce some new as well as well-known results from our investigations.

1. Introduction

Let \( \mathcal{U} \) denotes the class of analytic functions \( f(z) \) in the open unit disk \( \mathcal{U} = \{z : |z| < 1\} \) such that

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}
\]

Subordination of two functions \( f \) and \( g \) is denoted by \( f \prec g \) and defined as \( f(z) = g(w(z)) \), where \( w(z) \) is schwarz function in \( \mathcal{U} \). Let \( S, S^* \) and \( C \) denotes the subclasses of \( \mathcal{U} \) of univalent functions, starlike functions and convex functions respectively. For \( 0 \leq \delta < 1 \), \( S^*(\delta) \) and \( C(\delta) \) are the subclasses of \( S \) of functions \( f \) satisfies;

\[
\frac{zf'(z)}{f(z)} < \frac{1 + (1 - 2\delta) z}{1 - z}, \quad (z \in \mathcal{U}), \tag{1.2}
\]

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\[
\frac{(zf'(z))'}{f'(z)} < \frac{1 + (1 - 2\delta)z}{1 - z}, \quad (z \in \mathbb{U}),
\]
respectively. Mocanu [13] introduced the class \( M_\alpha \) of \( \alpha \)-convex functions \( f \in S \) satisfies:
\[
\left( (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} \right) < \frac{1 + z}{1 - z}, \quad (z \in \mathbb{U}), \quad (1.4)
\]
where \( \alpha \in [0, 1], \frac{zf'(z)}{f(z)} \neq 0 \) and \( z \in \mathbb{U} \). We see that \( M_0 = S^* \) and \( M_1 = C \). This class is vastly studied by several authors. See [4, 15, 17–19].

For \( k \in [0, \infty) \), Kanas and Wisniowska [8, 9] introduced the classes \( k-UCV \) of \( k \)-uniformly convex functions and \( k-ST \) of \( k \)-starlike functions. The analytic conditions for these classes are given [6–9] as:
\[
k-UCV = \left\{ f \in S : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > k \left| \frac{zf''(z)}{f'(z)} \right| \right\}, \quad (z \in \mathbb{U}).
\]
\[
k-ST = \left\{ f \in S : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\}, \quad (z \in \mathbb{U}).
\]
We can rewrite the above relations easily as;
\[
\text{Re} \left( p(z) \right) > k |p(z) - 1|,
\]
where \( p(z) = 1 + \frac{zf''(z)}{f'(z)} \) or \( p(z) = \frac{zf'(z)}{f(z)} \). It is clear that \( p(\mathbb{U}) \) is conic domain defined as;
\[
\Omega_k = \{ w \in \mathbb{C} : \text{Re} \left( w \right) > k |w - 1| \}, \quad (1.8)
\]
or
\[
\Omega_k = \left\{ u + iv : u > k \sqrt{(u - 1)^2 + v^2} \right\}, \quad (0 \leq k < \infty). \quad (1.9)
\]
These conic domains are being studied by several authors. See [2, 6, 14, 16]. Sokol and Nomukawa [23] introduced the class defined as:
\[
MN = \left\{ f \in S : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > k \left| \frac{zf''(z)}{f'(z)} \right| - 1 \right\}, \quad (z \in \mathbb{U}).
\]
It is obvious that \( MN \subset C \). Recently S. Sivasubramanian et al. [22] extend the Sokol and Nomukawa’s work in terms of conic domains. They introduced a new class \( k-MN \) of functions \( f \in S \) such that
\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad (z \in \mathbb{U}).
\]
In motivation of the work [23], A. Rasheed et al. [21], introduced an interesting class \( k-UM_\alpha \) \( (0 \leq \alpha \leq 1) \) of functions \( f \in S \) such that
\[
\text{Re} \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} \right)' \right] > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad (z \in \Omega). \tag{1.12}
\]

Obviously, we can see \(k - UM_1 = k - MN\) and \(1 - UM_1 = MN\).

