CONTROLLED ∗-G-FRAMES AND ∗-G-MULTIPLIERS IN HILBERT PRO-C∗-MODULES

ZAHRA AHMADI MOOSAVI1,* AND AKBAR NAZARI2

1Department of Mathematics Faculty of Mathematics and computer Shahid Bahonar University of Kerman, 76169-14111, Kerman, Iran

2Department of Mathematics Faculty of Mathematics and computer Shahid Bahonar University of Kerman, 76169-14111, Kerman, Iran

*Corresponding author: nazari@uk.ac.ir

Abstract. A generalization of multiplier, controlled g-frames and g-Bessel sequences to ∗-g-frames and ∗-g-Bessel sequences in Hilbert pro-C∗-modules is presented. It is demonstrated that controlled ∗-g-frames are equivalent to ∗-g-frames in Hilbert pro-C∗-modules.

1. Introduction

Frame theory is an application of harmonic analysis. This theory has been rapidly generalized to Hilbert spaces and Hilbert C∗-modules. In 2005, Sun [22] introduced the notion of g-frames as a generalization of frames for bounded operators on Hilbert spaces. Frank-Larson [5] have extended the theory for elements of C∗-algebras and (finitely or countably generated) Hilbert C∗-modules have been considered in [1].

It is well known that Hilbert C∗-modules are a generalization of Hilbert spaces where the inner product takes values in a C∗-algebra rather than in the field of complex numbers. The theory of Hilbert C∗-modules

Received 2018-04-13; accepted 2018-06-22; published 2019-01-04.

2010 Mathematics Subject Classification. 42C15, 46L08.

Key words and phrases. Hilbert pro-C∗-modules; ∗-g-frames; ∗-g-Bessel sequences; controlled ∗-g-frames; (C,C′)-controlled ∗-g-frames.

©2019 Authors retain the copyrights of their papers, and all open access articles are distributed under the terms of the Creative Commons Attribution License.
has applications in the study of locally compact quantum groups, complete maps between $C^*$-algebras, non-commutative geometry and KK-theory. Not all properties of Hilbert spaces hold in Hilbert $C^*$-modules. For instance, the Riesz representation theorem for continuous linear functionals on Hilbert spaces can not be extended to Hilbert $C^*$-modules [23] and there exist closed subspaces in Hilbert $C^*$-modules that have no orthogonal complement [16]. Moreover, as known, every bounded operator on a Hilbert space has an adjoint whereas there are bounded operators on Hilbert $C^*$-modules which do not have this property [17]. So, it is to be expected that frames and $*$-frames in Hilbert $C^*$-modules are more complicated than those in Hilbert spaces. The properties of $g$-frames for Hilbert $C^*$-modules have been widely investigated in the literature (see [1,5,12,25], and the references therein).

The paper is organized as follows. In the next section, we give a brief survey of the fundamental definitions and notations of Hilbert pro-$C^*$-modules.

Section 3 is devoted to investigating $*$-$g$-frames with $\mathcal{A}$-valued bounds and analyzing their elementary properties. In Section 4 we define the concept of controlled $*$-$g$-frames and we show that a controlled $*$-$g$-frame is equivalent to a $*$-$g$-frame in Hilbert pro-$C^*$-modules. Finally, in section 5 we define multipliers of controlled $*$-$g$-frame operators in Hilbert pro-$C^*$-modules.

2. Preliminaries

In this section, we recall some of the basic definitions and properties of pro-$C^*$-algebras and Hilbert modules over them [7,15,18].

A pro-$C^*$-algebra is a complete Hausdorff complex topological $*$-algebra $\mathcal{A}$ whose topology is determined by its continuous $C^*$-seminorms in the sense that a net $\{a_\lambda\}$ converges to 0 iff $\rho(a_\lambda) \to 0$ for any continuous $C^*$-seminorm $\rho$ on $\mathcal{A}$ and we have:

\begin{align*}
(1) \quad & \rho(ab) \leq \rho(a)\rho(b); \\
(2) \quad & \rho(a^*a) = \rho(a)^2;
\end{align*}

for all $C^*$-seminorms $\rho$ on $\mathcal{A}$ and $a, b \in \mathcal{A}$.

If the topology of pro-$C^*$-algebra is determined by only countably many $C^*$-seminorms, then it is called a $\sigma$-$C^*$-algebra.

Let $\mathcal{A}$ be a unital pro-$C^*$-algebra with unit $1_\mathcal{A}$ and let $a \in \mathcal{A}$. Then spectrum $\text{sp}(a)$ of $a \in \mathcal{A}$ is the set $\{\lambda \in \mathbb{C} : \lambda 1_\mathcal{A} - a \text{ is not invertible}\}$. If $\mathcal{A}$ is not unital, then the spectrum is taken with respect to its unitization $\hat{\mathcal{A}}$.

