LINEAR FUNCTIONALS CONNECTED WITH STRONG DOUBLE CESARO SUMMABILITY

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Abstract. D. Borwein characterized linear functionals on the normed linear spaces $w_p$ and $W_p$. In this paper we extend his results by presenting definitions for the double strong Cesaro mean. Using these new notions of strongly $p$-Cesaro summable double sequence and strongly $p$-Cesaro summable bivariate function we present extensions of D. Borwein’s results.

1. Introduction

The first definitions and investigations of the convergence of double sequences are usually attributed to F. Pringsheim [12], who studied such sequences and series more than hundred years ago. Pringsheim defined what we call the P limit and gave examples of convergence (P convergence) of double sequences with and without the usual convergence of rows and columns. G. H. Hardy [4], considered in more details the case of convergence of double sequences where, besides the existence of the P limit, rows and columns converge. F. Moricz [6–8] discovered an alternative approach to the Hardy convergence, which significantly influenced the whole theory.

The following notion of convergence for double sequences was presented by Pringsheim in [11]. A double sequence $x = \{x_{nm}\}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim’s sense if for any $\varepsilon > 0$,
there exists \( N_\varepsilon \in \mathbb{N} \) such that \(|x_{nm} - L| < \varepsilon\), whenever \( n, m > N_\varepsilon \). In this case we denote such limit as follow:

\[
P - \lim_{n,m \to \infty} x_{nm} = L.
\]

A classical notion of sequence space is the following:

\[
w_p = \{ x = (x_n) : \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |x_n - \ell|^p = 0 \}.
\]

In [2], D. Borwein extended the sequence space \( w_p \) to the function space \( W_p \), the space of real valued functions \( x \), measurable (in the Lebesque sense) in the interval \((1, \infty)\) for which there is a number \( \ell = \ell_x \) such that

\[
\lim_{T,R \to \infty} \frac{1}{TR} \int_{1}^{T} \int_{1}^{R} |x(t,r) - \ell|^p drdt = 0.
\]

By a linear functional we mean one that is real-valued, additive, homogeneous and continuous. It is to be supposed throughout that \( 1 \leq p < \infty \) and that \( \frac{1}{p} + \frac{1}{q} = 1 \).

2. MAIN RESULTS

We begin to the main results with following definitions:

**Definition 2.1.** Let \( x = \{x_{nm}\} \) be a real double sequence. Then the double sequence \( x \) is said to be strongly \( p\)-Cesaro summable to \( \ell \) if

\[
P - \lim_{N,M \to \infty} \frac{1}{NM} \sum_{n=1}^{N} \sum_{m=1}^{M} |x_{nm} - \ell|^p = 0.
\]

The space of all strongly \( p\)-Cesaro summable double sequences will be denote by \( w^2_p \). Observe that this space is normed by

\[
\|x\|_2 = \sup_{N,M \geq 1} \left( \frac{1}{NM} \sum_{n=1}^{N} \sum_{m=1}^{M} |x_{nm} - \ell|^p \right)^{\frac{1}{p}}.
\]

**Definition 2.2.** Let \( x \) be a real valued bivariate function, measurable (in the Lebesque sense) in the \((1, \infty) \times (1, \infty)\). Then the bivariate function \( x \) is said to be strongly \( p\)-Cesaro summable to \( \ell \) if

\[
\lim_{T,R \to \infty} \frac{1}{TR} \int_{1}^{T} \int_{1}^{R} |x(t,r) - \ell|^p drdt = 0.
\]

The space of all strongly \( p\)-Cesaro summable bivariate functions will be denote by \( W^2_p \). Observe that this space is normed by

\[
\|x\|_2 = \sup_{T \geq 1, R \geq 1} \left( \frac{1}{TR} \int_{1}^{T} \int_{1}^{R} |x(t,r) - \ell|^p drdt \right)^{\frac{1}{p}}.
\]

