ON GENERALIZED LOCAL PROPERTY OF $|A; \delta|_k$-SUMMABILITY OF FACTORED FOURIER SERIES

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Abstract. The convergence of Fourier series of a function at a point depends upon the behaviour of the function in the neighborhood of that point and it leads to the local property of Fourier series. In the proposed paper a new result on local property of $|A; \delta|_k$-summability of factored Fourier series has been established that generalizes a theorem of Sarıogöl [13] (see [M. A. Sarıogöl, On local property of $|A|_k$-summability of factored Fourier series, J. Math. Anal. Appl. 188 (1994), 118-127]) on local property of $|A|_k$-summability of factored Fourier series.

1. Introduction and Motivation

Suppose $\sum a_n$ be a given infinite series with sequence of partial sum $(s_n)$ and let $A = (a_{nv})$ be a lower triangular matrix with nonzero diagonal entries. Then $A$ defines the sequence-to-sequence transformation...
from the sequence $s = (s_n)$ to $A(s) = (A_n(s))$, with

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v. \quad (1.1)$$

A series $\sum a_n$ is summable $|A|_k \ (k \geq 1)$ if, (see [13])

$$\sum_{n=1}^{\infty} |a_{nn}|^{1-k}|A_n(s) - A_{n-1}(s)|^k < \infty, \quad (1.2)$$

and the series $\sum a_n$ is summable $|A; \delta|_k \ (k \geq 1)$ if, (see [6])

$$\sum_{n=1}^{\infty} |a_{nn}|^{1-k-\delta k}|A_n(s) - A_{n-1}(s)|^k < \infty. \quad (1.3)$$

Let us consider two lower triangular matrices $\bar{A}$ and $\hat{A}$ associated with $A$ as follows:

$$\bar{a}_{nv} = \sum_{r=v}^{n} a_{nr}, \quad (n, v = 0, 1, 2, \ldots)$$

and

$$\hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}. \quad (n = 1, 2, 3, \ldots).$$

In special case, when $A = (\bar{N}, p_n)$ then $|A, \delta|_k$-summability reduces to $|\bar{N}, p_n; \delta|_k$-summability and for $k = 1, (|\bar{N}, p_n; \delta|)$ is equivalent to $|\mathcal{R}, p_n; \delta|_1$-summability (see [2]). Also, if we take $A = (C, \alpha)$ with ($\alpha > -1$), then $|A, \delta|_k$-summability becomes $|C, \alpha, (\alpha - 1)(1 - 1/k)\delta|_k$ in Flett’s notation. Furthermore, for double absolute factorable summability matrix (see [11]).

We use the notations

$$\Delta c_n = c_n - c_{n+1} \text{ and } \bar{\Delta}c_{n,v} = c_{nv} - c_{n-1,v}, \quad c_{-1,0} = 0, \quad (n, v = 0, 1, 2, \ldots).$$

A sequence $(\lambda_n)$ is called a convex sequence if,

$$\Delta^2(\lambda_n) \geq 0 \text{ for every } n \in \mathbb{Z}_+,$$

where
\[ \Delta^2(\lambda_n) = \Delta(\lambda_n) - \Delta(\lambda_{n+1}) \quad \text{and} \quad \Delta(\lambda_n) = \lambda_n - \lambda_{n+1}. \]

Let \( f(t) \in L(-\pi, \pi) \) be a \( 2\pi \) periodic function. Without loss of generality let us consider that \( a_0 = 0 \) in the Fourier series expansion of \( f(t) \) that is,

\[
\int_{-\pi}^{\pi} f(t) dt = 0. \quad (1.4)
\]

Thus the Fourier series expansion of \( f(t) \) becomes:

\[
f(t) = \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \quad (1.5)
\]

It is well known that the convergence of the Fourier series at \( t = x \) is a local property of \( f \) [16] (i.e., it depends only on the behavior of \( f \) in an arbitrarily small neighborhood of \( x \)) and hence the summability of the Fourier series at \( t = x \) by any regular linear summability method is also a local property of \( f \). Moreover, as regards to the approximation of Fourier series of functions see the recent results [9], [10] and [5].

