FIXED POINT THEOREMS FOR GENERALIZED F-CONTRACTIONS AND GENERALIZED F-SUZUKI-CONTRACTIONS IN COMPLETE DISLOCATED \( S_b \)-METRIC SPACES

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ABSTRACT. In this paper, first we describe the notion of dislocated \( S_b \)-metric space and then we introduce the new notions of generalized \( F \)-contraction and generalized \( F \)-Suzuki-contraction in the setup of dislocated \( S_b \)-metric spaces. We establish some fixed point theorems involving these contractions in complete dislocated \( S_b \)-metric spaces. We also furnish some examples to verify the effectiveness and applicability of our results.

1. INTRODUCTION AND PRELIMINARIES

Bakhtin [1] and Czerwik [2] introduced \( b \)-metric spaces and proved the contraction principle in this framework. In recent times, many authors obtained fixed point results for single-valued or set-valued functions, in the setting of \( b \)-metric spaces.

In 2012, Sedghi et al. [11] introduced the concept of \( S \)-metric space by modifying \( D \)-metric and \( G \)-metric spaces and proved some fixed point theorems for a self-mapping on a complete \( S \)-metric space. After that Özgür and TaŞ studied some generalizations of the Banach contraction principle on \( S \)-metric spaces in [8]. They also obtained some fixed point theorems for the Rhoades’ contractive condition on \( S \)-metric spaces [7]. Sedgh et al. [10] introduced the concept of \( S_b \)-metric space as a generalization of \( S \)-metric space and proved some coupled common fixed point theorems in \( S_b \)-metric space. Kishore et al. [4] proved some fixed point
Theorems for generalized contractive conditions in partially ordered complete $S_b$-metric spaces and gave some applications to integral equations and homotopy theory.

On the other hand, Wardowski [12] introduced a new contraction, the so-called $F$-contraction, and obtained a fixed point result as a generalization of the Banach contraction principle. Thereafter, Dung and Hang [3] studied the notion of a generalized $F$-contraction and established certain fixed point theorems for such mappings. Recently, Piri and Kumam [6] extended the fixed point results of [12] by introducing a generalized $F$-Suzuki-contraction in $b$-metric spaces.

Motivated by the aforementioned works, in this paper, we first introduce the notion of dislocated $S_b$-metric space and then we describe some fixed point results of [3], [6] by introducing generalized $F$-contractions and generalized $F$-Suzuki-contractions in dislocated $S_b$-metric spaces. We begin with some basic well-known definitions and results which will be used further on.

Throughout this paper $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ denote the set of all real numbers, the set of all nonnegative real numbers and the set of all positive integers, respectively.

**Definition 1.1.** [11] Let $X$ be a nonempty set. An $S$-metric on $X$ is a function $S : X^3 \rightarrow \mathbb{R}_+$ that satisfies the following conditions:

- ($S_1$) $0 < S(x, y, z)$ for each $x, y, z \in X$ with $x \neq y \neq z \neq x$,
- ($S_2$) $S(x, y, z) = 0$ if and only if $x = y = z$,
- ($S_3$) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for each $x, y, z, a \in X$.

Then the pair $(X, S)$ is called an $S$-metric space.

**Definition 1.2.** [10] Let $X$ be a nonempty set and $b \geq 1$ be a given real number. Suppose that a mapping $S_b : X^3 \rightarrow \mathbb{R}_+$ satisfies:

- ($S_b1$) $0 < S_b(x, y, z)$ for all $x, y, z \in X$ with $x \neq y \neq z \neq x$,
- ($S_b2$) $S_b(x, y, z) = 0$ if and only if $x = y = z$,
- ($S_b3$) $S_b(x, y, z) \leq b(S_b(x, x, a) + S(y, y, a) + S(z, z, a))$ for all $x, y, z, a \in X$.

Then $S_b$ is called an $S_b$-metric on $X$ and the pair $(X, S_b)$ is called an $S_b$-metric space.

**Definition 1.3.** [10] If $(X, S_b)$ is an $S_b$-metric space, a sequence $\{x_n\}$ in $X$ is said to be:

1. **Cauchy sequence** if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S_b(x_n, x_n, x_m) < \varepsilon$ for all $m, n \geq n_0$.
2. **convergent to a point** $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer $n_0$ such that $S_b(x_n, x_n, x) < \varepsilon$ or $S_b(x, x, x_n) < \varepsilon$ for all $n \geq n_0$, and we denote by $\lim_{n \rightarrow \infty} x_n = x$.

**Definition 1.4.** [10] An $S_b$-metric space $(X, S_b)$ is called complete if every Cauchy sequence is convergent in $X$. 

Example 1.1. [9] Let $X = \mathbb{R}$. Define $S_b : X^3 \to \mathbb{R}_+$ by $S_b(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in X$. Then $(X, S_b)$ is a complete $S_b$-metric space with $b = 2$.

Definition 1.5. Let $(X, S_b)$ be an $S_b$-metric space. Then $S_b$ is called symmetric if

$$S_b(x, x, y) = S_b(y, y, x) \quad (1.1)$$

for all $x, y \in X$.

It is easy to see that the symmetry condition (1.1) is automatically satisfied by an $S$-metric [11].

We conclude this section recalling the following fixed point theorems of Dung and Hang [3] and Piri and Kumam [6]. For this, we need some preliminaries.

Definition 1.6. [12] Let $F$ be the family of all functions $F : (0, +\infty) \to \mathbb{R}$ such that:

(F1) $F$ is strictly increasing, that is for all $\alpha, \beta \in (0, +\infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$,

(F2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n \to +\infty} \alpha_n = 0$ if and only if $\lim_{n \to +\infty} F(\alpha_n) = -\infty$,

(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

In 2014, Piri and Kumam [5] described a large class of functions by replacing the condition (F3) in the above definition with the following one:

(F3') $F$ is continuous on $(0, +\infty)$.

They denote by $\mathfrak{F}$ the family of all functions $F : (0, +\infty) \to \mathbb{R}$ which satisfy conditions (F1), (F2), and (F3').

Example 1.2. (see [5], [13]) The following functions $F : (0, +\infty) \to \mathbb{R}$ are the elements of $\mathfrak{F}$.

(1) $F(\alpha) = -\frac{1}{\sqrt{\alpha}}$,

(2) $F(\alpha) = -\frac{1}{\alpha} + \alpha$,

(3) $F(\alpha) = \frac{1}{1-e^\alpha}$,

(4) $F(\alpha) = \ln \alpha$,

(5) $F(\alpha) = \ln (\alpha + \alpha)$.

Definition 1.7. [3] Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is said to be a generalized $F$-contraction on $(X, d)$ if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(N(x, y)),$$

in which

$$N(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}, \frac{d(T^2x, x) + d(T^2x, Ty)}{2}, d(T^2x, Tx), d(T^2x, y), d(T^2x, Ty) \right\}.$$
Theorem 1.1. [3] Let $(X,d)$ be a complete metric space and let $T : X \to X$ be a generalized $F$-contraction mapping. If $F$ or $F$ is continuous, then $T$ has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence \( \{T^n x\} \) converges to $x^*$.