If $\mathcal{A}^+$ denotes the set of all positive elements of $\mathcal{A}$, then $\mathcal{A}^+$ is a closed convex $C^*$-seminorms on $\mathcal{A}$. We denote by $S(\mathcal{A})$, the set of all continuous $C^*$-seminorms on $\mathcal{A}$.

**Example 2.1.** Every $C^*$-algebra is a pro-$C^*$-algebra.
Example 2.2. A sub-closed \(\ast\)-algebra of a pro-C\(^\ast\)-algebra is a pro-C\(^\ast\)-algebra.

Proposition 2.1 ([6]). Let \(\mathcal{A}\) be a unital pro-C\(^\ast\)-algebra with an identity \(1_\mathcal{A}\). Then for any \(\rho \in \mathcal{S}(\mathcal{A})\), we have:

1. \(\rho(a) = \rho(a^\ast)\) for all \(a \in \mathcal{A}\);
2. \(\rho(1_\mathcal{A}) = 1\);
3. If \(a, b \in \mathcal{A}^+\) and \(a \leq b\), then \(\rho(a) \leq \rho(b)\);
4. If \(1_\mathcal{A} \leq b\), then \(b\) is invertible and \(b^{-1} \leq 1_\mathcal{A}\);
5. If \(a, b \in \mathcal{A}^+\) are invertible and \(0 \leq a \leq b\), then \(b^{-1} \leq a^{-1}\);
6. If \(a, b, c \in \mathcal{A}\) and \(a \leq b\) then \(c^\ast ac \leq c^\ast bc\);
7. If \(a, b \in \mathcal{A}^+\) and \(a^2 \leq b^2\), then \(0 \leq a \leq b\).

Definition 2.1. A pre-Hilbert module over pro-C\(^\ast\)-algebra \(\mathcal{A}\), is a complex vector space \(E\) which is also a left \(\mathcal{A}\)-module compatible with the complex algebra structure, equipped with an \(\mathcal{A}\)-valued inner product \(\langle \cdot, \cdot \rangle : E \times E \to \mathcal{A}\) which is \(\mathbb{C}\)-and \(\mathcal{A}\)-linear in its first variable and satisfies the following conditions:

1. \(\langle x, y \rangle^* = \langle y, x \rangle\);
2. \(\langle x, x \rangle \geq 0\);
3. \(\langle x, x \rangle = 0\) iff \(x = 0\);

for every \(x, y \in E\). We say \(E\) is a Hilbert \(\mathcal{A}\)-module (or Hilbert pro-C\(^\ast\)-module over \(\mathcal{A}\)) if \(E\) is complete with respect to the topology determined by the family of seminorms

\[
\rho_E(x) = \sqrt{\rho(\langle x, x \rangle)} \quad x \in E, \rho \in \mathcal{S}(\mathcal{A}).
\]

Let \(E\) be a pre-Hilbert \(\mathcal{A}\)-module. By [6], for \(\rho \in \mathcal{S}(\mathcal{A})\) and for all \(x, y \in E\), the following Cauchy-Bunyakovskii inequality holds:

\[
\rho(\langle x, y \rangle)^2 \leq \rho(\langle x, x \rangle)\rho(\langle y, y \rangle).
\]

Consequently, for each \(\rho \in \mathcal{S}(\mathcal{A})\), we have:

\[
\rho_E(ax) \leq \rho(a)\rho_E(x), \quad a \in \mathcal{A}, x \in E.
\]

Let \(\mathcal{A}\) be a pro-C\(^\ast\)-algebra and \(E\) and \(F\) be two Hilbert \(\mathcal{A}\)-modules. An \(\mathcal{A}\)-module map \(T : E \to F\) is said to bounded if for each \(\rho \in \mathcal{S}(\mathcal{A})\), there is \(C_\rho > 0\) such that:

\[
\rho_F(Tx) \leq C_\rho \rho_E(x) \quad (x \in E),
\]

where \(\rho_E\), respectively \(\rho_F\), are continuous seminorms on \(E\), respectively \(F\). A bounded \(\mathcal{A}\)-module map from \(E\) to \(F\) is called an operators from \(E\) to \(F\). We denote the set of all operators from \(E\) to \(F\) by \(\text{Hom}_\mathcal{A}(E, F)\), and we set \(\text{Hom}_\mathcal{A}(E, F) = \text{End}_\mathcal{A}(E)\).
Proposition 2.2. Let $T^* \in \mathsf{Hom}_A(E,F)$. We say $T$ is adjointable if there exists an operator $T^* \in T \in \mathsf{Hom}_A(F,E)$ such that:

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

holds for all $x \in E, y \in F$.

