ON INTEGRATED AND DIFFERENTIATED $C_2$-SEQUENCE SPACES

LAKSHMI NARAYAN MISHRA$^{1,2,*}$, SUKHDEV SINGH$^3$, VISHNU NARAYAN MISHRA$^4$

1 Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, University, Vellore 632014, TN, India

2 L. 1627 Awadh Puri Colony Beniganj, Phase -III, Opposite - Industrial Training Institute (I.T.I.), Ayodhya Main Road Faizabad-224 001, UP, India

3 Department of Mathematics, Lovely Professional University, Jalandhar-Delhi Road, Phagwara-144411, Punjab, India

4 Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484887, India

*Corresponding author: lakshminarayannishra04@gmail.com

Abstract. The integrated and differentiated $C_2$-sequence spaces are defined and studied by using the norm on the bicomplex space $C_2$, infinite matrices of the bicomplex number and the Orlicz functions. We also studied some topological properties of the $C_2$-sequence spaces We define the $\alpha$-duals of the integrated and differentiated $C_2$-sequence spaces.

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1. Introduction

The set of bicomplex numbers \([8]\) is denoted by \(\mathbb{C}_2\) and sets of real and complex numbers are denoted as \(\mathbb{C}_0\) and \(\mathbb{C}_1\), respectively. The set of bicomplex number is defined as (cf. \([8], [9]\))

\[
\mathbb{C}_2 := \{a_1 + i_1 a_2 + i_2 a_3 + i_1 i_2 a_4 : a_k \in \mathbb{C}_0, 1 \leq k \leq 4\}
\]

\[
:= \{w_1 + i_2 w_2 : w_1, w_2 \in \mathbb{C}_1\}
\]

where \(i_1^2 = i_2^2 = -1, i_1 i_2 = i_2 i_1\).

The set of bicomplex numbers \(\mathbb{C}_2\) have exactly two non-trivial idempotent elements denoted by \(e_1\) and \(e_2\) give as \(e_1 = (1 + i_1 i_2)/2\) and \(e_2 = (1 - i_1 i_2)/2\). Note that \(e_1 + e_2 = 1\) and \(e_1 e_2 = 0\). The number \(\xi = w_1 + i_2 w_2\) can be uniquely expressed as a complex combination of \(e_1\) and \(e_2\) \([8]\).

\[
\xi = w_1 + i_2 w_2 = 1\xi e_1 + 2\xi e_2, \quad (1.1)
\]

where \(1\xi = w_1 - i_1 w_2\) and \(2\xi = w_1 + i_1 w_2\). The complex coefficients \(1\xi\) and \(2\xi\) are called the idempotent components of \(\xi\), and \(1\xi e_1 + 2\xi e_2\) is known as idempotent representation of bicomplex number \(\xi\).

The auxiliary complex spaces \(A_1\) and \(A_2\) are defined as follows:

\[
A_1 = \{1\xi : \xi \in \mathbb{C}_2\} \quad \text{and} \quad A_2 = \{2\xi : \xi \in \mathbb{C}_2\}.
\]

The norm in \(\mathbb{C}_2\) is defined as follows:

\[
||\xi|| = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2} = \sqrt{|w_1|^2 + |w_2|^2} = \sqrt{\frac{|1\xi|^2 + |2\xi|^2}{2}} \quad (1.2)
\]

Further, the norm of the product of two bicomplex numbers and the product of their norms are connected by means of the following inequality:

\[
||\xi \cdot \eta|| \leq \sqrt{2} \cdot ||\xi|| \cdot ||\eta|| \quad (1.3)
\]

The inequality given in (1.3) is the best possible relation. For this reason, we call \(\mathbb{C}_2\) as modified complex Banach algebra \([8]\).

Throughout the paper, the \(\omega_4, c, c_0\) and \(\ell^\infty_\mathbb{C}\) denote the space of all bicomplex sequences, convergent sequences, null sequences and all bounded sequences. We denote the zero sequence \((0, 0, 0, \ldots, 0, \ldots)\) by \(\pi\). Refer the book by Mursaleen \([?]\) for details about summability methods.