We use $\mathfrak{F}_G$ to denote the set of all functions $F : (0, +\infty) \to \mathbb{R}$ which satisfy conditions $(F1)$ and $(F3')$ and $\Psi$ to denote the set of all functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi$ is continuous and $\psi(t) = 0$ if and only if $t = 0$ (see [6]).

Definition 1.8. [6] Let $(X,d)$ be a $b$-metric space. A self-mapping $T : X \to X$ is said to be a generalized $F$-Suzuki-contraction if there exists $F \in \mathfrak{F}_G$ such that, for all $x, y \in X$ with $x \neq y$,

$$\frac{1}{2s} d(x,Tx) < d(x,y) \Rightarrow F(s^2d(Tx,Ty)) \leq F(M_T(x,y)) - \psi(M_T(x,y)),$$

in which $\psi \in \Psi$ and

$$M_T(x,y) = \max \left\{ d(x,y), d(T^2x,y), \frac{d(Tx,y) + d(x,Ty)}{2s}, \frac{d(T^2x,x) + d(T^2x,Ty)}{2s}, \right.$$

$$d(T^2x,Ty) + d(T^2x,Tx), d(T^2x,Ty) + d(Tx,x), d(Tx,y) + d(y,Ty) \right\}.$$

Theorem 1.2. [6] Let $(X,d)$ be a complete $b$-metric space and $T : X \to X$ be a generalized $F$-Suzuki-contraction. Then $T$ has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence \( \{T^n x\} \) converges to $x^*$.

2. Main results

In this section, we first introduce the concept of dislocated $S_b$-metric space and then we demonstrate some fixed point results for generalized $F$-contractions and generalized $F$-Suzuki-contractions in such spaces. Our results are remarkable for two reasons: first dislocated $S_b$-metric is more general, second the contractivity condition involves auxiliary functions form a wider class.

Definition 2.1. Let $X$ be a nonempty set and $b \geq 1$ be a given real number. A mapping $S_b : X^3 \to \mathbb{R}_+$ is a dislocated $S_b$-metric if, for all $x, y, z, a \in X$, the following conditions are satisfied:

$(dS_b1)$ $S_b(x,y,z) = 0$ implies $x = y = z$,

$(dS_b2)$ $S_b(x,y,z) \leq b(S_b(x,x,a) + S_b(y,y,a) + S_b(z,z,a))$.

A dislocated $S_b$-metric space is a pair $(X,S_b)$ such that $X$ is a nonempty set and $S_b$ is a dislocated $S_b$-metric on $X$. In the case that $b = 1$, $S_b$ is denoted by $S$ and it is called dislocated $S$-metric, and the pair $(X,S)$ is called dislocated $S$-metric space.

Definition 2.2. Let $(X, S_b)$ be a dislocated $S_b$-metric space, $\{x_n\}$ be any sequence in $X$ and $x \in X$. Then:

(i) The sequence $\{x_n\}$ is said to be a Cauchy sequence in $(X, S_b)$ if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S_b(x_n, x_m, x_m) < \varepsilon$ for each $m, n \geq n_0$. 

(ii) The sequence \( \{x_n\} \) is said to be convergent to \( x \) if, for each \( \varepsilon > 0 \), there exists a positive integer \( n_0 \) such that \( S_b(x, x, x_n) < \varepsilon \) for all \( n \geq n_0 \) and we denote it by \( \lim_{n \to \infty} x_n = x \).

(iii) \((X, S_b)\) is said to be complete if every Cauchy sequence is convergent.

The following example shows that a dislocated \( S_b \)-metric need not be a dislocated \( S \)-metric.

**Example 2.1.** Let \( X = \mathbb{R}_+ \), then the mapping \( S_b : X^3 \to \mathbb{R}_+ \) defined by
\[
S_b(x, y, z) = x + y + 4z
\]
is a complete dislocated \( S_b \)-metric on \( X \) with \( b = 2 \). However, it is not a dislocated \( S \)-metric space. Indeed, we have
\[
4 = S_b(0, 0, 1) \not\leq 2S_b(0, 0, 0) + S_b(1, 1, 0) = 2.
\]

**Definition 2.3.** Suppose that \((X, S_b)\) is a dislocated \( S_b \)-metric space. A mapping \( T : X \to X \) is said to be a generalized \( F \)-contraction on \((X, S_b)\) if there exist \( F \in \mathfrak{F} \) and \( \tau > 0 \) such that for all \( x, y \in X \),
\[
S_b(Tx, Tx, Ty) > 0 \Rightarrow \tau + F(b^2 S_b(Tx, Tx, Ty)) \leq F(N(x, y)), \tag{2.1}
\]
where
\[
N(x, y) = \max \{ S_b(x, x, y), S_b(Tx, Tx, Ty), \frac{S_b(y, y, Tx)}{10^a}, \frac{S_b(x, x, Ty)}{10^b}, \frac{S_b(y, y, T^2x)}{10^4} \}.
\]

Our first main result is the following.

**Theorem 2.1.** Let \((X, S_b)\) be a complete dislocated \( S_b \)-metric space and \( T : X \to X \) be a generalized \( F \)-contraction mapping satisfying the following condition:
\[
\max \left\{ \frac{S_b(y, y, Ty)}{5b^2} + \frac{S_b(Tx, Tx, Ty)}{10^a}, \frac{S_b(x, x, Ty)}{10^b}, \frac{S_b(y, y, T^2x)}{10^4} \right\} \leq S_b(Tx, Tx, Ty)
\]
for all \( x, y \in X \).

Then \( T \) has a unique fixed point \( v \in X \).