We denote by $\mathsf{Hom}^*_A(E,F)$, the set of all adjointable operators from $E$ to $F$ and $\mathsf{End}^*_A(E) = \mathsf{Hom}^*_A(E,E)$.

Proposition 2.3 ([6]). Let $T : E \to F$ and $T^* : F \to E$ be two maps such that the equality

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

holds for all $x \in E, y \in F$. Then $T \in \mathsf{Hom}^*_A(E,F)$.

It is easy to see that for any $\rho \in S(A)$, the map defined by:

$$\hat{\rho}_{E,F}(T) = \sup \{ \rho_F(T(x) : x \in E, \rho_E(x) \leq 1}, \quad T \in \mathsf{Hom}_A(E,F),$$

is a seminorm on $\mathsf{Hom}_A(E,F)$.

Definition 2.2. Let $E$ and $F$ be two Hilbert modules over pro-$C^*$-algebra $A$. Then the operator $T : E \to F$ is called uniformly bounded (below), if there exists $C > 0$ such that:

$$\overline{\rho}_F(Tx) \leq C \overline{\rho}_E(x). \quad (2.1)$$

$$C \overline{\rho}_E(x) \leq \overline{\rho}_F(Tx)) \quad (2.2)$$

The number $C$ in (2.1) is called an upper bound for $T$ and we set:

$$\|T\|_\infty = \inf \{ C : C \text{ is an upper bound for } T \}.$$

Clearly, in this case we have:

$$\hat{\rho}(T) \leq \|T\|_\infty, \quad \forall \rho \in S(A).$$

Let $T$ be an invertible element in $\mathsf{End}^*_A(E)$ such that both are uniformly bounded. Then by [2, Proposition 3.2], for each $x \in E$ we have the inequality

$$\|T^{-1}\|_\infty^{-2} \langle x, x \rangle \leq \langle Tx, Tx \rangle \leq \|T\|_\infty^2 \langle x, x \rangle. \quad (2.3)$$

The following proposition will be used in the next section.

Proposition 2.4 ([6]). Let $T$ be an uniformly bounded below operator in $\mathsf{Hom}_A(E,F)$. then $T$ is closed(range) and injective.
3. *-G-frames in Hilbert pro-$C^*$-modules

Throughout this section $A$ is a pro-$C^*$-algebra, $U$ and $V$ are two Hilbert $A$-modules. Also $\{V_j\}_{j \in J}$ is a countable sequence of closed submodules of $V$.

**Definition 3.1.** A sequence $\Lambda = \{A_j \in \text{Hom}_A^*(U, V_j)\}_{j \in J}$ is called a *-g-frame for $U$ with respect to $\{V_j\}_{j \in J}$ if

$$C(f,f)C^* \leq \sum_{j \in J} \langle A_j f, A_j f \rangle \leq D(f,f)D^*$$

for all $f \in U$ and strictly nonzero elements $C, D \in A$.

The number $C$ and $D$ are called *-g-frame bounds for $\Lambda$. The *-g-frame is called tight if $C = D$ and a Parseval if $C = D = 1$. If in the above we only have the upper bound, then $\Lambda$ is called a *-g-Bessel sequence. Also if for each $j \in J, V_j = V$, we call $\Lambda$ a *-g-frame for $U$ with respect to $V$.

We mentioned that the set of all g-frames in Hilbert pro-$C^*$-modules are a subset of the family of *-g-frames. To illustrate this, let $\Lambda = \{A_j\}_{j \in J}$ be a g-frame for $U$ with respect to $\{V_j\}_{j \in J}$. Note that for $f \in U$,

$$(\sqrt{C})1_A (f,f)(\sqrt{C})1_A \leq \sum_{j \in J} \langle A_j f, A_j f \rangle (\sqrt{D})1_A (f,f)(\sqrt{D})1_A$$

Therefore, every g-frame for $U$ with real bounds $C$ and $D$ is a *-g-frame for $U$ with $A$-valued *-g-frame bounds $(\sqrt{C})1_A$ and $(\sqrt{D})1_A$.