We recall a Hurwitz-Lerch Zeta function \(\Phi(s, b; z)\) \[25\] defined by
\[
\Phi(s, a; z) = \sum_{n=2}^{\infty} \frac{z^n}{(n + b)^s}, \tag{1.13}
\]
where \(b \in \mathbb{C} \setminus \mathbb{Z}_0\) and \(s \in \mathbb{C}\) when \(|z| < 1\); \(\text{Re}(s) > 1\) when \(|z| = 1\),

Srivastava and Attiya \[24\] introduced the linear operator \(J_{s,b} : \chi \to \chi\) defined in terms of the convolution (or Hadamard product), by
\[
J_{s,b}f(z) = G_{s,b}(z) * f(z), \tag{1.14}
\]
where
\[
G_{s,b}(z) = (1 + b)^s \Phi(s, b; z) - b^s, \tag{1.15}
\]
with \(z \in \Omega, b \in \mathbb{C} \setminus \mathbb{Z}_0\) and \(s \in \mathbb{C}\). Therefore, using (1.13) to (1.15), we have
\[
J_{s,b}f(z) = z + \sum_{n=2}^{\infty} \left( \frac{1+b}{n+b} \right)^s a_n z^n, \tag{1.16}
\]
where \(z \in \Omega, b \in \mathbb{C} \setminus \mathbb{Z}_0\) and \(s \in \mathbb{C}\).

The Srivastava-Attiya operator generalizes the integral operators introduced by Alexandar \[1\], Libera \[10\], Bernardi \[3\] and Jung et al. \[5\].

In 2007, Raducanu and Srivastava \[20\] introduced and studied the class \(S^*_s,b(\delta)\) of functions \(f \in \chi\) satisfies \(J_{s,b}f(z) \in S^*(\delta)\).

Now by using concepts of conic domains and Srivastava-Attiya integral operator, we introduce new classes as following.

**Definition 1.1.** Let \(k \in [0, \infty)\) and \(\alpha, \beta \in [0, 1]\). Then \(f \in k - UM(\alpha, \beta)\) if and only if
\[
\text{Re} \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} \right)' \right] > k \left| (1 - \beta) \frac{zf'(z)}{f(z)} + \beta \left( \frac{zf'(z)}{f(z)} \right)' - 1 \right|, \quad (z \in \Omega). \]

Some of the special cases are given below and we refer to \[8,9,21–23\].

**Special cases:**
(i) For \(\beta = 0\), the class \(k - UM(\alpha, \beta)\) reduces to the class \(k - UM_\alpha\). See [21].
(ii) For \(\alpha = 1\) and \(\beta = 0\), the class \(k - UM(\alpha, \beta)\) reduces to the class \(k - MN\). See [22].
(iii) For \(\alpha = 1, \beta = 0\) and \(k = 1\), the class \(k - UM(\alpha, \beta)\) reduces to the class \(MN\). See [23].
(iv) For \(\alpha = 1\) and \(\beta = 1\), the class \(k - UM(\alpha, \beta)\) reduces to the class \(k - UCV\). See [9].
(v) For $\alpha = 0$ and $\beta = 0$, the class $k – UM(\alpha, \beta)$ reduces to the class $k – ST$. See [8].

**Definition 1.2.** Let $\alpha, \beta \in [0, 1], k \in [0, \infty), b \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $s \in \mathbb{C}$. Then $f \in k – UM_s^b(\alpha, \beta)$ if and only if $J_s,b f(z) \in k – UM(\alpha, \beta)$.

Clearly, for $s = 0$ the classes $k – UM_s^b(\alpha, \beta)$ and $k – UM(\alpha, \beta)$ coincides.

2. **Preliminaries**

**Lemma 2.1.** [12] Let $h$ be an analytic function on $\mathring{U}$ except for at most one pole on $\partial U$ and univalent on $U$, $\wp$ be an analytic function in $U$ with $\wp(0) = h(0)$ and $\wp(z) \neq \wp(0), z \in \mathring{U}$. If $\wp$ is not subordinate to $h$, then there exist points $z_0 \in U, \xi_0 \in \partial U$ and $\varepsilon \geq 1$ for which

$$\wp(|z| < |z_0|) \subset h(U), \quad \wp(z_0) = h(\xi_0), \quad z_0 \wp'(z_0) = \varepsilon \xi_0 \wp'(\xi_0).$$