2. PRELIMINARIES

Dealing with Riesz summability and local property of Fourier series, Mohanty [12] has established that \(|R, \log(n), 1|\)-summability of a factored Fourier series

\[
\sum \frac{A_n}{\log(n+1)} \quad (2.1)
\]

of a function \( f(t) \) at any point \( t = x \) is a local property of the generating function of \( f(t) \) but the summability \(|C, 1|\) of this series is not. Subsequently, replacing the series (2.1) by

\[
\sum \frac{A_n(t)}{(\log \log(n+1))^\delta} \quad (\delta > 1). \quad (2.2)
\]

Matsumoto [7] as obtained a new result on local property of \(|R, p_n, 1|\)-summability.

Generalizing the above result Bhatt [1] proved the following theorem:

**Theorem 2.1.** Suppose \( (\lambda_n) \) is a convex sequence such that \( \sum \frac{\lambda_n}{n} \) is convergent, then the \(|R, \log(n), 1|\)-summability of a factored Fourier series \( \sum A_n(t) \lambda_n \log(n) \) at any point \( t = x \) is a local property of \( f(t) \).
By replacing the factor $\lambda_n \log(n)$ in a most general form, Mishra [8] has proved the following theorem.

**Theorem 2.2.** Suppose $(p_n)$ be a sequence satisfying following conditions:

$$P_n = O(np_n),$$

$$P_n \Delta p_n = O(p_n p_{n+1}).$$

Then the $|N,p_n|$-summability of a factored Fourier series

$$\sum_{n=1}^{\infty} A_n(t) \lambda_n P_n (np_n)^{-1}$$

at any point $t = x$ is a local property of $f(t)$, where $(\lambda_n)$ is a convex sequence.

Replacing $|N,p_n|$-summability in Mishra’s result, Bor [3] proved a more general form on $|N,p_n|_k$-summability method. Quite recently, Bor [4] introduced the following result on $|N,p_n|_k$-summability of a factored Fourier series at any point $t = x$ as a local property of $f(t)$ under more appropriate conditions than those given in the theorem.

**Theorem 2.3.** Let the positive sequence $(p_n)$ and a sequence $(\lambda_n)$ be such that

$$\Delta X_n = O(n^{-1});$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \{|\lambda_n|^k + |\lambda_{n+1}|^k\} X_n^{k-1} \leq \infty;$$

$$\sum_{n=1}^{\infty} (X_n^k + 1) |\Delta \lambda_n| \leq \infty,$$

where $X_n = (np_n)^{-1} P_n$. Then the $|N,p_n|_k$-summability of a factored Fourier series $\sum_{n=1}^{\infty} \lambda_n X_n A_n(t)$ at any point $t = x$ is a local property of $f(t)$.

Later Sarigöl (see [13]) has proved the following
Theorem 2.4. Suppose that $A = (a_{nv})$ is a positive normal matrix satisfying

\[ a_{n-1, v} \geq a_{nv}, \quad (n \leq v + 1) \]

\[ \tilde{a}_{n,0} = 1, \quad (n = 0, 1, 2, \ldots, ) \]

\[ \sum_{v=1}^{n-1} a_{vn} \tilde{a}_{n,v-1} = O(a_{nn}), \]

\[ \Delta x_n = O(n^{-1}), \]

where $X_n = \frac{1}{a_{nn}}$. If a sequence $(\lambda_n)$ satisfying following conditions

\[ \sum_{n=1}^{\infty} n^{-1} \{ |\lambda_n^k| + |\lambda_{n+1}|^k \} X_{n-1}^k \leq \infty, \]

\[ \sum_{n=1}^{\infty} (X_n^k + 1)|\Delta \lambda_n| \leq \infty. \]

Then the $|A|_k$-summability of a factored Fourier series $\sum_{n=1}^{\infty} \lambda_n X_n A_n(t)$ at any point $t = x$ is a local property of $f(t)$.