Given any real double sequence \( \alpha = \{\alpha_{nm}\} \). We define a double sequence \( \{m_{nm}(\alpha, p)\} \) by
Given any real real valued bivariate function \( \alpha(t, r) \) measurable in \((1, \infty) \times (1, \infty)\). We define a double sequence \( \{M_{nm}(\alpha, p)\} \) by

\[
M_{nm}(\alpha, p) = \begin{cases} 
\sup_{2^n \leq v < 2^{n+1}} \{ |v \alpha(v)| \}, & \text{if } p = 1 \\
\left( \frac{1}{2^{n+1}} \sum_{u=2n}^{2^{n+1}-1} \frac{1}{2^{m+1}} \sum_{u=2m}^{2^{m+1}-1} |v \alpha(v)|^q \right)^{\frac{1}{q}}, & \text{if } p > 1.
\end{cases}
\]

Theorem 2.1.

(i) If \( f \) is a linear functional on \( W^2_p \), then there is a real number \( a \) and a real valued bivariate function \( \alpha \), measurable in \((1, \infty) \times (1, \infty)\) such that

\[
f(x) = a \ell + \int_1^\infty \int_1^\infty \alpha(t, r)x(t, r)drdt
\]

for every \( x \in W^2_p \) and

\[
\sum_{n=0}^\infty \sum_{m=0}^\infty M_{nm}(\alpha, p) < \infty.
\]

(ii) If \( a \) is a real number and \( \alpha \) is a real valued bivariate function, measurable in \((1, \infty) \times (1, \infty)\), satisfying \( (2.2) \), then \( (2.1) \) defines a linear function on \( W^2_p \) with

\[
\|f\|_2 \leq |a| + 2 \pi \sum_{n=0}^\infty \sum_{m=0}^\infty M_{nm}(\alpha, p)
\]

and the integral in \( (2.1) \) is absolutely convergent for every \( x \in W^2_p \).

Proof. Let \( L^2_p \) be the linear space of real valued bivariate functions \( x \) measurable in \((1, \infty) \times (1, \infty)\) for which

\[
\int_1^\infty \int_1^\infty |x(t, r)|^pdrdt < \infty,
\]

with norm

\[
\|x\|_{L^2_p} = \left( \int_1^\infty \int_1^\infty |x(t, r)|^pdrdt \right)^{\frac{1}{p}}.
\]

Clearly, if \( x \in L^2_p \), then \( x \in W^2_p \), \( \ell = 0 \) and \( \|x\|_2 = \|x\|_{W^2_p} \leq \|x\|_{L^2_p} \). Consequently the restriction to \( L^2_p \) of the given linear functional \( f \) on \( W^2_p \) is linear on \( L^2_p \). It follows from standard results that there is a real valued bivariate function \( \alpha \), measurable in \((1, \infty) \times (1, \infty)\), such that

\[
f(x) = \int_1^\infty \int_1^\infty \alpha(t, r)x(t, r)drdt
\]
for all $x \in L^2_p$ and either

$$\text{ess.sup } \{|\alpha(t,r)|\} < \infty \quad \text{if } p = 1$$

$$1 \leq t < \infty$$

$$1 \leq r < \infty$$

or

$$\int_1^\infty \int_1^\infty |\alpha(t,r)|^q dr dt < \infty \quad \text{if } p > 1.$$

To show that $\alpha$ must necessarily satisfy (2.2) we consider the cases $p = 1$ and $p > 1$ separately. If $p = 1$, let $M_{nm} = M_{nm}(\alpha, 1)$. There is a measurable set $e_{nm}$ of positive measure $|e_{nm}|$ in the $(2^n, 2^{n+1}) \times (2^m, 2^{m+1})$ such that

$$|tr\alpha(t,r)| > M_{nm} - \frac{1}{2^{n+m}}$$

for all $(t, r) \in e_{nm}$.