**Lemma 2.2.** [6] If $f \in S^*(\alpha)$ for some $\alpha \in [\frac{1}{2}, 0)$, then

$$Re \left( \frac{f(z)}{z} \right) > \frac{1}{3 - 2\alpha}.$$

**Lemma 2.3.** [6] If $Re \left( \sqrt{f'(z)} \right) > \alpha$ for some $\alpha \in [\frac{1}{2}, 0)$, then

$$Re \left( \frac{f(z)}{z} \right) > \frac{2\alpha^2 + 1}{3}.$$

3. **Main Results**

**Theorem 3.1.** Let $k \in [0, \infty)$ and $\alpha, \beta \in [0, 1]$. Also, let $p$ be a function analytic in the unit disk such that $p(0) = 1$. If

$$Re \left[ p(z) + \alpha \frac{zp'(z)}{p(z)} - k \right] p(z) - 1 + \beta \frac{zp'(z)}{p(z)} > 0,$$

then

$$p(z) < \frac{1 + (1 - 2\gamma)z}{1 - z} := h(z),$$

where $\gamma = \gamma(k, \alpha, \beta)$ is given by

$$\gamma(k, \alpha, \beta) = \frac{1}{4} \left[ \sqrt{\frac{(\alpha - 2k + \beta k)^2}{(1 + k)^2} + \frac{8(\alpha + \beta k)}{(1 + k)} - \frac{(\alpha - 2k + \beta k)}{(1 + k)}} \right]. \quad (3.1)$$

**Proof.** We may assume that $\gamma \geq \frac{1}{2}$ since the condition $Re \left( p(z) + \frac{zp'(z)}{p(z)} \right) > 0$ implies at least $Re(p(z)) > \frac{1}{2}$. (See [11]). Suppose now, on the contrary that $p \neq h$. Then, by Lemma 2.1, there exist $z_0 \in \mathring{U}, \xi_0 \in \partial U$ and $m \geq 1$ such that
\[ p(z_0) = \gamma + ix, \ z_0 p'(z_0) = my, \text{ where } y \leq -\frac{(1 - \gamma)^2 + x^2}{2(1 - \gamma)}, \ (x, y \in \mathbb{R}). \]

Using these relations, we have

\[ \text{Re} \left[ p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right] - k \left| p(z_0) - 1 + \beta \frac{z_0 p'(z_0)}{p(z_0)} \right| > 0, \]

or

\[ 0 < \text{Re} \left[ p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} \right] - k \left| p(z_0) - 1 + \beta \frac{z_0 p'(z_0)}{p(z_0)} \right| = \text{Re} \left[ \gamma + ix + \alpha \frac{my}{\gamma + ix} \right] - k \left| \gamma + ix - 1 + \beta \frac{my}{\gamma + ix} \right| = \gamma + \frac{amy \gamma}{\gamma^2 + x^2} - k \left| \frac{(\gamma + ix)^2 - (\gamma + ix) + \beta my}{\gamma + ix} \right| \leq \gamma - \frac{\alpha \gamma}{2(1 - \gamma)} \left( \frac{(1 - \gamma)^2 + x^2}{\gamma^2 + x^2} \right) - k \frac{\sqrt{(X + Y x)^2 + T x^2}}{\gamma^2 + x^2} = R(x), \]

where \( X = \frac{(2\gamma + \beta)(1 - \gamma)}{2(1 - \gamma)}, \ Y = \frac{2(1 - \gamma) + \beta}{2(1 - \gamma)} \) and \( T = (2\gamma - 1)^2 \). The function \( R(x) \) is even in regard of \( x \). Now we have to show that \( R(x) \) has maximum value at \( x = 0 \) when \( \alpha, \beta \in [0, 1] \) and \( \gamma \in \left[ \frac{1}{2}, 1 \right) \). We can easily check

\[ R'(x) = -x \left[ \frac{\alpha \gamma (2\gamma - 1)}{(1 - \gamma) (\gamma^2 + x^2)} - k \left\{ 2Y(X + Y x^2) + T - \frac{2 \sqrt{(X + Y x)^2 + T x^2}}{\gamma^2 + x^2} \right\} \right]. \]