**Example 3.1.** Let $\ell^2(A)$ be the set of all sequences $(\alpha_n)_{n \in \mathbb{N}}$ of elements of a pro-$C^*$-algebra $A$ such that the series $\sum_{i \in \mathbb{N}} a_i a_i^*$ is convergent in $A$. Then, by [2, Example 3.2], $\ell^2(A)$ is a Hilbert module over $A$ with respect to pointwise operations and inner product defined by:

$$\langle (a_i)_{i \in \mathbb{N}}, (b_i)_{i \in \mathbb{N}} \rangle = \sum_{i \in \mathbb{N}} a_i b_i^*.$$

Let $a = (a_i)_{i \in \mathbb{N}}$ and $b = (b_i)_{i \in \mathbb{N}}$ in $\ell^2(A)$. We define $ab = (a_i b_i)_{i \in \mathbb{N}}$ and $\overline{a}(a) = \sqrt{\rho((a, a))}$ and $a^* := \{\overline{a_i}\}_{i \in \mathbb{N}}$ and $\langle a, b \rangle = ab^* = \sum_{i \in \mathbb{N}} a_i b_i^*$.

Now, let $j \in J := \mathbb{N}$ and define $f_j \in \ell^2(A)$ by $f_j = \{f_j^i\}_{i \in \mathbb{N}}$ such that

$$f_j^i = \begin{cases} \frac{1}{i} 1_A & i = j; \\ 0 & i \neq j, \forall j \in \mathbb{N}. \end{cases}$$

Set $\Lambda_j : \ell^2(A) \to A$ by $\Lambda_j(f_j) = \langle U, f_j \rangle$ for any $U \in \ell^2(A)$. We see that

$$\sum_{j \in J} \langle \Lambda_j(f_j), \Lambda_j(f_j) \rangle \leq \langle U, U \rangle.$$ 

Thus $\{\Lambda_j\}_{j \in J}$ is a *-g-Bessel sequence.
Definition 3.2. Let Λ = \{Λ_j ∈ End_A^*(U,V_j)\}_{j ∈ J} be a \ast-g-frame for U with respect to \{V_j\}_{j ∈ J} with bounds C and D. We define the corresponding \ast-g-frame transform as follows:

\[ T_Λ : U → ⊕_{j ∈ J} V_j , \quad T_Λ f = \{Λ_j f : j ∈ J\} , \quad \text{for all } f ∈ U. \]

Since Λ is a \ast-g-frame, hence for each f ∈ U we have:

\[ C (f,f) C^* ≤ \sum_{j ∈ J} (Λ_j f, Λ_j f) ≤ D (f,f) D^*, \]

So \( T_Λ \) is well-defined. Also for any \( ρ ∈ S(Λ) \) and \( f ∈ U \) the following inequality is obtained:

\[ ρ(C)^2 \rho_U(f) ≤ ρ ⊕_{j ∈ J} V_j(T_Λ f) ≤ ρ(D)^2 \rho_U(f). \]

From the above, it follows that the \ast-g-frame transform is an uniformly bounded below operator in End_A(U, ⊕_{j ∈ J} V_j).

Thus by Proposition 2.4, \( T_Λ \) is closed and injective.

Now, we define the synthesis operator for \ast-g-frame Λ as follows:

\[ T_Λ^* : ⊕_{j ∈ J} V_j → U , \quad T_Λ^*(y_j) = \sum_{j ∈ J} Λ_j^*(y_j), \quad (3.1) \]

where \( Λ_j^* \) is the adjoint operator of \( Λ_j \).

Proposition 3.1. The synthesis operator defined by (3.1) is well-defined, uniformly bounded and the adjoint of the transform operator.

Proof. Since \( Λ = \{Λ_j : j ∈ J\} \) is a \ast-g-frame for U with respect to \{V_j\}_{j ∈ J}, there exist \( C, D ∈ A \) such that for any \( f ∈ U \),

\[ C (f,f) C^* ≤ \sum_{j ∈ J} (Λ_j f, Λ_j f) ≤ D (f,f) D^*. \]

Let \( I \) be an arbitrary finite subset of \( J \). Using the Cauchy-Bunyakovskii inequality and [24, Lemma 2.2], for any \( ρ ∈ S(Λ) \) and \( (y_j)_{j ∈ J} \) we have:

\[ \rho(\sum_{j ∈ I} Λ_j^*(y_j)) = \sup\{ρ(\sum_{j ∈ I} Λ_j^*(y_j), f) : f ∈ U , \ \rho(f) ≤ 1\} \]

\[ = \sup\{ρ(\sum_{j ∈ I} (y_j, Λ_j f)) : f ∈ U , \ \rho(f) ≤ 1\} \]

\[ ≤ \sup_{\rho(f) ≤ 1} \left(ρ(\sum_{j ∈ I} (y_j, y_j))\right)^{1/2} \left(ρ(\sum_{j ∈ I} (Λ_j f, Λ_j f))\right)^{1/2} \]

\[ ≤ \sup_{\rho(f) ≤ 1} ρ(DD^*)^{1/2} \rho(\rho(\sum_{j ∈ I} (y_j, y_j)))^{1/2} \]