**Proof.** Let \( x_0 \) be an arbitrary point in \( X \) and let \( \{x_n\} \) be the Picard sequence of \( T \) based on \( x_0 \), that is, \( x_{n+1} = Tx_n \) for \( n = 0, 1, 2, \ldots \). If there exists \( n_0 \in \mathbb{N} \) such that \( S_b(x_{n_0}, x_{n_0}, x_{n_0+1}) = 0 \), then \( x_{n_0} \) is a fixed point of \( T \) and the existence part of the proof is finished. On the contrary case, assume that \( S_b(x_n, x_n, x_{n+1}) > 0 \) for all \( n \in \mathbb{N} \cup \{0\} \). Applying the contractivity condition (2.1), we get
\[
F(b^2 S_b(Tx_{n-1}, Tx_{n-1}, Tx_n)) \leq F(N(x_{n-1}, x_n)) - \tau. \tag{2.2}
\]
Using the definition of \( N(x, y) \) and the property \((dS_b, 2)\), we obtain that
\[
\max \left\{ S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_n, x_{n+1}) \right\} \leq N(x_{n-1}, x_n) \tag{2.3}
\]
\[
= \max \left\{ S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_n, Tx_n), \frac{S_b(x_n, x_n, Tx_n)}{10^a}, \frac{S_b(x_{n-1}, x_{n-1}, Tx_n)}{10^a}, \frac{S_b(x_n, x_n, Tx_n)}{10^b}, \frac{S_b(y, y, T^2x)}{10^4} \right\}
\]
\[
\leq \max \left\{ S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_n, Tx_n), 3S_b(x_n, x_n, x_{n+1}), \frac{S_b(x_n, x_n, Tx_n)}{10^a}, \frac{S_b(x_n, x_n, Tx_n)}{10^b}, \frac{S_b(x_{n-1}, x_{n-1}, Tx_n)}{10^a} \right\}
\]
\[
= \max \left\{ S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_n, x_{n+1}) \right\}.
\]
Then $N(x_{n-1}, x_n) = \max \{ S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_{n+1}) \}$ and so (2.2), becomes
\[
F(b^2 S_b(Tx_{n-1}, Tx_{n-1}, Tx_n)) \leq F(\max \{ S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_{n+1}) \}) - \tau.
\]
If we assume that
\[
\max \{ S_b(x_{n-1}, x_{n-1}, x_n), S_b(Tx_{n-1}, Tx_{n-1}, Tx_n) \} = S_b(Tx_{n-1}, Tx_{n-1}, Tx_n)
\]
for some $n$, then we have
\[
F(b^2 S_b(Tx_{n-1}, Tx_{n-1}, Tx_n)) \leq F(S_b(Tx_{n-1}, Tx_{n-1}, Tx_n)) - \tau < F(S_b(Tx_{n-1}, Tx_{n-1}, Tx_n)).
\]
Using condition (F1) we conclude that $S_b(x_n, x_{n+1}) < S_b(x_n, x_{n+1})$, which is a contradiction. Therefore, for each $n \in \mathbb{N}$ we have
\[
\max \{ S_b(x_{n-1}, x_{n-1}, x_n), S_b(x_n, x_{n+1}) \} = S_b(x_{n-1}, x_{n-1}, x_n).
\]
Applying again (2.2) and condition (F1), we deduce that
\[
S_b(x_n, x_{n+1}) < S_b(x_{n-1}, x_{n-1}, x_n)
\]
for each $n$. Thus $\{ S_b(x_{n}, x_{n+1}) \}$ is a nonnegative decreasing sequence of real numbers. Then there exists $A \geq 0$ such that
\[
\lim_{n \to +\infty} S_b(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} S_b(x_n, x_{n+1}) = A.
\]
We claim that $A = 0$. To support the claim, let it be untrue and $A > 0$. Then, for any $\varepsilon > 0$, it is possible to find a positive integer $m$ so that
\[
S_b(x_m, x_m, Tx_m) < A + \varepsilon.
\]
So, from (F1), we get
\[
F(S_b(x_m, x_m, Tx_m)) < F(A + \varepsilon). \tag{2.4}
\]
It follows from (2.1) that
\[
\tau + F(b^2 S_b(Tx_m, Tx_m, T^2 x_m)) \leq F(N(x_m, Tx_m)). \tag{2.5}
\]
By a similar argument as (2.3), it yields that
\[
N(x_m, Tx_m) = \max \{ S_b(x_m, x_m, Tx_m), S_b(Tx_m, Tx_m, T^2 x_m) \}.
\]
Hence (2.5), becomes
\[
F(b^2 S_b(Tx_m, Tx_m, T^2 x_m)) \leq F(\max \{ S_b(x_m, x_m, Tx_m), S_b(Tx_m, Tx_m, T^2 x_m) \}) - \tau. \tag{2.6}
\]
Now if, $\max \{ S_b(x_m, x_m, Tx_m), S_b(Tx_m, Tx_m, T^2 x_m) \} = S_b(Tx_m, Tx_m, T^2 x_m)$ for some $m$, then (2.6) gives us a contradiction. Thus, we infer that
\[
\max \{ S_b(x_m, x_m, Tx_m), S_b(Tx_m, Tx_m, T^2 x_m) \} = S_b(x_m, x_m, Tx_m),
\]
and therefore, we have

\[ F(b^2 S_b(T x_m, T x_m, T^2 x_m)) \leq F(S_b(x_m, x_m, T x_m)) - \tau. \]

It implies that

\[ F(b^2 S_b(T^2 x_m, T^2 x_m, T^3 x_m)) \leq F(S_b(T x_m, T x_m, T^2 x_m)) - \tau \]

\[ \leq F(b^2 S_b(T x_m, T x_m, T^2 x_m)) - \tau \]

\[ \leq F(S_b(x_m, x_m, T x_m)) - 2\tau. \]

Continuing the above process and taking (2.4) into account, we deduce that

\[ F(b^2 S_b(T^n x_m, T^n x_m, T^{n+1} x_m)) \leq F(S_b(T^{n-1} x_m, T^{n-1} x_m, T^n x_m)) - \tau \]

\[ \leq F(b^2 S_b(T^{n-1} x_m, T^{n-1} x_m, T^n x_m)) - \tau \]

\[ \leq F(S_b(T^{n-2} x_m, T^{n-2} x_m, T^{n-1} x_m)) - 2\tau \]

\[ \vdots \]

\[ \leq F(S_b(x_m, x_m, T x_m)) - n\tau \]

\[ < F(A + \varepsilon) - n\tau, \]

and by passing to the limit as \( n \to +\infty \) we obtain

\[ \lim_{n \to +\infty} F(b^2 S_b(T^n x_m, T^n x_m, T^{n+1} x_m)) = -\infty. \]

This fact together with the condition (F2) implies that

\[ \lim_{n \to +\infty} S_b(T^n x_m, T^n x_m, T^{n+1} x_m) = 0. \]

Thus \( S_b(T^n x_m, T^n x_m, T^{n+1} x_m) < A \) for \( n \) sufficiently large, which is a contradiction with the definition of \( A \). Then,

\[ \lim_{n \to +\infty} S_b(x_n, x_n, x_{n+1}) = 0. \quad (2.7) \]

Next, we intend to show that the sequence \( \{x_n\} \) is a Cauchy sequence in \( X \). Arguing by contradiction, we assume that there exist \( \varepsilon > 0 \), and subsequences \( \{x_{q(n)}\} \) and \( \{x_{p(n)}\} \) of \( \{x_n\} \) with \( n < q(n) < p(n) \) such that

\[ S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}) \geq \varepsilon \quad (2.8) \]

for each \( n \in \mathbb{N} \). Further, corresponding to \( q(n) \), we can choose \( p(n) \) in such a way that it is the smallest integer with \( q(n) < p(n) \) satisfying the above inequality, then

\[ S_b(x_{q(n)}, x_{q(n)}, x_{p(n)-1}) < \varepsilon \quad (2.9) \]

for all \( n \in \mathbb{N} \).