Then \( R'(x) = 0 \) if and only if, \( x = 0 \). Since \( \alpha, \beta \in [0, 1] \) and \( \gamma \in \left[ \frac{1}{2}, 1 \right) \). So one can see

\[ R''(x) = -\left[ \frac{\alpha (2\gamma - 1) - k \gamma (2\gamma + \beta)}{\gamma (1 - \gamma)} \right] \left\{ (2(1 - \gamma) + \beta) (2\gamma + \beta) + 2 (2\gamma - 1)^2 \right\} < 0. \]

Thus \( R(x) \) has maximum value at \( x = 0 \), that is

\[ R(x) \leq R(0) = \gamma - \frac{\alpha \gamma (1 - \gamma)}{2\gamma} - \frac{k(1 - \gamma)(2\gamma + \beta)}{2\gamma} = 0, \]

for \( \gamma = \gamma(k, \alpha, \beta) \) is given by (3.1), which contradicts the assumption. Hence

\[ p(z) \prec \frac{1 + (1 - 2\gamma)z}{1 - z} := h(z), \]

where \( \gamma = \gamma(k, \alpha, \beta) \) is given by (3.1).

\[ \square \]

**Theorem 3.2.** Let \( \alpha, \beta \in [0, 1], \ k \in [0, \infty), \ b \in \mathbb{C} \setminus \mathbb{Z}^{-} \) and \( s \in \mathbb{C} \). Then

\[ k - UM^*_s(\alpha, \beta) \subset S^*_{s,b}(\gamma), \]

where \( \gamma = \gamma(k, \alpha, \beta) \) is given by (3.1).
Proof. Let \( f \in k - UM_k^b(\alpha, \beta) \). Then, by Definition 1.2, \( J_{s,b}f(z) \in k - UM(\alpha, \beta) \), that is

\[
\text{Re} \left[ (1 - \alpha) \frac{z(J_{s,b}f(z))''}{J_{s,b}f(z)} + \alpha \frac{(J_{s,b}f(z))'}{J_{s,b}f(z)} \right] > k \left| (1 - \beta) \frac{z(J_{s,b}f(z))'}{J_{s,b}f(z)} + \beta \frac{(J_{s,b}f(z))'}{J_{s,b}f(z)} - 1 \right|, \quad (z \in \mathbb{U}).
\]

Putting \( p(z) = \frac{z(J_{s,b}f(z))'}{J_{s,b}f(z)} \), we have

\[
\text{Re} \left[ p(z) + \frac{zp'(z)}{p(z)} \right] - k \left| p(z) - 1 + \frac{zp'(z)}{p(z)} \right| > 0.
\]

Our required result follows easily by using Theorem 3.1. \( \square \)

When \( s = 0 \), then we have the following new result for class \( k - UM(\alpha, \beta) \)

**Theorem 3.3.** Let \( k \in [0, \infty) \) and \( \alpha, \beta \in [0, 1] \). Then

\[ k - UM(\alpha, \beta) \subset S^*(\gamma), \]

where \( \gamma = \gamma(k, \alpha, \beta) \) is given by (3.1).

The proof is straightforward by putting \( p(z) = \frac{zf'(z)}{f(z)} \) and using Theorem 3.1.

When \( k = 0 \), then we have the following result, proved in [21].

**Corollary 3.1.** Let \( f \in M_\alpha \). Then \( f \in S^*(\gamma) \), where

\[
\gamma(\alpha) = -\alpha + \sqrt{\alpha^2 + 8\alpha} \quad \frac{4}{\alpha^2 + 8\alpha}. \quad (3.2)
\]

When \( \beta = 0 \), then we have the following result, proved in [21].

**Corollary 3.2.** Let \( f \in k - UM(\alpha, 0) = k - UM_\alpha \). Then \( f \in S^*(\gamma) \), where

\[
\gamma(\alpha, k) = \frac{2\vartheta - \eta + \sqrt{(2\vartheta - \eta)^2 + 8\eta}}{4}, \quad (3.3)
\]

where \( \vartheta = \frac{k}{k + 1}, \eta = \frac{\alpha + k}{k + 1} \).

When \( \alpha = 1, \beta = 0, \) then we have the following result, proved in [22].