Again to improve upon and generalize Theorem 2.4, Sulaiman [14] has proved the following theorem for a normal matrix.

Theorem 2.5. Let $A = (a_{nv})$ is a normal matrix satisfying

\[ |\tilde{a}_{n,v+1}| \leq |a_{nn}|, \]

\[ \sum_{n=v+1}^{\infty} |\tilde{a}_{n,v+1}| \leq \infty, \]

\[ \sum_{v=1}^{n-1} |a_{vn}| |\tilde{a}_{n,v+1}| = O(|a_{nn}|), \]

\[ \Delta X_n = O\left(\frac{1}{n}\right), \]
where \( X_n = \frac{1}{(n a_{nn})} \). If a sequence \((\lambda_n)\) satisfying the following conditions

\[
\sum_{n=1}^{\infty} n^{-1} \{ |\lambda_n^k| + |\lambda_{n+1}|^k \} X_n^k n^{-1} \leq \infty,
\]

\[
\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| \leq \infty.
\]

Then the \(|A|_k\)-summability of a factored Fourier series \(\sum_{n=1}^{\infty} \lambda_n X_n A_n(t)\) at any point \(t = x\) is a local property of \(f(t)\).

3. Main result

In the present paper, we have established a new result on local property of \(|A, \delta|_k\)-summability of factored Fourier series \(\sum_{n=1}^{\infty} \lambda_n X_n A_n(t)\) in the form of a theorem as follows.

**Theorem 3.1.** Suppose \(A = (a_{nv})\) is a positive normal matrix such that

\[
a_{n-1,v} \geq a_{n,v} \ (n \leq v + 1);
\]

\[
\bar{a}_{n,0} = 1 \ (n = 0, 1, \ldots);
\]

\[
\sum_{v=1}^{n-1} a_{nv} \hat{a}_{n,v-1} = O(a_{nn});
\]

\[
\sum_{n=v+1}^{m+1} \hat{a}_{n,v+1} a_{nn}^{-\delta k} = O(v^{\delta k});
\]

\[
\sum_{n=v+1}^{m+1} a_{nn}^{-\delta k} |\Delta a_{nn}| = O(v^{\delta k});
\]

\[
\Delta X_n = O(n^{-1}),
\]

where \(X_n = \frac{1}{(n a_{nn})}\). If a sequence \((\lambda_n)\) satisfying the following conditions

\[
\sum_{n=1}^{\infty} n^{-1} \{ |\lambda| + |\lambda_{n+1}| \} X_n^k n^{\delta k} \leq \infty;
\]

\[
\sum_{n=1}^{\infty} (x_n^k + 1)|\Delta \lambda_n| n^{\delta k} \leq \infty.
\]
Then the $|A,\delta|_k$-summability of a factored Fourier series $\sum_{n=1}^{\infty} \lambda_n X_n A_n(t)$ at any point $t = x$ is a local property of $f(t)$.

**Remark 3.1.** The element $\hat{a}_{nv} \geq 0$ for each $n, v$. In fact, it is easily seen from the positiveness of the matrix, (3.1) and (3.2), that $\hat{a}_{00} = 1$,

$$\hat{a}_{nv} = a_{n0} - \bar{a}_{v-1,0} + \sum_{j=0}^{v-1} (a_{n-1,j} - a_{nj})$$

$$= \sum_{j=0}^{v-1} (a_{n-1,j} - a_{nj}) \geq 0 \ (1 \leq v \leq n)$$

and equal to zero otherwise.

In order to prove the above theorem we need the a lemma as follows.

**Lemma 3.1.** Suppose that the matrix $A$ and the sequence $(\lambda_n)$ satisfy the conditions of the theorem, and that $(s_n)$ is bounded. Then factored Fourier series $\sum_{n=1}^{\infty} \lambda_n X_n A_n(t)$ is summable to $|A,\delta|_k$ ($k \geq 1$, $\delta \geq 0$).