In the light of (2.8) and the condition (2.1), we conclude that

\[ F(b^2 S_b(T x_{q(n)-1}, T x_{q(n)-1}, T x_{p(n)-1})) \leq F(N(x_{q(n)-1}, x_{p(n)-1})) - \tau. \quad (2.10) \]
By our hypothesis and in view of \((dS_2)\), we get

\[
\max \left\{ S_b(x_{q(n)}-1, x_{p(n)}-1, x_{q(n)}), S_b(T x_{q(n)}-1, T x_{p(n)}-1) \right\}
\]

\[
\leq N(x_{q(n)}-1, x_{p(n)}-1)
\]

\[
= \max \left\{ S_b(x_{p(n)}-1, x_{p(n)}-1, x_{p(n)}-1), S_b(T x_{q(n)}-1, T x_{q(n)}-1, T x_{p(n)}-1),
S_b(x_{p(n)}-1, x_{p(n)}-1, T x_{q(n)}-1) \right\}
\]

\[
\frac{106^8}{106^4}
\]

\[
\leq \max \left\{ S_b(x_{p(n)}-1, x_{p(n)}-1, x_{p(n)}-1), S_b(T x_{q(n)}-1, T x_{q(n)}-1, T x_{p(n)}-1),
S_b(x_{p(n)}-1, x_{p(n)}-1, T x_{q(n)}-1) \right\}
\]

\[
\frac{106^7}{106^4}
\]

\[
\leq \max \left\{ S_b(x_{p(n)}-1, x_{p(n)}-1, x_{p(n)}-1), S_b(T x_{q(n)}-1, T x_{q(n)}-1, T x_{p(n)}-1) \right\}
\]

It enforces that

\[
N(x_{q(n)}-1, x_{p(n)}-1) = \max \left\{ S_b(x_{q(n)}-1, x_{q(n)}-1, x_{p(n)}-1), S_b(T x_{q(n)}-1, T x_{q(n)}-1, T x_{p(n)}-1) \right\}.
\]

Suppose that the maximum on the right-hand side is equal to \(S_b(T x_{q(n)}-1, T x_{q(n)}-1, T x_{p(n)}-1)\) for some \(n\), then from relation \((2.10)\) together with the condition \((F1)\) we get

\[
S_b(T x_{q(n)}-1, T x_{q(n)}-1, T x_{p(n)}-1) < S_b(T x_{q(n)}-1, T x_{q(n)}-1, T x_{p(n)}-1)
\]

which is a contradiction. Thus, we find that

\[
\max \left\{ x_{q(n)}-1, x_{q(n)}-1, x_{p(n)}-1 \right\} = S_b(x_{q(n)}-1, x_{q(n)}-1, x_{p(n)}-1)
\]

for all \(n\). Accordingly, \((2.10)\) becomes

\[
F(b^2 S_b(T x_{q(n)}-1, T x_{q(n)}-1, T x_{p(n)}-1)) \leq F(S_b(x_{q(n)}-1, x_{q(n)}-1, x_{p(n)}-1)) - \tau
\]

and so using \((F1)\) we get

\[
S_b(x_{q(n)}-1, x_{q(n)}-1, x_{p(n)}-1) < S_b(x_{q(n)}-1, x_{q(n)}-1, x_{p(n)}-1).
\]

Regarding to \((2.8)\), \((2.12)\) and employing \((dS_2)\) we observe that

\[
\varepsilon \leq S_b(x_{q(n)}-1, x_{q(n)}-1, x_{p(n)}-1)
\]

\[
< S_b(x_{q(n)}-1, x_{q(n)}-1, x_{p(n)}-1)
\]

\[
\leq 2b S_b(x_{q(n)}-1, x_{q(n)}-1, x_{q(n)}-1) + b S_b(x_{p(n)}-1, x_{p(n)}-1, x_{q(n)}-1)
\]

\[
\leq 2b S_b(x_{q(n)}-1, x_{q(n)}-1, x_{q(n)}-1) + 2b^2 S_b(x_{p(n)}-1, x_{p(n)}-1, x_{p(n)}-1)
\]

\[
+ b^2 S_b(x_{q(n)}-1, x_{q(n)}-1, x_{p(n)}-1)
\]

\[
\leq 2b S_b(x_{q(n)}-1, x_{q(n)}-1, x_{q(n)}-1) + 6b^3 S_b(x_{p(n)}-1, x_{p(n)}-1, x_{p(n)}-1)
\]

\[
+ b^2 S_b(x_{q(n)}-1, x_{q(n)}-1, x_{p(n)}-1).
\]

Combining this result with \((2.7)\) and \((2.9)\) we get

\[
\varepsilon \leq \limsup_{n \to +\infty} S_b(x_{q(n)}-1, x_{q(n)}-1, x_{p(n)}-1) \leq \limsup_{n \to +\infty} S_b(x_{q(n)}-1, x_{q(n)}-1, x_{p(n)}-1) \leq b^2 \varepsilon.
\]
In view of (2.13) and (2.11) and applying the conditions \((F1)\) and \((F3')\), we have
\[
F(b^2 \varepsilon) \leq F \left( b^2 \limsup_{n \to +\infty} S_b(x_{q(n)}, x_{p(n)}) \right) \\
\leq F \left( \limsup_{n \to +\infty} S_b(x_{q(n)-1}, x_{p(n)-1}) \right) - \tau \\
\leq F(b^2 \varepsilon) - \tau.
\]
It is a contradiction with \(\tau > 0\), and therefore it follows that \(\{x_n\}\) is a Cauchy sequence in \(X\). By completeness of \((X, S_b)\), \(\{x_n\}\) converges to some point \(v \in X\). Then, for each \(\varepsilon > 0\), there exists \(N_1 \in \mathbb{N}\) such that
\[
S_b(v, v, x_n) < \varepsilon,
\]
for all \(n \geq N_1\). We are going to show that \(v\) is a fixed point of \(T\). For this aim, we consider two following cases:

Case 1. If \(S_b(Tv, Tv, Tx_n) = 0\) for some \(n \geq N_1\), then from \((dS_b2)\) we find that
\[
S_b(Tv, Tv, v) \leq 2bS_b(Tv, Tv, Tx_n) + bS_b(v, v, Tx_n) \leq b\varepsilon.
\]
Case 2. If \(S_b(Tv, Tv, Tx_n) > 0\) for all \(n \geq N_1\), then using (2.1), we get
\[
F(b^2 S_b(Tv, Tv, Tx_n)) \leq F(N(v, x_n)) - \tau.
\]
From our assumptions, and using \((dS_b2)\), it follows that
\[
\max \{ S_b(v, v, x_n), S_b(Tv, Tv, Tx_n) \} \\
\leq N(v, x_n) \\
= \max \{ S_b(v, v, x_n), S_b(Tv, Tv, Tx_n), S_b(x_n, x_n, Tv), S_b(Tv, Tv, Tx_n), S_b(x_n, x_n, Tx_n) \} \\
= \max \{ S_b(v, v, x_n), S_b(Tv, Tv, Tx_n) \}.
\]
It enforces that \(N(v, x_n) = \max \{ S_b(v, v, x_n), S_b(Tv, Tv, Tx_n) \}\). Now, if we assume that the maximum on the right-hand side of this equality is equal to \(S_b(Tv, Tv, Tx_n)\), then by replacing it in (2.15), we obtain \(S_b(Tv, Tv, Tx_n) < S_b(Tv, Tv, Tx_n)\) which is a contradiction. Consequently, for each \(n \in \mathbb{N}\) we have
\[
\max \{ S_b(v, v, x_n), S_b(Tv, Tv, Tx_n) \} = S_b(v, v, x_n).
\]
Hence, (2.15) turns into
\[
F(b^2 S_b(Tv, Tv, Tx_n)) \leq F(S_b(v, v, x_n)) - \tau \\
< F(S_b(v, v, x_n)).
\]
Employing the condition \((F1)\), we get
\[
S_b(Tv, Tv, Tx_n) < S_b(v, v, x_n).
\]
From \((dS_b)\), (2.16) and (2.14), we deduce that
\[
S_b(Tv, Tv, v) \leq 2bS_b(Tv, Tv, Tx_{N_1}) + bS_b(v, v, Tx_{N_1}) < 3b\varepsilon.
\]
From the arbitrariness of \(\varepsilon\) in each case, it follows that \(S_b(Tv, Tv, v) = 0\) which implies that \(Tv = v\). Hence, \(v\) is a fixed point of \(T\).