The Orlicz function \(M\) is defined as \(M : [0, \infty) \rightarrow [0, \infty)\). It is continuous, non-decreasing and \(M(0) = 0, M(x) > 0\) for \(x > 0\). Also, for \(\lambda \in (0, 1)\) it satisfies the condition

\[
M(\lambda x + (1 - \lambda)y) \leq \lambda M(x) + (1 - \lambda)M(y) \quad (1.4)
\]

and if the condition of convexity of the Orlicz function \(M\) is replaced by \(M(x + y) \leq M(x) + M(y)\), then the function \(M\) is called the modulus function.
The notations \((X : Y)\) denote the class of all matrices \(M\), such that \(M : X \rightarrow Y\). Therefore, \(M \in (X : Y)\) if and only if \(M(x) = \{(Mx)_n\}_{n \in \mathbb{N}} \in Y\).

A sequence \(\{\xi_n\}\) in \(\mathbb{C}_2\) is said to be \(M\)-summable to the bicomplex number \(\xi\) if \(M(\xi_n)\) converges to \(\xi\) which is called \(M\)-limit of \(\{\xi_n\}\).

In [1], the sequence space \(bv_p\) is defined, which have all sequences such that their \(\Delta\)-transform is in \(\ell_p\), where \(\Delta\) denotes the matrix \(\Delta = \{\delta_{nm}\}\) as

\[
\delta_{nm} := \begin{cases} (-1)^{n-k}, & n - 1 \leq m \leq n \\ 0, & 0 \leq m \leq n - 1 \text{ or } k > n \end{cases}
\]  

(1.5)

We consider following matrices for our \(\mathbb{C}_2\)-sequence spaces.

\[
\omega_{nm} := \begin{cases} \xi, & 1 \leq m \leq n \xi \\ 0; & \xi \prec_{ld} \eta \end{cases}
\]  

(1.6)

\[
\gamma_{nm} := \begin{cases} \xi, & m = n \\ -\xi, & n - 1 = m \\ 0, & \text{otherwise} \end{cases}
\]  

(1.7)

\[
\pi_{nm} := \begin{cases} \xi^{-1}, & 1 \leq m \leq n \\ 0, & m > n \end{cases}
\]  

(1.8)

and

\[
\pi_{nm} := \begin{cases} \xi^{-1}, & n = m \\ -\xi^{-1}, & n - 1 = m \\ 0, & \text{otherwise} \end{cases}
\]  

(1.9)

Here we must note that \(\xi^{-1}\) exists if and only if \(\xi \in \mathbb{C}_2/\mathbb{D}_2\).

The integrated and differentiated sequence space were first studied by Goes and Goes [3]. In this paper, we define and study some \(\mathbb{C}_2\)-sequence space. In the last section we studied the \(\alpha\)-dual of these sequence spaces.

2. BICOMPLEX INTEGRATED (\(int\)) AND DIFFERENTIATED (\(diff\)) \(\mathbb{C}_2\)-SEQUENCE SPACES

Goes and Goes [3] has given the concept of the integrated sequence space. In this section we will obtain the matrix domains of the sequence space \(\ell_1\) by using the bicomplex matrices. We shall show that the integrated and differentiated \(\mathbb{C}_2\)-sequence spaces are Banach Spaces, \(BK\)-spaces, norm isomorphic to \(\ell_1\), separable these
spaces have AK-property. The spaces $\ell^1$ and $\ell_1$ have monotone norms and therefore the spaces $\ell^1$ and $\ell_1$ have AK-property. Let $\omega_4$ denote the family of bicomplex sequences.

Now we are giving the definitions of some $C_2$-sequence spaces as follows:

**Definition 2.1** (Integrated $C_2$-sequence spaces).

$$
\overline{\ell}_1(C_2, \mathcal{M}, \| \cdot \|) = \{ \xi_n \in \omega_4 : \sum_{n=1}^{\infty} \mathcal{M}\left( \frac{\|n\xi_n\|}{K} \right) < \infty, \text{for some } K > 0 \}
$$

and

$$
\overline{bv}(C_2, \mathcal{M}, \| \cdot \|) := \{ \xi = \{ \xi_n \} \in \omega_4 : \sum_{n=2}^{\infty} \mathcal{M}\left( \frac{\|\Delta(n \xi_n)\|}{K} \right) < \infty, \text{for some } K > 0 \}
$$