**Proof.** Let $(T_n)$ be an $A-$ transformation of $\sum_{i=1}^{n} \lambda_i X_i A_n(t)$, then

$$T_n = \sum_{i=0}^{n} a_{ni} s_i = \sum_{i=1}^{n} a_{ni} \sum_{v=1}^{i} \lambda_v X_v = \sum_{v=1}^{n} \lambda_v X_v \sum_{i=v}^{n} a_{ni} = \sum_{v=1}^{n} \bar{a}_{nv} \lambda_v X_v$$

$$\Delta T_n = T_n - T_{n-1} = \sum_{v=1}^{n} (\bar{a}_{nv} - \bar{a}_{n-1,v}) \lambda_v X_v = \sum_{v=1}^{n} \bar{a}_{nv} \lambda_v X_v$$

$$\Delta T_n = \sum_{v=1}^{n-1} (\bar{a}_{nv} \lambda_v X_v) s_v + a_{nn} \lambda_n X_n s_n$$

but, $\Delta (\bar{a}_{nv} \lambda_v X_v) = \lambda_v X_v \Delta \bar{a}_{nv} + \Delta (\lambda_v X_v) \bar{a}_{n,v+1}$

$$= \lambda_v X_v \Delta \bar{a}_{nv} + (X_v \Delta \lambda_v + \Delta X_v \lambda_{v+1}) \bar{a}_{n,v+1}.$$
\[ \Delta T_n = \sum_{v=1}^{n-1} a_{n,v+1} X_v \Delta \lambda_v s_v + \sum_{v=1}^{n-1} \tilde{a}_{n,v+1} \lambda_{v+1} \Delta X_v s_v + \sum_{v=1}^{n-1} \tilde{a}_{n,v} \lambda_v s_v + a_{nn} \lambda_n X_n s_n \\
= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{(say)}. \]

To complete the proof, it is sufficient to show that by using Minkowski’s inequality

\[ \sum_{n=1}^{\infty} a_{nn}^{1-k-\delta_k} |T_{n,m}|^k < \infty \quad (m = 1, 2, 3, 4). \]

Using Hölder inequality and (3.1), (3.2), (3.8),

Let

\[ I_1 = \sum_{n=2}^{m+1} a_{nn}^{1-k-\delta_k} |T_{n,1}|^k \]

\[ \leq \sum_{n=2}^{m+1} a_{nn}^{1-k-\delta_k} \left\{ \sum_{v=1}^{n-1} \tilde{a}_{n,v+1} X_v |\Delta \lambda_v||s_v| \right\}^k \]

\[ = O(1) \sum_{n=2}^{m+1} a_{nn}^{1-k-\delta_k} \left\{ \sum_{v=1}^{n-1} \tilde{a}_{n,v+1} X_v |\Delta \lambda_v| \right\}^k \]

\[ = O(1) \sum_{n=2}^{m+1} a_{nn}^{-\delta_k} \sum_{v=1}^{n-1} \tilde{a}_{n,v+1} X_v^k |\Delta \lambda_v| \left\{ (a_{nn})^{-1} \sum_{v=1}^{n-1} \tilde{a}_{n,v+1} |\Delta \lambda_v| \right\}^{k-1}. \]

Since,

\[ \tilde{a}_{n,v+1} = \sum_{r=v+1}^{n} (a_{nr} - a_{n-1,r}) = \sum_{r=0}^{n} (a_{n-1,r} - a_{n,r}) \]

\[ \leq \sum_{r=0}^{n-1} (a_{n-1,r} - a_{nr}) = \tilde{a}_{n-1,0} - \tilde{a}_{n0} + a_{nn} = a_{nn}. \]

\[ \Rightarrow \sum_{v=1}^{n-1} \tilde{a}_{n,v+1} |\Delta \lambda_v| \leq a_{nn} \sum_{v=1}^{n-1} |\Delta \lambda_v| = O(a_{nn}). \]
\[ I_1 = O(1) \sum_{n=2}^{m+1} a_{nn}^{-k} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} X_v^k |\Delta \lambda_v| \]

\[ = O(1) \sum_{v=1}^{m} X_v^k |\Delta \lambda_v| \sum_{n=v+1}^{m+1} a_{n,v+1}^{-\delta k} \]

\[ = O(1) \sum_{v=1}^{m} X_v^k |\Delta \lambda_v| v^{\delta k} \]

\[ = O(1). \]