Finally, we show that \(T\) has at most one fixed point. Indeed, if \(v_1, v_2 \in X\) are two fixed points of \(T\), such that \(v_1 \neq v_2\), then we obtain
\[
F(b^2 S_b(Tv_1, Tv_2)) \leq F(N(v_1, v_2)) - \tau,
\]
(2.17)

From our hypothesis and by using \((dS_b)\), it follows that
\[
S_b(v_1, v_1, v_2) \leq N(v_1, v_2)
\leq \max \left\{ S_b(v_1, v_1, v_2), S_b(Tv_1, Tv_1, Tv_2), \frac{S_b(v_2, v_2, Tv_2)}{10b^2}, \frac{S_b(Tv_1, Tv_1, Tv_2)}{100b^2} \right\},
\]
\[
= \max \left\{ S_b(v_1, v_1, v_2), S_b(Tv_1, Tv_1, Tv_2) \right\}
= S_b(v_1, v_1, v_2).
\]

Then (2.17) becomes
\[
F(b^2 S_b(v_1, v_1, v_2)) \leq F(S_b(v_1, v_1, v_2)) - \tau.
\]

It gives us a contradiction. Therefore, \(v_1 = v_2\) and the fixed point is unique. \(\square\)

Now we illustrate our result contained in Theorem 2.1 with help of two examples.

**Example 2.2.** Let \((X, S_b)\) be as in Example 2.1 and let \(\tau > 0\) be an arbitrary fixed number. Define the mapping \(T : X \rightarrow X\) by \(T(x) = e^{-\tau}x\) and take \(F(\alpha) = \ln \alpha + \alpha \frac{1}{\alpha} (\alpha > 0)\). It is easily verified that \(N(x, y) = S_b(x, x, y) = 2x + 4y\). Assume that \(x\) or \(y\) is nonzero, then \(S_b(Tx, Tx, Ty) > 0\) and we have
\[
\tau + F(b^2 S_b(Tx, Tx, Ty)) = \tau + \ln(e^{-\tau}(x + 2y)) + e^{-\tau}(x + 2y)
= \ln(x + 2y) + e^{-\tau}(x + 2y)
\leq \ln(2x + 4y) + 2x + 4y
= F(S_b(x, x, y))
= F(N(x, y)).
\]

Hence, \(T\) is a generalized \(F\)-contraction. On the other hand, if we assume that \(0 < \tau \leq 0.0250587314\), then the following estimate holds:
\[
\max \left\{ \frac{S_b(y, y, Ty)}{5b^2}, \frac{S_b(Tx, Tx, Ty)}{10b^2}, \frac{S_b(x, x, Ty)}{10b^2}, \frac{S_b(y, y, T^2x)}{10b^4} \right\}
= \max \left\{ \frac{2y + e^{-\tau} y}{5 \times 2^2} + \frac{e^{-\tau}(\frac{y}{4} + \frac{y}{2})}{10 \times 2^2} \times \frac{2x + e^{-\tau} y}{10 \times 2^2} + \frac{2y + e^{-2\tau} x}{10 \times 2^4} \right\}
\leq e^{-\tau}(\frac{y}{4} + \frac{y}{2}) = S_b(Tx, Tx, Ty).
\]

Thus all conditions of Theorem 2.1 hold and 0 is a unique fixed point of \(T\).
Example 2.3. Let $X = \mathbb{R}$, and $S_b : X^3 \to \mathbb{R}_+$ be a mapping defined by $S_b(x, y, z) = x^2 + y^2 + 2z^2$. Then $(X, S_b)$ is a complete dislocated $S_b$-metric with $b = 2$. Define the mapping $T : X \to X$ by $T(x) = \frac{x}{3}$ and take $F(\alpha) = \ln \alpha$ ($\alpha > 0$). It is easily checked that $N(x, y) = S_b(x, x, y) = x^2 + 2y^2$. Assume that $x$ or $y$ is nonzero, then $S_b(Tx, Tx, Ty) > 0$ and we have

$$
\tau + F(\sqrt[3]{b} S_b(Tx, Tx, Ty)) \leq F(N(x, y)) \Leftrightarrow \ln(\frac{\tau}{4}) \geq \tau.
$$

Also, we observe that

$$
\max \left\{ \frac{S_b(y, y, Ty)}{5b}, \frac{S_b(Tx, Tx, Ty)}{16b^3}, \frac{S_b(x, x, Ty)}{10b^3}, \frac{S_b(y, y, T^2x)}{10b^4} \right\}
$$

$$
= \max \left\{ \frac{48g^2 + 2x^2}{23040}, \frac{18x^2 + 4y^2}{92160}, \frac{162y^2 + 4x^2}{25920} \right\}
$$

$$
\leq \frac{10240x^2 + 20480y^2}{92160}
$$

$$
= S_b(Tx, Tx, Ty)
$$

for all $x, y \in X$. Now, if we assume that $0 < \tau \leq \ln(\frac{\tau}{4})$, then all the conditions of Theorem 2.1 hold and 0 is a unique fixed point of $T$.

Now, we describe the concept of generalized $F$-Suzuki-contraction in the framework of dislocated $S_b$-metric spaces.

Definition 2.4. Let $(X, S_b)$ be a dislocated $S_b$-metric space. A mapping $T : X \to X$ is said to be a generalized $F$-Suzuki-contraction if there exists $F \in \mathcal{F}$ such that for all $x, y \in X$

$$
\frac{1}{2b} S_b(x, x, Tx) < S_b(x, x, y) \Rightarrow F(2b S_b(Tx, Tx, Ty)) \leq F(M_T(x, y)) - \psi(M_T(x, y)),
$$

(2.18)

where $\psi \in \Psi$ and

$$
M_T(x, y) = \max \left\{ S_b(x, x, y), \frac{S_b(y, y, Ty)}{10}, \frac{S_b(x, x, Tx)}{10}, S_b(Tx, Tx, Ty), \frac{S_b(y, y, Tx)}{186}, \frac{S_b(Tx, T^2x)}{2} \right\}.
$$

Our second main result is the following.

Theorem 2.2. Let $(X, S_b)$ be a complete dislocated $S_b$-metric space and $T : X \to X$ be a generalized $F$-Suzuki-contraction satisfying the following condition:

$$
\max \left\{ \frac{S_b(y, y, Ty)}{10}, \frac{S_b(x, x, Tx)}{10}, \frac{S_b(y, y, Ty)}{9}, \frac{S_b(Tx, Tx, Ty)}{18}, \frac{S_b(Tx, Tx, T^2x)}{2} \right\} \leq S_b(Tx, Tx, Ty)
$$

for all $x, y$ in $X$. Then $T$ has a unique fixed point in $X$.