**Corollary 3.3.** Let \( f \in k - UM(1, 0) = k - MN \). Then \( f \in S^*(\gamma) \), where

\[
\gamma(k) = \frac{1}{4} \left[ \sqrt{\left( \frac{1 - 2k}{1 + k} \right)^2 + \frac{8}{(1 + k) - \left( \frac{1 - 2k}{1 + k} \right)}} \right]. \quad (3.4)
\]

When \( \alpha = 1, \beta = 0, \) and \( k = 1 \), then we have the following result, proved in [22].

**Corollary 3.4.** Let \( f \in 1 - UM(1, 0) = MN \). Then \( f \in S^*(\gamma) \), where \( \gamma \approx 0.6403 \).

When \( \alpha = \beta = 1 \), then we have the following result, proved in [6].
Corollary 3.5. Let \( f \in k - UM(1, 1) = k - UCV \). Then \( f \in S^*(\gamma) \), where
\[
\gamma(k) = \frac{1}{4} \left[ \sqrt{\left(\frac{1 - k}{1 + k}\right)^2 + 8 - \left(\frac{1 - k}{1 + k}\right)^2} \right].
\] (3.5)

Theorem 3.4. Let \( f \in k - UM^b_k(\alpha, \beta) \). Then
\[
\frac{J_{s,b}f(z)}{z} \prec \frac{1 + (1 - 2\eta)z}{1 - z},
\]
where \( \eta = \frac{1}{3 - 2\gamma} \) and \( \gamma = \gamma(k, \alpha, \beta) \) is given by (3.1).

Proof. Let \( f \in k - UM^b_k(\alpha, \beta) \). Then by Theorem 3.2 we have
\[
\frac{z (J_{s,b}f(z))'}{J_{s,b}f(z)} \prec \frac{1 + (1 - 2\gamma)z}{1 - z},
\]
where \( \gamma = \gamma(k, \alpha, \beta) \) is given by (3.1). Using Lemma 2.2, we get
\[
\frac{J_{s,b}f(z)}{z} \prec \frac{1 + (1 - 2\eta)z}{1 - z},
\]
where \( \eta = \frac{1}{3 - 2\gamma} \). \( \Box \)

When \( s = 0 \), then one can prove the following result by using Theorem 3.3 together with Lemma 2.2.

Theorem 3.5. Let \( f \in k - UM(\alpha, \beta) \). Then
\[
\frac{f(z)}{z} \prec \frac{1 + (1 - 2\eta)z}{1 - z},
\]
where \( \eta = \frac{1}{3 - 2\gamma} \) and \( \gamma = \gamma(k, \alpha, \beta) \) is given by (3.1).

When \( \alpha = 1, \beta = 0 \), then we have the following result, proved in [22].

Corollary 3.6. Let \( f \in k - UM(1, 0) = k - MN \). Then
\[
\frac{f(z)}{z} \prec \frac{1 + (1 - 2\eta)z}{1 - z},
\]
where \( \eta = \frac{1}{3 - 2\gamma} \) and \( \gamma = \gamma(k) \) is given by (3.4).

When \( \alpha = 1, \beta = 0 \) and \( k = 1 \), then we have the following result, proved in [22].

Corollary 3.7. Let \( f \in 1 - UM(1, 0) = MN \). Then
\[
\frac{f(z)}{z} \prec \frac{1 + (1 - 2\eta)z}{1 - z}, \text{ where } \eta \simeq 0.58159.
\]

When \( \alpha = \beta = 1 \), then we have the following result, proved in [6].

Corollary 3.8. Let \( f \in k - UM(1, 1) = k - UCV \). Then
\[
\frac{f(z)}{z} \prec \frac{1 + (1 - 2\eta)z}{1 - z},
\]
where \( \eta = \frac{1}{3 - 2\gamma} \) and \( \gamma = \gamma(k) \) is given by (3.5).
When $\alpha = \beta = k = 1$, then we have the following result, proved in [6].