\[ ≤ \left(ρ(D) (\rho(\sum_{j ∈ I} (y_j, y_j)))^{1/2}\right). \]
Now, since the series $\sum_{j \in J} \langle y_j, y_j \rangle$ converges in $A$, the above inequality shows that $\sum_{j \in J} \Lambda_j^* (y_j)$ is convergent. Hence $T_A^*$ is well-defined. On the other hand, for any $f \in U$ and $(y_j)_{j \in J} \in \bigoplus_{j \in J} V_j$, we have:

$$\langle T_A(f), (y_j)_j \rangle = \langle (\Lambda_j f)_j, (y_j)_j \rangle = \sum_{j \in J} \langle \Lambda_j f, y_j \rangle = \sum_{j \in J} (f, \Lambda_j^* y_j) = \langle f, \sum_{j \in J} \Lambda_j^* y_j \rangle = \langle f, T_A^*(y_j)_{j \in J} \rangle.$$ 

Therefore by Proposition 2.2 it follows that the synthesis operator is the adjoint of the transform operator. Also, for any $\rho \in S(A)$ we have:

$$\rho_U(T_A^*(y)) \leq \rho(D) \rho_{\bigoplus_{j \in J} V_j} (y), \quad y = (y_j)_j \in \bigoplus_{j \in J} V_j.$$ 

Hence the synthesis operator is uniformly bounded. \hfill \Box

Let $\Lambda = \{ \Lambda_j , \ j \in J \}$ be a $\ast$-g-frame for $U$ with respect to $\{ V_j \}_{j \in J}$. Define the corresponding $\ast$-g-frame operator $S_\Lambda$ as follows:

$$S_\Lambda = T_A^* T_A : U \to U \quad S_\Lambda(f) = \sum_{j \in J} \Lambda_j^* \Lambda_j f.$$ 

Since $S_\Lambda$ is a combination of two bounded operators, it is a bounded operator.

**Theorem 3.1.** Let $\Lambda = \{ \Lambda_j \}_{j \in J}$ be a $\ast$-g-frame for $U$ with respect to $\{ V_j \}_{j \in J}$ and with bounds $C, D$. Then $S_\Lambda$ is an invertible positive operator. Also it is a self-adjoint operator such that:

$$C I_U C^* \leq S_\Lambda \leq D I_U D^*.$$ 

Here $I_U$ is the identity function on $U$.

**Proof.** According to the definition of the transform operator, for any $f \in U$ we can write:

$$\langle T_\Lambda(f), T_\Lambda(f) \rangle = \langle \{ \Lambda_j f \}_{j \in J}, \{ \Lambda_j f \}_{j \in J} \rangle = \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle.$$ 

Since $\Lambda$ is a $\ast$-g-frame for $U$ with bounds $C$ and $D$, for each $f \in U$ it follows that

$$C \langle f, f \rangle C^* \leq \langle T_\Lambda(f), T_\Lambda(f) \rangle \leq D \langle f, f \rangle D^*.$$ 

On the other hand,

$$\langle S_\Lambda(f), f \rangle = \langle T_A^* T_A(f), f \rangle = \langle T_A(f), T_A(f) \rangle = \langle f, T_A^* T_A(f) \rangle = \langle f, S_\Lambda(f) \rangle.$$ 

Consequently, $S_\Lambda$ is a self-adjoint operator. Also, for any $f \in U$, we obtain
\[ C\langle f, f \rangle C^* \leq \langle S_\Lambda(f), f \rangle \leq D\langle f, f \rangle D^*. \]

It follows that \( * \)-g-frame operator is positive and (3.2) also holds. Moreover, since \( S_\Lambda \) is one-to-one it follows that \( S_\Lambda \) is invertible. \( \Box \)

According to (3.2) and Proposition 2.1 we have the following Lemma

**Lemma 3.1.**

\[ D^{-1}I_U(D^{-1})^* \leq S_\Lambda^{-1} \leq C^{-1}I_U(C^{-1})^*. \]

Hence the \( * \)-g-frame operator and its inverse belong to \( \text{End}_A(U) \).