**Definition 2.2** (Differentiated $C_2$-sequence spaces).

$$
\ell_1(C_2, \mathcal{M}, \| \cdot \|) := \{ \xi = \{ \xi_n \} \in \omega_4 : \sum_{n=1}^{\infty} \mathcal{M}\left( \frac{\|n \xi_n\|}{K} \right) < \infty, \text{for some } K > 0 \}
$$

$$
\overline{bv}(C_2, \mathcal{M}, \| \cdot \|) := \{ \xi = \{ \xi_n \} \in \omega_4 : \sum_{n=2}^{\infty} \mathcal{M}\left( \frac{\|\Delta(n \xi_n)\|}{K} \right) < \infty, \text{for some } K > 0 \}
$$

we can redefine the spaces $\ell(C_2, \mathcal{M}, \| \cdot \|), \overline{bv}(C_2, \mathcal{M}, \| \cdot \|), \ell(C_2, \mathcal{M}, \| \cdot \|)$ and $\overline{bv}(C_2, \mathcal{M}, \| \cdot \|)$ by

$$(\ell_1)^{\Omega} = \overline{\ell}_1(C_2, \mathcal{M}, \| \cdot \|), \quad (\ell_1)^{\Gamma} = \overline{bv}(C_2, \mathcal{M}, \| \cdot \|), \quad (\ell_1)^{\Pi} = \ell_1(C_2, \mathcal{M}, \| \cdot \|), \quad (\ell_1)^{\lambda} = \overline{bv}(C_2, \mathcal{M}, \| \cdot \|).$$

Let $\xi = \{ \xi_n \} \in \overline{\ell}_1(C_2, \mathcal{M}, \| \cdot \|)$. Then the $\Omega$-transform of $\xi$ is defined as

$$
\zeta_n := (\Omega(\xi))_n = \sum_{m=1}^{n} \mathcal{M}\left( \frac{\|m \xi_m\|}{K} \right) \quad \text{for some } K > 0
$$

or equivalently,

$$
1\xi_n := (\Omega^{(1)}(\xi))_n = \sum_{m=1}^{n} \mathcal{M}\left( \frac{\|m \xi_m\|}{K} \right) \quad \text{and} \quad 2\xi_n := (\Omega^{(2)}(\xi))_n = \sum_{m=1}^{n} \mathcal{M}\left( \frac{\|m \xi_m\|}{K} \right)
$$

Let $\xi = \{ \xi_n \} \in \overline{bv}(C_2, \mathcal{M}, \| \cdot \|)$. The $\Gamma$-transform of $\{ \xi_n \}$ is defined as

$$
\zeta_n := (\Gamma(\xi))_n = \begin{cases} 
\xi_1, & p = 1 \\
\Delta(p \xi_p), & p \geq 2
\end{cases}
$$

Let $\xi = \{ \xi_n \} \in \ell_1(C_2, \mathcal{M}, \| \cdot \|)$. The $\Pi$-transform of $\{ \xi_n \}$ is defined as

$$
\zeta_n = (\Pi(\xi))_n = \sum_{p=1}^{n} \mathcal{M}\left( \frac{\|p \xi_p\|}{K} \right) \quad \text{for some } K > 0
$$

Let $\xi = \{ \xi_n \} \in \overline{bv}(C_2, \mathcal{M}, \| \cdot \|)$. The $\Sigma$-transform of $\{ \xi_n \}$ is defined as
Proof. Let \( \zeta_n := (\Sigma(\xi))_n = \begin{cases} \xi_1, & p = 1 \\ \Delta(p^{-1} \xi_p), & p \geq 2 \end{cases} \)

For the convenience, we use the following notations.

\[
K_1 = \overline{\ell}_1(\mathbb{C}_2, \mathcal{M}, ||||), \quad K_2 = \overline{b_0}(\mathbb{C}_2, \mathcal{M}, ||||), \quad K_3 = \ell_1(\mathbb{C}_2, \mathcal{M}, ||||), \quad K_4 = b_0(\mathbb{C}_2, \mathcal{M}, ||||).
\]

**Proposition 2.1.** A sequence \( \{\xi_n\} \) is in \( X(\mathbb{C}_2, \mathcal{M}, ||||) \) if and only if \( \{1 \xi_n\} \in S(\mathcal{A}_1, \mathcal{M}, ||||) \) and \( \{2 \xi_n\} \in S(\mathcal{A}_2, \mathcal{M}, ||||) \), where \( X = K_1, K_2, K_3 \) and \( K_4 \).