**Corollary 3.9.** Let $f \in 1 - UM(1,1) = 1 - UCV$. Then

$$\text{Re} \left( \frac{f(z)}{z} \right) > 0.6289.$$ 

**Theorem 3.6.** Let $\alpha, \beta \in [0,1]$, $k \in [0,\infty)$, $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $s \in \mathbb{C}$. If

$$\text{Re} \left[ \sqrt{(J_{s,b,f(z)}')'} + \alpha \frac{z(J_{s,b,f(z)}'')''}{2(J_{s,b,f(z)})'} \right] > k \left| (J_{s,b,f(z)}')' + \beta \frac{z(J_{s,b,f(z)}'')''}{2(J_{s,b,f(z)})'} - 1 \right|,$$

then

$$\sqrt{(J_{s,b,f(z)}')'} \prec \frac{1 + (1 - 2\gamma)z}{1 - z} \Rightarrow \frac{J_{s,b,f(z)}}{z} \prec \frac{1 + (1 - 2\eta)z}{1 - z}$$

where $\eta = \frac{\gamma^2 + 1}{3}$ and $\gamma = \gamma(k,\alpha,\beta)$ is given by (3.1).

**Proof.** If we put $p(z) = \sqrt{(J_{s,b,f(z)}')'}$, then

$$\frac{zp'(z)}{p(z)} = \frac{z(J_{s,b,f(z)}'')''}{2(J_{s,b,f(z)})'}.$$ 

The proof follows easily by using Theorem 3.1 along with Lemma 2.3. \hfill $\Box$

We can deduce the following result from Theorem 3.6 by choosing $s = 0$.

**Theorem 3.7.** Let $k \in [0,\infty)$ and $\alpha, \beta \in [0,1]$. If

$$\text{Re} \left[ \sqrt{f'(z)} + \alpha \frac{zf''(z)}{2f'(z)} \right] > k \left| \sqrt{f'(z)} + \beta \frac{zf''(z)}{2f'(z)} - 1 \right|,$$

then

$$\sqrt{f'(z)} \prec \frac{1 + (1 - 2\gamma)z}{1 - z} \Rightarrow \frac{f(z)}{z} \prec \frac{1 + (1 - 2\eta)z}{1 - z}$$

where $\eta = \frac{\gamma^2 + 1}{3}$ and $\gamma = \gamma(k,\alpha,\beta)$ is given by (3.1).

When $\alpha = 1$, $\beta = 0$, then we have the following result, proved in [22].

**Corollary 3.10.** If

$$\text{Re} \left[ \sqrt{f'(z)} + \frac{zf''(z)}{2f'(z)} \right] > k \left| \sqrt{f'(z)} - 1 \right|,$$

then

$$\text{Re} \left( \sqrt{f'(z)} \right) > \gamma \Rightarrow \text{Re} \left( \frac{f(z)}{z} \right) > \eta,$$

where $\eta = \frac{\gamma^2 + 1}{3}$ and $\gamma = \gamma(k)$ is given by (3.4).

When $k = 1$, then we have the following result, proved in [22].
Corollary 3.11. If
\[
\text{Re} \left[ \sqrt{f'(z)} + \frac{zf''(z)}{2f'(z)} \right] > \left| \sqrt{f'(z)} - 1 \right|,
\]
then
\[
\text{Re} \left( \sqrt{f'(z)} \right) > \gamma \simeq 0.64 \Rightarrow \text{Re} \left( \frac{f(z)}{z} \right) > \eta \simeq 0.60.
\]

For \( k = 0 \), we have the following result, refer to [22].

Corollary 3.12. If
\[
\text{Re} \left[ \sqrt{f'(z)} + \frac{zf''(z)}{2f'(z)} \right] > 0,
\]
then
\[
\text{Re} \left( \sqrt{f'(z)} \right) > \gamma \simeq 0.64 \Rightarrow \text{Re} \left( \frac{f(z)}{z} \right) > \eta \simeq 0.60.
\]

Theorem 3.8. Let \( \alpha, \beta \in [0, 1], k \in [0, \infty) \), \( b \in \mathbb{C} \setminus \mathbb{Z}^+ \) and \( s \in \mathbb{C} \). If
\[
\text{Re} \left\{ \frac{J_{s,b}f(z)}{z} + \alpha \left( \frac{z}{J_{s,b}f(z)} \right)' - 1 \right\} > k \left| \frac{J_{s,b}f(z)}{z} + \beta \left( \frac{z}{J_{s,b}f(z)} \right)' - 1 \right|,
\]
then
\[
\frac{J_{s,b}f(z)}{z} < \frac{1 + (1 - 2\gamma)z}{1 - z},
\]
where \( \gamma = \gamma(k, \alpha, \beta) \) is given by (3.1).