Let

$$x(t, r) = \begin{cases} \frac{2^{n+m}}{e_{nm}} \text{sign}(\alpha(t,r)), & \text{if } (t, r) \in e_{nm}, n \leq s, m \leq u \\ 0, & \text{otherwise.} \end{cases}$$

Then $x \in L^2_1$ and so, by (2.3),

$$\|f\|_2 ||x||_2 \geq f(x) = \int_1^\infty \int_1^\infty x(t, r)\alpha(t, r) dr dt$$

$$= \sum_{n=0}^s \sum_{m=0}^u \int_{e_{nm}} 2^{n+m} |\alpha(t,r)| dr dt$$

$$\geq \frac{1}{4} \sum_{n=0}^s \sum_{m=0}^u \frac{1}{|e_{nm}|} \int_{e_{nm}} |tr\alpha(t,r)| dr dt$$

$$\geq \frac{1}{4} \sum_{n=0}^s \sum_{m=0}^u (M_{nm} - \frac{1}{2^{n+m}}).$$

Furthermore, for $2^s \leq T < 2^{s+1} \leq 2^{s+1}, 2^h \leq R < 2^{h+1} \leq 2^{h+1},$

$$\frac{1}{TR} \int_1^T \int_1^R |x(t, r)| dr dt \leq \frac{1}{2^{s+h}} \int_1^{2^{s+1}} \int_1^{2^{h+1}} |x(t, r)| dr dt$$

$$= \frac{1}{2^{s+h}} \sum_{n=0}^z \sum_{m=0}^h \int_{e_{nm}} x(t, r) dr dt$$

$$\leq \frac{1}{2^{s+h}} \sum_{n=0}^z \sum_{m=0}^h 2^{n+m} < 4,$$

and for $T > 2^{s+1}, R > 2^{h+1}$

$$\frac{1}{TR} \int_1^T \int_1^R |x(t, r)| dr dt \leq \frac{1}{2^{s+1}2^{h+1}} \int_1^{2^{s+1}} \int_1^{2^{h+1}} |x(t, r)| dr dt < 1.$$
Hence \( \|x\|_2 < 4 \) and so, by (2.4),
\[
4\|f\|_2 + \frac{1}{4} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{2^{m+n}} = 4\|f\|_2 + 1 \geq \frac{1}{4} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm},
\]
which establishes (2.2) in this case. If \( p > 1 \), let \( M_{nm} = M_{nm}(\alpha, p) \) and let
\[
x(t, r) = \begin{cases} \frac{(tr)^y}{\alpha(t, r)} \frac{1}{z} \text{sign}(\alpha(t, r)), & \text{if } 2^n \leq t < 2^{n+1} \leq 2^{r+1}; \\
0, & \text{otherwise.}
\end{cases}
\]
Then \( x \in L^2_p \) and so, by (2.3),
\[
f(x) = \int_1^{2^{r+1}} \int_1^{2^{r+1}} |\alpha(t, r)x(t, r)| \, dr \, dt = \sum_{n=0}^{z} \sum_{m=0}^{u} \int_1^{2^{n+1}} \int_2^{2^{m+1}} |\alpha(t, r)x(t, r)| \, dr \, dt
\]
\[
= \sum_{n=0}^{z} \sum_{m=0}^{u} M_{nm}. \quad (2.5)
\]
Furthermore, for \( 2^z \leq T < 2^{z+1} \leq 2^h, 2^h \leq R < 2^{h+1} \leq 2^u \),
\[
\frac{1}{TR} \int_1^T \int_1^R |x(t, r)|^p \, dr \, dt \leq \frac{1}{2^{z+h}} \int_1^{2^{z+1}} \int_1^{2^{h+1}} |x(t, r)|^p \, dr \, dt
\]
\[
= \frac{1}{2^{z+h}} \sum_{n=0}^{z} \sum_{m=0}^{h} \int_1^{2^{n+1}} \int_1^{2^{m+1}} |x(t, r)|^p \, dr \, dt
\]
\[
\leq \frac{2^p}{2^{z+h}} \sum_{n=0}^{z} \sum_{m=0}^{h} 2^{n+m} < 2^{2p+2},
\]
and for \( T > 2^{z+1}, R > 2^{h+1} \)
\[
\frac{1}{TR} \int_1^T \int_1^R |x(t, r)|^p \, dr \, dt \leq \frac{1}{2^{z+1}2^{h+1}} \int_1^{2^{z+2}} \int_1^{2^{h+2}} |x(t, r)|^p \, dr \, dt < 4^p.
\]
Hence \( \|x\|_2 < 2^{2+\frac{p}{2}} \) and so, by (2.5),
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm} \leq 2^{2+\frac{p}{2}} \|f\|_2,
\]
which established (2.2) in this case.