**Theorem 2.1.** The space \( \ell_1(\mathbb{C}_2, \mathcal{M}, ||||) \) is a linear space over \( \mathbb{C}_0 \).

Proof. Let \( \{\xi_n\}, \{\eta_n\} \in \ell_1(\mathbb{C}_2, \mathcal{M}, ||||) \). Then there exist \( P_1 > 0 \) and \( P_2 > 0 \) such that

\[
\sum_{n=1}^{\infty} \mathcal{M}(\frac{||n \xi_n||}{P_1}) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \mathcal{M}(\frac{||n \eta_n||}{P_2}) < \infty.
\]

Now let \( \alpha, \beta \in \mathbb{C}_2 \setminus \mathcal{O}_2 \) and \( P = \max\{2 ||\alpha||P_1, 2 ||\beta||P_2\} \). Then

\[
\sum_{k=1}^{\infty} \mathcal{M}\left(\frac{||\alpha \Delta(k \xi_k) + \beta \Delta(k \eta_k)||}{P}\right) \leq \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{||\alpha \Delta(k \xi_k)||}{P_1}\right) + \sum_{k=1}^{\infty} \mathcal{M}\left(\frac{||\beta \Delta(k \eta_k)||}{P_2}\right).
\]

Therefore, \( \{\alpha \xi_n + \beta \eta_n\} \in \ell_1(\mathbb{C}_2, \mathcal{M}, ||||) \). Hence, the space \( \ell_1(\mathbb{C}_2, \mathcal{M}, ||||) \) is a linear space over \( \mathbb{C}_2 \setminus \mathcal{O}_2 \). \( \square \)

**Lemma 2.1.** The functions \( ||\xi||_{\overline{\ell}_1(\mathbb{C}_2, \mathcal{M}, ||||)} = \sum_{m=1}^{\infty} ||\omega_{nm} \xi_m|| \) and \( ||\xi||_{\ell_1(\mathbb{C}_2, \mathcal{M}, ||||)} = \sum_{m=1}^{\infty} ||\pi_{nm} \xi_m|| \) are norms on the spaces \( \overline{\ell}_1(\mathbb{C}_2, \mathcal{M}, ||||) \) and \( \ell_1(\mathbb{C}_2, \mathcal{M}, ||||) \), respectively.

**Theorem 2.2.** The spaces \( \overline{\ell}_1(\mathbb{C}_2, \mathcal{M}, ||||) \) and \( \ell_1(\mathbb{C}_2, \mathcal{M}, ||||) \) are Banach spaces with norms \( ||\xi||_{\overline{\ell}_1(\mathbb{C}_2, \mathcal{M}, ||||)} = \sum_{m=1}^{\infty} ||\omega_{nm} \xi_m|| \) and \( ||\xi||_{\ell_1(\mathbb{C}_2, \mathcal{M}, ||||)} = \sum_{m=1}^{\infty} ||\pi_{nm} \xi_m|| \), respectively.

Proof. Let \( \{\xi^n_k\} \) be a Cauchy sequence in \( \overline{\ell}_1(\mathbb{C}_2, \mathcal{M}, ||||) \). Then for given \( \epsilon > 0, \exists m_0 \in \mathbb{N} \) such that

\[
||\xi^n_k - \xi^m_k|| < \epsilon, \quad \forall n, m > m_0 \tag{2.1}
\]

Therefore,

\[
\sum_k ||\Omega(\xi^m_k) - \Omega(\xi^n_k)|| < \epsilon, \quad \forall n, m > m_0
\]

\( \Rightarrow \{\Omega(\xi^1_k), \Omega(\xi^2_k), \Omega(\xi^3_k), \ldots, \Omega(\xi^n_k), \ldots\} \) is a Cauchy Sequence of bicomplex numbers. Since, \( \mathbb{C}_2 \) is a modified Banach space. Therefore, \( \{\Omega(\xi^n_k)\} \) is convergence in \( \mathbb{C}_2 \). Suppose that

\[
\Omega(\xi^n_k) \rightarrow \Omega(\xi), \quad n \rightarrow \infty, \forall k
\]

Using all these limits, we define a sequence \( \{\Omega(\xi_1), \Omega(\xi_2), \Omega(\xi_3), \ldots\} \).
and from equation (2.1), we have
\[ \sum_{k=1}^{p} \|\Omega(\xi^n)_k - \Omega(\xi^m)_k\| < \epsilon \quad (2.2) \]

For any \( n > m_0 \), by letting \( m \to \infty \) and \( p \to \infty \), we have
\[ \|\xi^n - \xi\|_{\overline{T}_1(\mathcal{C}_2, \mathcal{M}, \|\|)} \leq \epsilon \]