**Theorem 3.2.** Let \( \{\Lambda_j \in \text{End}_A(U, V_j)\}_{j \in J} \) and \( \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \) converge in the semi-norm for \( f \in U \). Then \( \Lambda = \{\Lambda_j\}_{j \in J} \) is a \( * \)-g-frame for \( U \) with respect to \( \{V_j\}_{j \in J} \) if and only if there are two strictly nonzero elements \( C, D \in A \) such that for every \( f \in U \),

\[
(\rho(C^{-1})^{-1} \rho(\langle f, f \rangle) \rho(C^{*-1})^{-1}) \leq \rho(D) \rho(\langle f, f \rangle) \rho(D^*). \quad (3.3)
\]

**Proof.** If \( \{\Lambda_j \in \text{End}_A(U, V_j)\}_{j \in J} \) is a \( * \)-g-frame for \( U \) with respect to \( \{V_j\}_{j \in J} \), then

\[
(\langle f, f \rangle) \leq C^{-1}(\sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle)(C^{*-1})^{-1}
\]

and

\[
(\sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle) \leq D\langle f, f \rangle D^*.
\]

Therefore, by Proposition 2.1,

\[
(\rho(C^{-1})^{-1} \rho(\langle f, f \rangle) \rho(C^{*-1})^{-1}) \leq \rho(D) \rho(\langle f, f \rangle) \rho(D^*). \quad (3.4)
\]

For the converse, let (3.3) hold. Then we define a linear operator as follows:

\[
M : U \to \bigoplus_{j \in J} V_j, \quad M(f) = \{\Lambda_j f\}_{j \in J}, \quad \forall f \in U,
\]

\[
\langle Mf, Mf \rangle = \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle, \quad \forall f \in U.
\]

Hence, by (3.3), we have

\[
\rho_U(M(f)) \leq \rho(D)^{\frac{1}{2}} \rho_U(f) \rho(D^*)^{\frac{1}{2}}.
\]
This shows that $M$ is uniformly bounded. We write $M^*M = K$. Then $(M(f), M(f)) = (M^*M(f), f) = (K(f), f)$. Therefore, $K$ is positive. As, $K^* = (M^*M), K$ is self-adjoint. On the other hand,

$$
\langle K^{\frac{1}{2}}f, K^{\frac{1}{2}}f \rangle = \langle Kf, f \rangle = \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle.
$$

Now, according to Proposition 2.4 and (3.3), $K^{\frac{1}{2}}$ is invertible and uniformly bounded; therefore, by [2, Proposition 3.2], we have:

$$
\|K^{-\frac{1}{2}}\|^{-1}_\infty \langle f, f \rangle \leq \|K^{\frac{1}{2}}\| \langle f, f \rangle \|K^{\frac{1}{2}}\|_\infty
$$

Hence $\{\Lambda_j\}_{j \in J}$ is a $*$-g-frame. 

4. CONTROLLED $*$-G-FRAMES IN HILBERT PRO-\textit{C}*-MODULES

In this section, we define the concept of multipliers for $*$-g-Bessel sequences and we show that controlled $*$-g-frames are equivalent to $*$-g-frames.

Let $\mathcal{A}$ be a pro-$C^*$-algebra, $U$ and $V$ be two Hilbert $\mathcal{A}$-modules. also, let $\{V_j\}_{j \in J}$ be a countable sequence of closed submodules of $V$, $L(U, V)$ and $L(U)$ the collection of all bounded linear operators from $U$ into $V$ and $U$ respectively. $gl(U)$ the set of all bounded operators with a bounded inverse and $gl^+(U)$ be the set of positive operators in $gl(U)$.

**Proposition 4.1.** Let $\Lambda = \{\Lambda_j \in L(U, V_j) : j \in J\}$ and $\theta = \{\theta_j \in L(U, V_j) : j \in J\}$ be $*$-g-Bessel sequences with bounds $B_\Lambda$ and $B_\theta$. If for $m = \{m_j\}_{j \in J} \subseteq \ell^\infty(R)$, the operator

$$
M = M_{m, \Lambda, \theta} : U \to U
$$

$$
M(f) = \sum_j m_j \Lambda_j^* \theta_j f,
$$

is well-defined, then $M$ is called the $*$-g-multiplier of $\Lambda, \theta$ and $m$.

**Proof.** Let $I$ be an arbitrary finite subset of $J$. Using the Cauchy-Bunyakovskii inequality and [24, Lemma 2.2], for any $\rho \in S(\mathcal{A})$ and $f \in U$ we have:

$$
\rho(\sum_{j \in I} m_j \Lambda_j^* \theta_j f) = \sup \{\rho(\sum_{j \in I} m_j \Lambda_j^* \theta_j f, g) : g \in U, \rho(g) \leq 1\}
$$

$$
= \sup \{\rho(\sum_{j \in I} \langle m_j \theta_j f, \Lambda_j g \rangle) : g \in U, \rho(g) \leq 1\}
$$

$$
\leq \sup_{\rho(g) \leq 1} \left(\rho(\sum_{j \in I} \langle m_j \theta_j f, m_j \theta_j f \rangle)^{1/2} \left(\rho(\sum_{j \in I} \langle \Lambda_j g, \Lambda_j g \rangle)^{1/2}\right)\right).
$$
Since
\[\sum_j (m_j \theta_j f, m_j \theta_j f) = \sum_j m_j \langle \theta_j f, \theta_j f \rangle m_j^* \]
\[= \sum_j (\rho(m_j))^2 \langle \theta_j f, \theta_j f \rangle \]
\[\leq \|m\|_\infty^2 B_\theta \langle f, f \rangle B_\theta^*, \]
so by Proposition 2.1 we have:
\[\rho(\sum_j (m_j \theta_j f, m_j \theta_j f)) \leq \|m\|^2_\infty (\rho(f))^2 \rho(B_\theta)^2.\]