Suppose now \( p \geq 1, M_{nm} = M_{nm}(\alpha, p) \) and \( x \in W^2_p \). Then by Hölder inequality
\[
\int \int |\alpha(t, r)x(t, r)| \, dr \, dt = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_1^{2^{n+1}} \int_2^{2^{m+1}} |\alpha(t, r)x(t, r)| \, dr \, dt
\]
\[
\leq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm} \left( 2^p(1-\frac{1}{p})(n+m) \int_1^{2^{n+1}} \int_2^{2^{m+1}} |x(t, r)|^p \, dr \, dt \right)^{\frac{1}{p}}
\]
\[
\leq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm} \left( 2^{-(n+m)} \int_1^{2^{n+1}} \int_2^{2^{m+1}} |x(t, r)|^p \, dr \, dt \right)^{\frac{1}{p}}.
\]
\[
\leq 2^{\frac{2}{p}} \|x\|_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm}. \tag{2.6}
\]

It follows that
\[
\int_1^\infty \int_1^\infty |\alpha(t,r)x(t,r)| dr dt < \infty
\]
whenever \(x \in W^2_p\), and in particular since the characteristic function of \((1, \infty) \times (1, \infty)\) is in \(W^2_p\), that
\[
\int_1^\infty \int_1^\infty |\alpha(t,r)| dr dt < \infty.
\]
Suppose next that \(x \in W^2_p\) and \(\ell = \ell_x\). Let
\[
y(t,r) = x(t,r) - \ell
\]
and so, by (2.3),
\[
\int_1^\infty \int_1^\infty |\alpha(t,r)x(t,r)| dr dt < \infty.
\]
Then \(y \in W^2_p\), \(y_{nm} \in L^2_p\) and
\[
\|y_{nm} - y\|_2 = \sup_{T \geq n, R \geq m} \left( \frac{1}{TR} \int_1^T \int_1^R |x(t,r) - \ell|^p \right)^{\frac{1}{p}} = o(1) \text{ as } n, m \to \infty.
\]
But
\[
|f(y_{nm} - y)| = |f(y_{nm}) - f(y)| \leq \|y_{nm} - y\|_2 \|f\|_2,
\]
and so, by (2.3),
\[
f(y) = P - \lim_{n,m \to \infty} f(y_{nm}) = P - \lim_{n,m \to \infty} \int_1^\infty \int_1^\infty y(t,r) \alpha(t,r) dr dt
\]
\[
= \int_1^\infty \int_1^\infty x(t,r) \alpha(t,r) dr dt - \ell \int_1^\infty \int_1^\infty \alpha(t,r) dr dt.
\]
Since both integrals on the right hand side have been shown to be absolutely convergent. Taking \(\delta\) to be characteristic function of \((1, \infty) \times (1, \infty)\) we see that
\[
f(x) = f(y + \ell \delta) f(y) + \ell f(\delta) = \int_1^\infty \int_1^\infty x(t,r) \alpha(t,r) dr dt + a \ell
\]
where \(a = f(\delta) - \int_1^\infty \int_1^\infty \alpha(t,r)\). This completes the proof of part (i).

(ii) It follows from (2.6) that if \(x \in W^2_p\), \(\ell = \ell_x\) and \(M_{nm} = M_{nm}(\alpha, p)\), then
\[
|f(x)| = \left| \int_1^\infty \int_1^\infty x(t,r) \alpha(t,r) dr dt + a \ell \right| \leq \|x\|_2 2^{\frac{2}{p}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm} + |a| \|. \tag{2.7}
\]
Further, by Minkowski’s inequality
\[
\left( 1 - \frac{1}{TR} \right)^{\frac{1}{p}} \|\ell\| \leq \left( \frac{1}{TR} \int_1^T \int_1^R |x(t,r) - \ell|^p dr dt \right)^{\frac{1}{p}} + \left( \frac{1}{TR} \int_1^T \int_1^R |x(t,r)|^p dr dt \right)^{\frac{1}{p}}
\]
and the first term on the right hand side is \(o(1)\). Hence \(|\ell| \leq \|x\|_2\) and consequently, by (2.7),
\[
|f(x)| \leq \|x\|_2 \left| a \right| + 2^{\frac{2}{p}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm}
\]
for every \( x \in W_p^2 \). The additive and homogenous functional \( f \) defined by (2.1) is therefore also continuous on \( W_p^2 \) and

\[
|f(x)| \leq |a| + 2^{2p} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm}.
\]