In particular,
\[ \|\xi\|_{\overline{T}_1(\mathcal{C}_2, \mathcal{M}, \|\|)} \leq K + \|\xi^n\|_{\overline{T}_1(\mathcal{C}_2, \mathcal{M}, \|\|)}, \quad \text{for some} \quad K \geq \epsilon. \]

Hence, \( \xi \in \overline{T}_1(\mathcal{C}_2, \mathcal{M}, \|\|) \). Further, \( \xi^n \to \xi \). Therefore, \( \overline{T}_1(\mathcal{C}_2, \mathcal{M}, \|\|) \) is complete. \( \Box \)

**Corollary 2.1.** The space \( \ell_1(\mathcal{C}_2, \mathcal{M}, \|\|) \) is a Banach space.

**Theorem 2.3.** The spaces \( \overline{T}_1(\mathcal{C}_2, \mathcal{M}, \|\|) \) and \( \ell_1(\mathcal{C}_2, \mathcal{M}, \|\|) \) are BK-spaces with norms \( \|\xi\|_{\overline{T}_1(\mathcal{C}_2, \mathcal{M}, \|\|)} = \sum_{m=1}^{\infty} \|\xi_m\| \) and \( \|\xi\|_{\ell_1(\mathcal{C}_2, \mathcal{M}, \|\|)} = \sum_{m=1}^{\infty} \|\pi_m\xi_m\| \), respectively.

**Proof.** Let \( \{\xi_n\} \in \overline{T}_1(\mathcal{C}_2, \mathcal{M}, \|\|) \). Define \( f_p(\xi_n) = \xi_p, \forall n \in \mathbb{N} \). Then
\[ \|\xi_n\|_{\overline{T}_1(\mathcal{C}_2, \mathcal{M}, \|\|)} = \sum \|n\xi_n\| \]

So that \( \|n\xi_n\| \leq \|\xi_n\|_{\overline{T}_1(\mathcal{C}_2, \mathcal{M}, \|\|)} \) \( \Rightarrow \|\xi_n\| \leq K\|\xi_n\|_{\overline{T}_1(\mathcal{C}_2, \mathcal{M}, \|\|)} \) \( \Rightarrow \|f_n(\xi_p)\| \leq K\|\xi_n\|_{\overline{T}_1(\mathcal{C}_2, \mathcal{M}, \|\|)}. \)

Therefore, \( f_n \) is a continuous linear functional for each \( n \). So, \( \overline{T}_1(\mathcal{C}_2, \mathcal{M}, \|\|) \) is a BK-space. \( \Box \)

In the similar manner, we can prove that \( \ell_1(\mathcal{C}_2, \mathcal{M}, \|\|) \) is a BK-space.

**Theorem 2.4.** The space \( \overline{bv}(\mathcal{C}_2, \mathcal{M}, \|\|) \) is a BK-space with the norm \( \|\xi\|_{\overline{bv}(\mathcal{C}_2, \mathcal{M}, \|\|)} = \sum_{m=1}^{\infty} \|\Delta(m\xi_m)\| \).

**Proof.** As we know, \( \overline{bv}(\mathcal{C}_2, \mathcal{M}, \|\|) = (\ell_1)_{\Sigma} \) is true and \( \ell_1 \) is a BK-space with respect to the norm \( \|\xi\|_{\ell_1} \) and also the matrix \( \Sigma \) is a triangular matrix. Then by Wilansky [?], the space \( \overline{bv} \) is a BK-space. \( \Box \)

**Theorem 2.5.** The function \( \|\xi\|_{\overline{bv}(\mathcal{C}_2, \mathcal{M}, \|\|)} = \sum_{m=1}^{\infty} \|\Delta(m\xi_m)\| \) is a norm on \( \overline{bv}(\mathcal{C}_2, \mathcal{M}, \|\|) \).

**Theorem 2.6.** The spaces \( \overline{bv}(\mathcal{C}_2, \mathcal{M}, \|\|) \) and \( \overline{bv} \) have AK-property.