Proof. Let $x_0$ be arbitrary. Define $x_n = T^{n-1}$ for each $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $S_b(x_n, x_n, Tx_n) = 0$, then $x_n = Tx_n$ and $x_n$ becomes a fixed point of $T$, which completes the proof. Therefore, we assume that $S_b(x_n, x_n, Tx_n) > 0$ for all $n \in \mathbb{N}$. Taking into account (2.18), we deduce

$$
F(2b S_b(Tx_n, Tx_n, Tx_{n+1})) \leq F(M_T(x_n, x_{n+1})) - \psi(M_T(x_n, x_{n+1})).
$$

(2.19)
Using \((dS_b2)\) we get
\[
\max \left\{ S_b(x_n, x_n, x_{n+1}), S_b(x_{n+1}, x_{n+1}, T x_{n+1}) \right\}
\]
\[
\leq M_T(x_n, x_{n+1})
\]
\[
\leq \max \left\{ S_b(x_n, x_n, x_{n+1}), \frac{S_b(x_{n+1}, x_{n+1}, T x_{n+1})}{10}, \frac{S_b(x_n, x_n, T x_n)}{10}, S_b(T x_n, T x_n, T x_{n+1}), \frac{S_b(T x_n, T x_n, T x_{n+1})}{2}, \frac{S_b(x_{n+1}, x_{n+1}, x_{n+2})}{6} \right\}
\]
\[
= \max \left\{ S_b(x_n, x_n, x_{n+1}), S_b(x_{n+1}, x_{n+1}, x_{n+2}) \right\}
\]
and combining it with the relation \((2.19)\) we derive
\[
F(2b^3 b(Tx_n, Tx_n, Tx_{n+1})) \leq F \left( \max \left\{ S_b(x_n, x_n, x_{n+1}), S_b(x_{n+1}, x_{n+1}, x_{n+2}) \right\} \right) - \psi \left( \max \left\{ S_b(x_n, x_n, x_{n+1}), S_b(x_{n+1}, x_{n+1}, x_{n+2}) \right\} \right).
\]
\[
(2.20)
\]
If \(\max \{S_b(x_n, x_n, x_{n+1}), S_b(x_{n+1}, x_{n+1}, x_{n+2})\} = S_b(x_{n+1}, x_{n+1}, x_{n+2})\), then \((2.20)\) becomes
\[
F(2b^3 b(Tx_n, Tx_n, Tx_{n+1})) \leq F(S_b(x_{n+1}, x_{n+1}, x_{n+2})) - \psi(S(x_{n+1}, x_{n+1}, x_{n+2})).
\]
By the property of \(\psi\) and using condition \((F1)\), we obtain
\[
2b^3 b(Tx_n, Tx_n, Tx_{n+1}) < S_b(Tx_n, Tx_n, Tx_{n+1}),
\]
which is a contradiction. Hence \(\max \{S_b(x_n, x_n, x_{n+1}), S_b(x_{n+1}, x_{n+1}, x_{n+2})\} = S_b(x_n, x_{n+1})\), then \((2.20)\) becomes
\[
F(2b^3 b(Tx_n, Tx_n, Tx_{n+1})) \leq F(S_b(x_n, x_{n+1})) - \psi(S(x_n, x_{n+1})).
\]
\[
(2.21)
\]
This together with condition \((F1)\) implies that \(S_b(Tx_n, Tx_n, Tx_{n+1}) < S_b(x_n, x_{n+1})\) for each \(n \in \mathbb{N}\). Then \(\{S_b(x_n, x_n, x_{n+1})\}\) is a nonnegative decreasing sequence of real numbers. Therefore, there exists \(A \geq 0\) such that \(\lim_{n \to +\infty} S_b(x_n, x_{n+1}) = A\).

Letting \(n \to +\infty\) in \((2.21)\) and using \((F3')\) and continuity of \(\psi\), we get
\[
F(2b^3 A) \leq F(A) - \psi(A).
\]
It gives us \(\psi(A) = 0\). By property of \(\psi\) we deduce that \(A = 0\). Consequently, we have
\[
\lim_{n \to +\infty} S_b(x_n, x_{n+1}) = 0.
\]
\[
(2.22)
\]
Next, we prove that \(\{x_n\}\) is a Cauchy sequence in \(X\). If it is not true, then there exist \(\varepsilon > 0\) and increasing sequences of natural numbers \(\{p(n)\}\) and \(\{q(n)\}\) such that
\[
n < q(n) < p(n),
\]
\[
S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}) \geq \varepsilon,
\]
\[
S_b(x_{q(n)}, x_{q(n)}, x_{p(n)}-1) < \varepsilon
\]
\[
(2.23)
\]
for all \(n \in \mathbb{N}\).

Owing to \((2.22)\), there exists \(N_1 \in \mathbb{N}\) such that
\[
S_b(x_{q(n)}, x_{q(n)}, T x_{q(n)}) < \varepsilon
\]
\[
(2.24)
\]
for all \( n \geq N_1 \). Hence, from (2.23) and (2.24) it follows that

\[
\frac{1}{2b} S_b(x_q(n), x_q(n), Tx_q(n)) < \frac{1}{2b} \varepsilon < S_b(x_q(n), x_q(n), x_p(n))
\]

for all \( n \geq N_1 \). By using (2.18) we obtain

\[
F(2b^3 S_b(Tx_q(n), Tx_q(n), Tx_p(n))) \leq F(M_T(x_q(n), x_p(n))) - \psi(M_T(x_q(n), x_p(n))).
\] (2.25)

From our assumptions and regarding \((dS_2)\), we get

\[
\max \left\{ S_b(x_q(n), x_q(n), x_p(n)), S_b(Tx_q(n), Tx_q(n), Tx_p(n)) \right\}
\]

\[
\leq M_T(x_q(n), x_p(n))
\]

\[
\leq \max \left\{ S_b(x_q(n), x_q(n), x_p(n)), \frac{S_b(x_p(n), x_p(n), x_p(n)+1)}{10}, S_b(Tx_q(n), Tx_q(n), Tx_p(n)) \right\}
\]

\[
\leq \max \left\{ S_b(x_q(n), x_q(n), x_p(n)), \frac{S_b(x_p(n), x_p(n), x_p(n)+1)}{10}, \frac{S_b(x_p(n), x_p(n), x_p(n)+1)}{2}, \frac{S_b(x_p(n), x_p(n), x_p(n)+1)}{9} \right\}
\]

\[
\leq \max \left\{ S_b(x_q(n), x_q(n), x_p(n)), S_b(x_q(n)+1, x_q(n)+1, x_p(n)+1) \right\}.
\]

Then (2.25) becomes

\[
F(2b^3 S_b(Tx_q(n), Tx_q(n), Tx_p(n))) \leq F\left( \max \left\{ S_b(x_q(n), x_q(n), x_p(n)), S_b(x_q(n)+1, x_q(n)+1, x_p(n)+1) \right\} \right)
\]

\[
- \psi\left( \max \left\{ S_b(x_q(n), x_q(n), x_p(n)), S_b(x_q(n)+1, x_q(n)+1, x_p(n)+1) \right\} \right).
\]