Finally, by (2.6), the integral in (2.1) is absolutely convergent. Thus the proof is completed. \( \square \)

**Theorem 2.2.**

(i) If \( f \) is a linear functional on \( w_p^2 \), then there is a real number \( a \) and a real double sequence \( \alpha = \{ \alpha_{nm} \} \) such that

\[
f(x) = a \ell + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{nm} x_{nm}
\]

for every \( x = \{ x_{nm} \} \in w_p^2 \) and

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m_{nm}(\alpha, p) < \infty.
\]

(ii) If \( a \) is a real number and \( \alpha = \{ \alpha_{nm} \} \) is a real double sequence satisfying (2.9), then (2.8) defines a linear function on \( w_p^2 \) with

\[
\|f\|_2 \leq |a| + 2^{2p} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m_{nm}(\alpha, p)
\]

and the series in (2.8) is absolutely convergent for every \( x = \{ x_{nm} \} \in w_p^2 \).

**Proof.** Given any real double sequence \( x = \{ x_{nm} \} \), define a bivariate function \( x^* \) by

\[
x^*(t, r) = x_{nm} \text{ for } n < t \leq n + 1; m < r \leq m + 1, n = 1, 2, 3, ..., m = 1, 2, 3, ....
\]

It is easily verified that this defines a one to one correspondence between \( w_p^2 \) and a linear subspace \((W_p^2)^*\) of \( W_p^2 \) such that

\[
\ell_{x^*} = \ell_x \text{ and } \|x^*\|_2 \leq \|x\|_2 \leq 2^{2p} \|x^*\|_2.
\]

Hence given a linear functional on \( W_p^2 \), the functional \( f^* \) defined by

\[
f^*(x^*) = f(x)
\]

is linear on \((W_p^2)^*\). Consequently, by the Hahn-Banach theorem and Theorem 2.1, there is a real number \( a \) and a real valued bivariate function \( \alpha^* \), integrable over \((1, \infty) \times 1, \infty)\), such that

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm}(\alpha^*, p) < \infty
\]

and, for every \( x \in w_p^2 \),

\[
f(x) = f^*(x^*) = a \ell_{x^*} + \int_1^{\infty} \int_1^{\infty} \alpha^*(t, r)x^*(t, r) dr dt = a \ell_x + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{nm} x_{nm}
\]
where $\alpha_{nm} = \int_n^{n+1} \int_m^{m+1} \alpha^*(t,r) dr dt$. Furthermore, for $\alpha = \{\alpha_{nm}\}$,
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m_{nm}(\alpha, p) \leq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} M_{nm}(\alpha^*, p);
\]
and this completes the proof of (i).

(ii) If $x = \{x_{nm}\} \in w_p^2$, $m_{nm} = m_{nm}(\alpha, p)$ and $\ell = \ell_x$ then by Hölder’s and Minkowski’s inequalities, as in the proof of (ii) of Theorem 2.1,
\[
f(x) = a\ell + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{nm} x_{nm} \leq |a\ell| + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\alpha_{nm} x_{nm}|
\]
\[
\leq |a\ell| + 2^\frac{2}{p} \|x\|_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m_{nm} \leq \|x\|_2 \left(|a| + 2^\frac{2}{p} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m_{nm}\right).
\]
The functional $f$ defined by (2.8) is therefore linear on $w_p^2$,
\[
\|f\|_2 \leq |a| + 2^\frac{2}{p} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m_{nm}
\]
and the series in (2.8) absolutely convergent. This completes the proof. \qed

References