**Proof.** Let \( \{\xi^n_k\} \in \overline{bv}(\mathcal{C}_2, \mathcal{M}, \|\|) \) and \( \{\xi^n_k\} = \{\xi^n_1, \xi^n_2, \xi^n_3, \ldots, \xi^n_k, 0, 0, 0, \ldots\} \).

\[ \xi^n_k - [\xi^n_k] = \{0, 0, 0, \ldots, 0, \xi^n_{k+1}, 0, 0, 0, \ldots\}. \]

\[ \Rightarrow \|\xi^n_k - [\xi^n_k]\|_{\overline{bv}(\mathcal{C}_2, \mathcal{M}, \|\|)} = \|0, 0, 0, \ldots, \xi^n_{k+1}, 0, 0, 0, \ldots\|_{\overline{bv}(\mathcal{C}_2, \mathcal{M}, \|\|)}. \]

\[ = \sum_{p \geq k+1} \mathcal{M} \left( \frac{\|\xi^n_p\|}{K} \right) \to 0, \quad \text{as} \quad p \to 0. \]

\[ \Rightarrow [\xi^n_k] \to \xi^n_k \quad \text{as} \quad k \to \infty \]
Then, the space $bv(C_2, M, \|\cdot\|)$ has $AK$-property. \hfill \Box

**Theorem 2.7.** The spaces $\ell_1(C_2, M, \|\cdot\|)$, $bv(C_2, M, \|\cdot\|)$, $\ell_1(C_2, M, \|\cdot\|)$ and $bv(C_2, M, \|\cdot\|)$ are norm isomorphic to $\ell_1$.

**Proof.** We must show that there is a one-one and onto linear mapping between $bv(C_2, M, \|\cdot\|)$ and $\ell_1$.

Suppose that $T : bv(C_2, M, \|\cdot\|) \rightarrow \ell_1$ be a mapping defined as $\xi \mapsto T\xi$.

Clearly, for $\xi = \theta \Rightarrow T\xi = \theta$.

Now, let $\eta \in \ell_1$. Define a sequence $\{\xi_k\} \in bv(C_2, M, \|\cdot\|)$ by

$$\xi_k = \frac{1}{k} \sum_{p=1}^{k} y_p$$

Then

$$\|\xi_k\|_{bv(C_2, M, \|\cdot\|)} = \sum_{k} \Delta(k \xi_k) = \sum_{k} \left(\sum_{p=1}^{k} p \eta_p - (p - 1) \sum_{p=1}^{k-1} \eta_p\right) = \sum_{k} \|\eta_k\| = \|\eta\|_{\ell_1}$$

Therefore, $\xi_n \in bv(C_2, M, \|\cdot\|)$. Hence, the spaces $bv(C_2, M, \|\cdot\|)$ and $\ell_1$ are isomorphic. \hfill \Box

In the similar way, we can prove the isomorphism of remaining spaces.

**Theorem 2.8.** The spaces $\ell_1(C_2, M, \|\cdot\|)$ and $bv(C_2, M, \|\cdot\|)$ have monotone norm.

**Proof.** Let $\{\xi_n\} \in bv(C_2, M, \|\cdot\|)$.

Define $\|\xi_n\|_{bv(C_2, M, \|\cdot\|)} = \sum_{k=1}^{n} \Delta(k \xi_k)$

and $\|\xi_p\|_{bv(C_2, M, \|\cdot\|)} = \sum_{k=1}^{p} \Delta(k \xi_k)$, $\forall \{\xi_k\} \in bv(C_2, M, \|\cdot\|)$.

Now, suppose $q > p$, then

$$\|\xi_p\|_{bv(C_2, M, \|\cdot\|)} = \sum_{k=1}^{p} \|\Delta(k \xi_k)\| \leq \sum_{k=1}^{q} \|\Delta(k \xi_k)\| \leq \|\xi_q\|_{bv(C_2, M, \|\cdot\|)}$$

Also,

$$\sup \|\xi_n\|_{bv(C_2, M, \|\cdot\|)} = \sup \left(\sum_{k=1}^{n} \|\Delta(k \xi_k)\|\right) = \|\xi_n\|_{bv(C_2, M, \|\cdot\|)}.$$

Therefore, the space $bv(C_2, M, \|\cdot\|)$ has the monotone norm. \hfill \Box
Remark 2.1. The spaces $\ell_1^b$ and $\text{bv}(C_2, \mathcal{M}, \|\|)$ have AB-property.