If \( \max \left\{ S_b(x_q(n), x_q(n), x_p(n)), S_b(x_q(n)+1, x_q(n)+1, x_p(n)+1) \right\} = S_b(x_q(n)+1, x_q(n)+1, x_p(n)+1) \) for some \( n \), then we have

\[
F(2b^3 S_b(Tx_q(n), Tx_q(n), Tx_p(n))) \leq F\left( S_b(x_q(n)+1, x_q(n)+1, x_p(n)+1) \right)
\]

\[- \psi\left( S_b(x_q(n)+1, x_q(n)+1, x_p(n)+1) \right).
\]

Obviously, \( S_b(x_q(n)+1, x_q(n)+1, x_p(n)+1) > 0 \) and by the property of \( \psi \) and (F1), we get

\[
S_b(Tx_q(n), Tx_q(n), Tx_p(n)) < S_b(x_q(n)+1, x_q(n)+1, x_p(n)+1),
\]

which is a contradiction. Due to this fact, we find that

\[
\max \left\{ S_b(x_q(n), x_q(n), x_p(n)), S_b(x_q(n)+1, x_q(n)+1, x_p(n)+1) \right\} = S_b(x_q(n), x_q(n), x_p(n))
\]

for all \( n \). Therefore

\[
F(2b^3 S_b(Tx_q(n), Tx_q(n), Tx_p(n))) \leq F\left( S_b(x_q(n), x_q(n), x_p(n)) \right) - \psi\left( S_b(x_q(n), x_q(n), x_p(n)) \right),
\] (2.26)

and by (F1), it follows that

\[
S_b(x_q(n)+1, x_q(n)+1, x_p(n)+1) < S_b(x_q(n), x_q(n), x_p(n)).
\]
In view of (2.23) and \((dS_b)^2\), we infer that

\[
\varepsilon \leq S_b(x_q(n), x_q(n), x_p(n)) \leq 2bS_b(x_q(n), x_q(n), x_p(n)) + bS_b(x_p(n), x_p(n) - 1) + \varepsilon
\]

\[
\leq 2bS_b(x_q(n), x_q(n), x_p(n)) + bS_b(x_p(n), x_p(n)) + b^2S_b(x_p(n) - 1, x_p(n) - 1, x_p(n))
\]

\[
\leq 2bS_b(x_q(n), x_q(n), x_p(n)) + b^3S_b(x_p(n), x_p(n), x_p(n) + 1)
\]

\[
+ b^3S_b(x_p(n) - 1, x_p(n) - 1, x_p(n)).
\]

Taking the limit as \(n \to +\infty\) in the above inequality and regarding (2.22) and (2.23), we deduce that

\[
\varepsilon \leq \lim_{n \to +\infty} S_b(x_q(n), x_q(n), x_p(n)) \leq 2b\varepsilon. \tag{2.27}
\]

On the other hand, we have

\[
\varepsilon \leq S_b(x_q(n), x_q(n), x_p(n)) \leq 2bS_b(x_q(n), x_q(n), x_q(n) + 1) + bS_b(x_p(n), x_p(n), x_q(n) + 1)
\]

\[
\leq 2bS_b(x_q(n), x_q(n), x_q(n) + 1) + 2b^2S_b(x_p(n), x_p(n), x_p(n) + 1)
\]

\[
+ b^3S_b(x_q(n) + 1, x_q(n) + 1, x_p(n) + 1).
\]

Taking the limit supremum as \(n \to +\infty\) in the above inequality. By using (2.22) we obtain

\[
\frac{\varepsilon}{b^3} \leq \limsup_{n \to +\infty} S_b(x_q(n) + 1, x_q(n) + 1, x_p(n) + 1). \tag{2.28}
\]

Taking the limit supremum as \(n \to +\infty\) on each side of (2.26) and using conditions (2.27) and (2.28) together with (F1) and (F3'), we deduce that

\[
F(2b\varepsilon) = F\left(\frac{2b^3\varepsilon}{b^3}\right) \leq F\left(2b^3\limsup_{n \to +\infty} S_b(x_q(n) + 1, x_q(n) + 1, x_p(n) + 1)\right)
\]

\[
\leq F\left(\limsup_{n \to +\infty} S_b(x_q(n), x_q(n), x_p(n))\right)
\]

\[
- \psi\left(\liminf_{n \to +\infty} S_b(x_q(n), x_q(n), x_p(n))\right)
\]

\[
\leq F(2b\varepsilon) - \psi(\varepsilon).
\]

It enforces that \(\psi(\varepsilon) = 0\), which leads to a contradiction. Therefore \(\{x_n\}\) is a Cauchy sequence in \(X\). Since \(X\) is a complete dislocated \(S_o\)-metric space, it follows that there exists \(v \in X\) in which for each \(\varepsilon > 0\), there exists \(N_2 \in \mathbb{N}\) such that

\[
S_b(v, v, x_n) < \varepsilon \tag{2.29}
\]

for all \(n > N_2\). Now, we prove that \(v\) is a fixed point of \(T\). To this end, we show that \(S_b(Tv, Tv, v) = 0\).

We consider the following cases:

**Case 1.** If \(S_b(v, v, x_n) = 0\) for sufficiently large \(n\), then \(v = x_n\). Thus, for sufficiently large \(n\), we can write

\[
S_b(Tv, Tv, v) = S_b(Tx_n, Tx_n, v) \leq 2bS_b(x_{n+1}, x_{n+1}, x_{n+2}) + bS_b(v, v, x_{n+2}).
\]
Letting \( n \to +\infty \) in the above inequality. From (2.22) and (2.29) we get \( S_b(Tv, Tv, v) = 0 \). Thus \( Tv = v \) and \( v \) is a fixed point of \( T \).

**Case 2.** If there exists \( n \geq N_2 \) such that \( S_b(v, v, x_n) > 0 \) and \( S_b(Tv, Tv, Tx_n) = 0 \), then from \((dSb)2\) we have

\[
S_b(Tv, Tv, v) \leq 2bS_b(Tv, Tv, Tx_n) + bS_b(v, v, x_{n+1}) \leq b\varepsilon,
\]

which implies that \( Tv = v \) by virtue of the arbitrariness of \( \varepsilon \).