Hence we have:
\[\rho(\sum_{j \in I} m_j \Lambda_j^* \theta_j f) \leq \|m\|_\infty \rho(f) \rho(B_\theta) \rho(B_\Lambda) \]

**Definition 4.1.** Let \(C, C' \in gl^+(U)\). The family \(\Lambda = \{\Lambda_j \in L(U, V_j) : j \in J\}\) is called a \((C, C')\)-controlled \(*\)-g-frame for \(U\) with respect to \(\{V_j\}_{j \in J}\) if \(\Lambda\) is a \(*\)-g-Bessel sequence and
\[A(f, f) A^* \leq \sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C' f \rangle \leq B(f, f) B^*, \tag{4.2}\]
for all \(f \in U\) and strictly nonzero elements \(A, B \in A\).

\(A, B\) are called controlled \(*\)-g-frame bounds. If \(C' = I\), we call \(\Lambda = \{\Lambda_j\}_{j \in J}\) a \(C\)-controlled \(*\)-g-frame for \(U\) with bounds \(A, B\). If only the second part of the above inequality holds, it is called a \((C, C')\)-controlled \(*\)-g-Bessel sequence with bound \(B\).

**Lemma 4.1** ([2]). Let \(X\) be a Hilbert module over \(C^*\)-algebra \(B\), \(S \geq 0\), i.e. this element is positive in \(C^*\)-algebra \(L(U)\). Then for each \(x \in X\),
\[\langle Sx, x \rangle \leq \|S\| \langle x, x \rangle.\]

**Proposition 4.2.** Let \(C \in gl^+(H)\). The family
\[\Lambda = \{\Lambda_j \in L(U, V_j) : j \in J\}\]
is a \(*\)-g-frame if and only if \(\Lambda\) is a \(C^2\)-controlled \(*\)-g-frame.

**Proof.** Let \(\Lambda\) be a \(C^2\)-controlled \(*\)-g-frame with bounds \(A, B\). Then
\[A(f, f) A^* \leq \sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C f \rangle \leq B(f, f) B^*, \quad \text{for } f \in U.\]
\[A(f, f) A^* = A(CC^{-1} f, CC^{-1} f) A^* \leq \|C\|^2 \langle C^{-1} f, C^{-1} f \rangle A^* \leq \|C\|^2 \sum_{j \in J} \langle \Lambda_j C C^{-1} f, C C^{-1} f \rangle.\]
Hence

\[ A\|C\|^{-1}\langle f, f \rangle A^*\|C\|^{-1} \leq \sum_{j \in J} \langle A_j f, A_j f \rangle. \]

On the other hand for every \( f \in U \)

\[
\sum_{j \in J} \langle A_j f, A_j f \rangle = \sum_{j \in J} \langle A_j C^{-1} f, C^{-1} f \rangle \\
\leq B\langle C^{-1} f, C^{-1} f \rangle B^* \\
\leq B\|C^{-1}\|^2\langle f, f \rangle B^*. 
\]

These inequalities yield that \( \Lambda \) is a \(*\)-g-frame with bounds \( A\|C^{-1}\|, B\|C^{-1}\| \). Conversely assume that \( \Lambda \) is a \(*\)-g-frame with bounds \( A', B' \). Then for all \( f \in U \),

\[ A'(f, f)A'^* \leq \sum_{j \in J} \langle A_j f, A_j f \rangle \leq B'(f, f)B'^*. \]

So for \( f \in U \),

\[
\sum_{j \in J} \langle A_j C f, A_j C f \rangle \leq B'(C f, C f)B'^* \leq B'\|C\|^2B'^*. 
\]

For the lower bound, since \( \Lambda \) is \(*\)-g-frame for any \( f \in U \),

\[
A'(f, f)A'^* = A'(C^{-1} C f, C^{-1} C f)A'^* \\
\leq A'\|C^{-1}\|^2\langle C f, C f \rangle A'^* \\
\leq \|C^{-1}\|^2 \sum_{j \in J} \langle A_j C f, A_j C f \rangle. 
\]

Therefore \( \Lambda \) is a \( C^2 \)-controlled \(*\)-g-frame with bounds \( A'\|C^{-1}\|, B'\|C^{-1}\| \) \( \square \)

**5. Multipliers of controlled \(*\)-G-frames in Hilbert pro-\( C^* \)-modules**

In this section, we define the multiplier of a controlled \(*\)-frame for \( C \)-controlled \(*\)-g-frames in Hilbert pro-\( C^* \)-modules. The definition of general case \((C, C')\)-controlled \(*\)-g-frames is similar.