Theorem 2.9. The following statements hold for $\text{bv}(C_2, \mathcal{M}, \|\|)$ and $\text{bv}(C_2, \mathcal{M}, \|\|)$ given as :

1. If $\zeta^{(m)} = \{\zeta^{(m)}_n\}$ is sequence where $\{\zeta^{(m)}_n\} \in \text{bv}(C_2, \mathcal{M}, \|\|)$ of elements of $\text{bv}(C_2, \mathcal{M}, \|\|)$, defined as

$$\zeta^{(m)}_n := \begin{cases} 1/m , & n \geq m \\ 0 , & n < m \end{cases}$$

This sequence is the basis for the space $\text{bv}(C_2, \mathcal{M}, \|\|)$ and select $B_m = (M\xi)_m$, for all $m \in \mathbb{N}$ and matrix $M$ defined in equation (??), then $\xi \in \text{bv}(C_2, \mathcal{M}, \|\|)$ has the unique representation of the type:

$$\xi = \sum_m (M\xi)_m \zeta^{(m)}_n$$

2. Define a sequence $\{\eta^m_n\}$ with $\eta^m_n \in \text{bv}(C_2, \mathcal{M}, \|\|)$ as

$$\eta^{(m)}_n := \begin{cases} m , & n \geq m \\ 0 , & n < m \end{cases}$$

Then this sequence $\zeta^{(m)}$ is a basis for the space $\text{bv}$ and for $E_m = (Ax)_m$, for all $m \in \mathbb{N}$, where the matrix $A$ is defined by $\Gamma = [\gamma_{nm}]$, every sequence $\xi \in \text{bv}$ have unique representation as

$$\xi = \sum_m E_m \zeta^{(m)}$$

Corollary 2.2. The spaces $\text{bv}(C_2, \mathcal{M}, \|\|)$ and $\text{bv}(C_2, \mathcal{M}, \|\|)$ are separable.

3. $\alpha$–Duals of the $C_2$–Sequence Spaces

In this section, we determine the $\alpha$–duals of the spaces $K_2$ and $K_4$.

Let $\xi = \{\xi_n\}$ and $\eta = \{\eta_n\}$ be sequences, and $A$ and $B$ be two subsets of $\omega_4$. Now let $M = (a_{mk})$ be an infinite matrix of bicomplex numbers. Define $\xi \eta = (\xi_n \eta_n)$,

$$\xi^{-1} \ast B = \{\zeta \in \omega_4 : \xi \xi \in B\}. \quad N(A, B) = \cap_{\xi \in A} \xi^{-1} \ast B = \{\zeta \in \omega_4 : \xi \xi \in B, \text{for } \xi \in A\}. \quad \text{In particular, for } B = \ell_1, cs \text{ or } bs, \text{ We have } \xi^a = \xi^{-1} \ast \ell_1, \xi^b = \xi^1 \ast cs \text{ and } \xi^c = \xi^{-1} \ast bs. \text{ The } \alpha- \text{dual of } A \text{ are given by } A^\alpha = M(A, \ell_1).$$

Suppose that $M_m = (a_{mk})_{k=0}^\infty$ denotes the $m$-th row of the matrix $M$. Let $M_m(\xi) = \sum_{k=0}^\infty a_{mk} \xi_k, \forall n = 0, 1, 2, \ldots$, and $M(\xi) = [M_m(\xi)]_{m=0}^\infty$, where $M_m \in \xi^b$.

Lemma 3.1. [?] Let $A_1, A_2$ be to BK-spaces, and $M = [\eta_{nm}]$ be a triangular matrix where $\xi_{nm} \in C_2/\mathcal{M}_2$, then for matrix $S_{A_1}^M = [\xi_{nm}]$ defined with $\nu = \{\nu_m\} \in A_1$ as
\[ \xi_{nm} = \sum_{i=1}^{n} \nu_i \eta_{nm} \mu_{im} \]

Then \( A_2 A_1(M) \subset A_1(M) \) holds if and only if the matrix \( S^M_{A_1} = M D_\nu M^{-1} \in (A_1 : A_1) \), where \( D_\nu \) is a diagonal matrix such that \( [D_\nu]_{nn} = \nu_n \), \( \forall n \in \mathbb{N} \).