**Case 3.** If \( S_b(v, v, x_n) > 0 \) and \( S_b(Tv, Tv, Tx_n) > 0 \) for all \( n \geq N_2 \), then using (2.18) we obtain

\[
F(2b^3S_b(Tv, Tv, Tx_n)) \leq F(M_T(v, x_n)) - \psi(M_T(v, x_n)). \tag{2.30}
\]

Thus, by using the hypothesis and taking into account \((dSb)2\), it yields

\[
\max \left\{ S_b(v, v, x_n), S_b(Tv, Tv, Tx_n) \right\} \leq M_T(v, x_n)
\]

\[
\leq \max \left\{ S_b(v, v, x_n), S_b(Tv, Tv, Tx_n), \frac{S_b(x_n, x_n, Tx_n)}{10} \right\}
\]

\[
+ \frac{10}{9} \frac{S_b(Tv, Tv, Tx_n)}{10} + \frac{S_b(Tv, Tv, T^2v)}{18} + \frac{S_b(Tv, Tv, T^3v)}{2}
\]

\[
\leq \max \left\{ S_b(v, v, x_n), S_b(Tv, Tv, Tx_n) \right\}.
\]

Then (2.30) becomes

\[
F(2b^3S_b(Tv, Tv, Tx_n)) \leq F(\max \left\{ S_b(v, v, x_n), S_b(Tv, Tv, Tx_n) \right\}) - \psi(\max \left\{ S_b(v, v, x_n), S_b(Tv, Tv, Tx_n) \right\}).
\]

If \( \max \left\{ S_b(v, v, x_n), S_b(Tv, Tv, Tx_n) \right\} = S_b(Tv, Tv, Tx_n) \), then we have

\[
F(2b^3S_b(Tv, Tv, Tx_n)) \leq F(S_b(Tv, Tv, Tx_n)) - \psi(S_b(Tv, Tv, Tx_n)).
\]

From this it follows that \( 2b^3S_b(Tv, Tv, Tx_n) < S_b(Tv, Tv, Tx_n) \), which is a contradiction. Therefore,

\[
\max \left\{ S_b(v, v, x_n), S_b(Tv, Tv, Tx_n) \right\} = S_b(v, v, x_n)
\]

and (2.30) becomes

\[
F(2b^3S_b(Tv, Tv, Tx_n)) \leq F(S_b(v, v, x_n)) - \psi(S_b(v, v, x_n)) < F(S_b(v, v, x_n)).
\]

Thus, from \((F1)\) we get

\[
S_b(Tv, Tv, Tx_n) < S_b(v, v, x_n). \tag{2.31}
\]

Applying (2.29), (2.31) and \((dSb)2\) we get

\[
S_b(Tv, Tv, v) \leq 2bS_b(Tv, Tv, x_n) + bS_b(v, v, x_{n+1}) < 3b\varepsilon
\]

for sufficiently large \( n \). It enforces that \( Tv = v \) by virtue of the arbitrariness of \( \varepsilon \). Then \( v \) is a fixed point of \( T \).
Next, we show the uniqueness. Indeed, if $v_1$, $v_2$ are two fixed points of $T$ such that $v_1 \neq v_2$, then in view of (2.18) we get

$$F(2b^3 S_b(Tv_1, Tv_1, Tv_2)) \leq F(M_T(v_1, v_2)) - \psi(M_T(v_1, v_2)).$$

(2.32)

According to our assumptions and by using $(dS_b2)$, we find that

$$S_b(v_1, v_1, v_2) \leq M_T(v_1, v_2)$$

$$\leq \max \left\{ S_b(v_1, v_1, v_2), S_b(Tv_1, Tv_1, Tv_2), \frac{S_b(v_2, v_2, Tv_2)}{10}, \frac{S_b(v_1, v_1, Tv_1)}{10}, \frac{S_b(v_2, v_2, Tv_2)}{9} + \frac{S_b(Tv_1, Tv_1, Tv_2)}{18} \right\}$$

$$\leq \max \{ S_b(v_1, v_1, v_2), S_b(Tv_1, Tv_1, Tv_2) \}$$

$$= S_b(v_1, v_1, v_2).$$

Then (2.32) becomes

$$F(2b^3 S_b(v_1, v_1, v_2)) \leq F(S_b(v_1, v_1, v_2)) - \psi(S_b(v_1, v_1, v_2)).$$

From this it follows that $2b^3 S_b(v_1, v_1, v_2) < S_b(v_1, v_1, v_2)$, which is a contradiction. Then $v_1 = v_2$ and so $T$ has a unique fixed point in $X$.

\[ \square \]

**Example 2.4.** Let $X = \{-1, 0, 1\}$. Define the mapping $S_b : X^3 \to \mathbb{R}_+$ by

$$S_b(x, y, z) = \begin{cases} 
\frac{3}{2}, & 0 = x = y \neq z = 1 \text{ or } -1 = x = y \neq z = 1 \\
\frac{10}{16}, & 1 = x = y \neq z \\
0, & x = y = z = -1 \text{ or } 1 \\
\frac{1}{5}, & \text{otherwise}
\end{cases}$$

for all $x, y, z \in X$. It is easy to show that $(X, S_b)$ is a complete dislocated $S_b$-metric space with $b = \frac{3}{2}$. Put $F(\alpha) = \ln \alpha$ ($\alpha > 0$) and $\psi(t) = t$ $(t \geq 0)$. Define $T : X \to X$ by

$$T(x) = \begin{cases} 
0, & x = 1 \\
-1, & x = -1, 0.
\end{cases}$$

Note that $S_b(x, x, y) > 0$ and $S_b(T(x), T(x), T(y)) > 0$ if and only if $x \in \{-1, 0\}$, $y = 1$ or $x = 1$, $y \in \{-1, 0\}$. Also, for each $x, y \in X$ we have $M_T(x, y) = S_b(x, x, y)$ and we find that

$$F(2b^3 S_b(T(x), T(x), T(y))) \leq F(S_b(x, x, y)) - \psi(S_b(x, x, y)) \iff \frac{S_b(x, x, y)}{2b^3 S_b(T(x), T(x), T(y))} \geq S_b(x, x, y).$$

Now, we consider two cases:
**Case 2.1.** Case 1. Let $x \in \{-1, 0\}$ and $y = 1$, then

\[ S_b(x, x, y) = \frac{3}{2}, \]

\[ S_b(T(x), T(x), T(y)) = S_b(-1, -1, 0) = \frac{1}{5}, \]

\[ S_b(0, 0, T(0)) = \frac{1}{5}, \quad S_b(-1, -1, T(-1)) = 0. \]

So, we have

\[ \ln \frac{S_b(x, x, y)}{2b^3 S_b(T(x), T(x), T(y))} = \ln \frac{\frac{3}{2}}{\frac{1}{5}} = \ln \frac{15}{2} = 3.0377 \geq S_b(x, x, y) = \frac{3}{2}. \]

**Case 2.** Let $x = 1$ and $y \in \{-1, 0\}$, then

\[ S_b(x, x, y) = \frac{10}{6}, \]

\[ S_b(T(x), T(x), T(y)) = \frac{1}{5}, \]

\[ S_b(x, x, T(x)) = \frac{10}{6}. \]

So, we have

\[ \ln \frac{S_b(x, x, y)}{2b^3 S_b(T(x), T(x), T(y))} = \ln \frac{\frac{10}{6}}{\frac{1}{5}} = \ln \frac{50}{6} = 3.4369 \geq S_b(x, x, y) = \frac{10}{6}. \]

On the other hand, for all $x, y \in X$ we have

\[ \max \left\{ \frac{S_b(y, y, T(y))}{10}, \frac{S_b(x, x, T(x))}{10}, \frac{S_b(y, y, T(y))}{9}, \frac{S_b(T(x), T(x), T(y))}{18}, \frac{S_b(T(x), T(x), T^2(x))}{2} \right\} \leq \frac{1}{5} = S_b(T(x), T(x), T(y)). \]

Hence, $T$ is a generalized $F$-Suzuki-contraction which satisfies the assumption of Theorem 2.2 and so it has a unique fixed point $-1$.

**References**