The proof follows easily by substituting \( p(z) = \frac{J_{s,b}f(z)}{z} \) in Theorem 3.1.

For \( s = 0 \), we can easily deduce the following result.

Theorem 3.9. Let \( k \in [0, \infty) \) and \( \alpha, \beta \in [0, 1] \). If
\[
\text{Re} \left\{ \frac{f(z)}{z} + \alpha \left( \frac{zf'}{f(z)} - 1 \right) \right\} > k \left| \frac{f(z)}{z} + \beta \left( \frac{zf'}{f(z)} - 1 \right) - 1 \right|,
\]
then
\[
\frac{f(z)}{z} < \frac{1 + (1 - 2\gamma)z}{1 - z},
\]
where \( \gamma = \gamma(k, \alpha, \beta) \) is given by (3.1).

When \( \alpha = 1, \beta = 0 \), then we have the following result, proved in [22].

Corollary 3.13. If
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} + \frac{f(z)}{z} - 1 \right\} > k \left| \frac{f(z)}{z} - 1 \right| \Rightarrow \frac{f(z)}{z} < \frac{1 + (1 - 2\gamma)z}{1 - z}
\]
where \( \gamma = \gamma(k) \) is given by (3.4).

When \( \alpha = 1, \beta = 0 \) and \( k = 1 \), then we have the following result, proved in [22].
Corollary 3.14. If
\[ \text{Re} \left[ \frac{zf'(z) + f(z)}{f(z)} - 1 \right] > \frac{|f(z)|}{z} - 1 \Rightarrow \text{Re} \left( \frac{f(z)}{z} \right) > \gamma \simeq 0.64. \]

When \( \alpha = 1, \beta = 0 \) and \( k = 0 \), then we have following result.

Corollary 3.15. If
\[ \text{Re} \left[ \frac{zf'(z) + f(z)}{f(z)} - 1 \right] > 0 \Rightarrow \text{Re} \left( \frac{f(z)}{z} \right) > \frac{1}{2}. \]

If we substitute \( p(z) = (J_{s,b}f(z))' \) in Theorem 3.1, then we have the following result.

Theorem 3.10. Let \( \alpha, \beta \in [0, 1] \), \( k \in [0, \infty) \), \( b \in \mathbb{C} \setminus \mathbb{Z} \) and \( s \in \mathbb{C} \). If
\[ \text{Re} \left[ (J_{s,b}f(z))' + \alpha z (J_{s,b}f(z))'' \right] > k \left| (J_{s,b}f(z))' + \beta z (J_{s,b}f(z))'' \right|, \]
then
\[ \text{Re} \left( (J_{s,b}f(z))' \right) > \gamma \]
where \( \gamma = \gamma(k, \alpha, \beta) \) is given by (3.1).

For \( s = 0 \), we have the following result.

Theorem 3.11. Let \( k \in [0, \infty) \) and \( \alpha, \beta \in [0, 1] \). If
\[ \text{Re} \left[ f'(z) + \alpha \frac{zf''(z)}{f'(z)} \right] > k \left| f'(z) + \beta \frac{zf''(z)}{f'(z)} \right|, \]
then
\[ \text{Re} \left( f'(z) \right) > \gamma \]
where \( \gamma = \gamma(k, \alpha, \beta) \) is given by (3.1).

When \( \alpha = 1, \beta = 0 \), then we have the following result, proved in [22].

Corollary 3.16. If
\[ \text{Re} \left[ f'(z) + \frac{zf''(z)}{f'(z)} \right] > k |f'(z)| - 1 \Rightarrow \text{Re} \left( f'(z) \right) > \gamma \simeq 0.64. \]

When \( \alpha = 1, \beta = k = 0 \), then we have the following result.

Corollary 3.17. If \( \text{Re} \left[ f'(z) + \frac{zf''(z)}{f'(z)} \right] > 0 \), then \( \text{Re} \left( f'(z) \right) > \frac{1}{2}. \)
References