**Lemma 3.2.** Let \( \{\gamma_k\} \) be a sequence in \( \omega_4 \) and \( M = [\eta_{nm}] \) be an invertible triangular matrix. Define a matrix \( S^M_{A_1} = [\xi_{nm}] \) as

\[ \xi_{nm} = \sum_{i=m}^{n} \eta_{i} \mu_{im} \]

Then

\[ A_2^\beta(M) = \{\eta_m \in \omega_4 : S(M) \in (A_1 : c)\} \]

and

\[ A_1^\gamma(M) = \{\eta_m \in \omega_4 : S(M) \in (A_1 : \ell_\infty)\} \]

**Lemma 3.3.** Let \( M = [\xi_{nm}] \) be an infinite matrix of bicomplex numbers. Then

1. \( M \in (\ell_1 : \ell_1) \iff \sup \sum_{k \in \mathbb{N}} \|\xi_{nm}\| < \infty. \)
2. \( M \in (\ell_1 : \ell_\infty) \iff \sup \|\xi_{nm}\| < \infty. \)
3. \( M \in (\ell_1, c) \iff \sup \|\xi_{nm}\| < \infty \) and for some sequence \( \{\kappa_m\} \) such that \( \lim_{n \to \infty} \xi_{nm} = \kappa_m \)

**Theorem 3.1.** For the space \( \text{bv}(\mathbb{C}_2, M, \|.\|) \), we have

\[ \text{bv}(\mathbb{C}_2, M, \|.\|)^{\alpha} = \alpha_1 \]

where

\[ \alpha_1 = \left\{ \xi = [\xi_n] \in \omega_4 : \sum_k \left\| \sum_{m=1}^{n} M \left( \frac{\|\Delta(\xi_m/n)\|}{K} \right) \eta_k \right\| < \infty, \ (\eta_k) \in \text{bv}(\mathbb{C}_2, M, \|.\|) \text{ for some } K > 0 \right\} \]

**Proof.** \( \{\xi_n\} \) be any sequence in \( \omega_4 \). Assume the following relation

\[ \xi_n \eta_n = \sum_{k=1}^{n} M \left( \frac{\|\Delta(\xi_m/n)\|}{K} \right) \eta_k = (E\eta)_k \]

where \( E = \{e_{nk}\} \) is defined by

\[ e_{nm} = \begin{cases} M \left( \frac{\|\Delta(\xi_m/n)\|}{K} \right), & 1 \leq m \leq n \\ 0, & n < m \end{cases} \tag{3.1} \]

Therefore, from the equation (3.1) and the Lemma (3.3) we have

\[ \left\{ M \left( \frac{\|\Delta(\xi_m/n)\|}{K} \right) \xi_n \right\} \in \ell_1 \text{ if and only if } E\eta \in \ell_1, \text{ whenever } \eta \in \ell_1. \]

So, \( \xi = [\xi_n] \in \text{bv}(\mathbb{C}_2, M, \|.\|)^{\alpha} \) if and only if \( E \in (\text{bv}(\mathbb{C}_2, M, \|.\|) : \ell_1) \).

Hence proved. \( \square \)
Analogously, we can prove the following theorems.

**Theorem 3.2.** For the space $bv(C_2, M, \|\cdot\|)$

$$bv(C_2, M, \|\cdot\|)^{\alpha_2} = \alpha_2$$

where

$$\alpha_1 = \left\{ \xi = \{\xi_n\} \in \omega_4 : \sum_k \sum_{m=1}^n M \left( \frac{\|\xi_m/m\|}{K} \right) \eta_k \right\} < \infty, (\eta_k) \in bv(C_2, M, \|\cdot\|) \text{ for some } K > 0 \right\}$$

**Theorem 3.3.** For the space $\ell_1(C_2, M, \|\cdot\|)$

$$\ell_1(C_2, M, \|\cdot\|)^{\alpha_1} = \alpha_1$$

where

$$\alpha_1 = \left\{ \xi = \{\xi_n\} \in \omega_4 : \sum_k \sum_{m=1}^n M \left( \frac{\|m\xi_m\|}{K} \right) \eta_k \right\} < \infty, (\eta_k) \in bv(C_2, M, \|\cdot\|) \text{ for some } K > 0 \right\}$$

**Theorem 3.4.** For the space $\ell_1(C_2, M, \|\cdot\|)$

$$\ell_1(C_2, M, \|\cdot\|)^{\alpha_2} = \alpha_2$$

where

$$\alpha_1 = \left\{ \xi = \{\xi_n\} \in \omega_4 : \sum_k \sum_{m=1}^n M \left( \frac{\|\Delta(m\xi_m)\|}{K} \right) \eta_k \right\} < \infty, (\eta_k) \in bv(C_2, M, \|\cdot\|) \text{ for some } K > 0 \right\}$$

**References**