**Lemma 5.1.** Let \( C, C' \in gl^+(U) \) and \( \Lambda = \{A_j \in L(U, V_j) : j \in J\}, \theta = \{\theta_j \in L(U, V_j) : j \in J\} \) be \( C'^2 \) and \( C^2 \)-controlled \(*\)-g-Bessel sequences for \( U \), respectively. Let \( m = \ell^\infty \). Then

\[
M_{m, C, \theta, \Lambda, C'} : U \rightarrow U, 
\]

defined by

\[ M_{m, C, \theta, \Lambda, C'} f := \sum_{j \in J} m_j C\theta_j^* A_j C' f, \]

is a well-defined bounded operator.
Proof. Let \( \Lambda = \{ \Lambda_j \in L(U, V_j) : j \in J \} \), \( \theta = \{ \theta_j \in L(U, V_j) : j \in J \} \) be \( C^2 \) and \( C^2 \)-controlled *-g-Bessel sequences for \( U \), with bounds \( B \), \( B' \), respectively. For any \( f, g \in U \) and finite subset \( I \subseteq J \),

\[
\overline{\rho}(\sum_{j \in I} m_j \theta^*_j \Lambda_j C' f) \leq \sup \{ \rho(\sum_{j \in I} m_j \theta^*_j \Lambda_j C' f, g) : g \in U, \overline{\rho}(g) \leq 1 \}
\]

\[
= \sup \{ \rho(\sum_{j \in I} \langle m_j \Lambda_j C' f, \theta_j C^* g \rangle) : g \in U, \overline{\rho}(g) \leq 1 \}
\]

\[
\leq \sup_{\overline{\rho}(g) \leq 1} \left( \rho(\sum_{j \in I} \langle m_j \Lambda_j C' f, m_j \Lambda_j C' f \rangle) \right)^{1/2} \left( \rho(\sum_{j \in I} \langle \theta_j C^* g, \theta_j C^* g \rangle) \right)^{1/2},
\]

since

\[
\sum_j \langle m_j \Lambda_j C' f, m_j \Lambda_j C' f \rangle = \sum_j m_j \langle \Lambda_j C' f, \Lambda_j C' f \rangle m_j^*
\]

\[
= \sum_j (\rho(m_j))^2 \langle \Lambda_j C' f, \Lambda_j C' f \rangle
\]

\[
\leq \|m\|_\infty^2 B(f, f) B^*.
\]

So by Proposition 2.1 we have:

\[
\rho(\sum_{j \in I} \langle m_j \Lambda_j C' f, m_j \Lambda_j C' f \rangle) = \rho(\sum_{j \in I} m_j \langle \Lambda_j C' f, \Lambda_j C' f \rangle m_j^*)
\]

\[
\leq \|m\|_\infty^2 \overline{\rho}(f)^2 \rho(B)^2.
\]

Hence

\[
\overline{\rho}(\sum_{j \in I} m_j \theta^*_j \Lambda_j C' f) \leq \|m\|_\infty \overline{\rho}(f) \rho(B) \rho(B')^*.
\]

This shows that \( M_{m,C,\theta,\Lambda,C'} \) is well-defined and

\[
\overline{\rho}(M_{m,C,\theta,\Lambda,C'}) \leq \|m\|_\infty \overline{\rho}(B) \rho(B')^*.
\]

The above Lemma provides a motivation for the following definition.

**Definition 5.1.** Let \( C, C' \in gl^+(U) \) and \( \Lambda = \{ \Lambda_j \in L(U, V_j) : j \in J \} \), \( \theta = \{ \theta_j \in L(U, V_j) : j \in J \} \) be \( C^2 \) and \( C^2 \)-controlled *-g-Bessel sequences for \( U \), respectively. Let \( m = \ell^\infty \). The operator

\[
M_{m,C,\theta,\Lambda,C'} : U \rightarrow U,
\]

defined by

\[
M_{m,C,\theta,\Lambda,C'} f := \sum_{j \in J} m_j \theta^*_j \Lambda_j C' f,
\]

is called \( (C, C') \)-controlled multiplier operator with symbol \( m \).
References