Using Hölder inequality, and (3.3), (3.4), (3.6), (3.7),

\[ I_2 = \sum_{n=2}^{m+1} a_{nn}^{1-k-\delta k} |T_{n,2}|^k \]

\[ \leq \sum_{n=2}^{m+1} a_{nn}^{1-k-\delta k} \left( \sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\lambda_{v+1}| |\Delta x_v||s_v| \right)^k \]

\[ = O(1) \sum_{n=2}^{m+1} a_{nn}^{1-k-\delta k} \left( \sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\lambda_{v+1}| a_{vv} X_v \right)^k \]

\[ = O(1) \sum_{n=2}^{m+1} (a_{nn})^{-\delta k} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\lambda_{v+1}|^k a_{vv} X_v \left( a_{nn} \right)^{-1} \sum_{v=1}^{n-1} a_{vv} \hat{a}_{n,v+1} \]

\[ = O(1) \sum_{n=2}^{m+1} (a_{nn})^{-\delta k} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\lambda_{v+1}|^k a_{vv} X_v \]

\[ = O(1) \sum_{n=2}^{m+1} a_{vv} X_v^k |\lambda_{v+1}|^{\delta k} \sum_{n=v+1}^{m} a_{nn}^{-\delta k} \hat{a}_{n,v+1} \]

\[ = O(1) \sum_{v=1}^{m} a_{vv} X_v^k |\lambda_{v+1}|^k v^{\delta k} \]

\[ = O(1) \sum_{v=1}^{m} \frac{1}{v} X_v^k |\lambda_{v+1}|^k v^{\delta k} \]

\[ = O(1). \]
Using Hölder inequality, and (3.1), (3.2),

\[ I_3 = \sum_{n=2}^{m+1} a_{nn}^{1-k-\delta k} |T_{n,3}|^k \]

\[ \leq \sum_{n=2}^{m+1} a_{nn}^{1-k-\delta k} \left\{ \sum_{v=1}^{n-1} |\Delta a_{nv}| |\lambda_v| X_v \right\}^k \]

\[ = O(1) \sum_{n=2}^{m+1} a_{nn}^{1-k-\delta k} \left\{ \sum_{v=1}^{n-1} |\Delta a_{nv}| |\lambda_v| X_v \right\}^k \]

\[ = O(1) \sum_{n=2}^{m+1} a_{nn}^{-\delta k} \sum_{v=1}^{n-1} |\Delta a_{nv}| |\lambda_v|^k X_v^{k \delta k} \left\{ (a_{nn})^{-1} \sum_{v=1}^{n-1} |\Delta a_{nv}| \right\}^{k-1}. \]

We know

\[ \sum_{v=1}^{n-1} |\Delta a_{nv}| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_{nv}) \]

\[ = \bar{a}_{n-1,0} - \bar{a}_{n,0} + a_{n0} - a_{n-1,0} + a_{nn} \]

\[ = a_{n0} - a_{n-1,0} + a_{nn} \leq a_{nn}. \]

\[ I_3 = O(1) \sum_{n=2}^{m+1} a_{nn}^{-\delta k} \sum_{v=1}^{n-1} |\Delta a_{nv}| |\lambda_v|^k X_v^{k \delta k} \]

\[ = O(1) \sum_{v=1}^{m} |\lambda_v|^k X_v^{k \delta k} \sum_{n=v+1}^{m+1} a_{nn}^{-\delta k} |\Delta a_{nv}| \]

\[ = O(1) \sum_{v=1}^{m} |\lambda_v|^k X_v^{k \delta k} a_{vv} \]

\[ = O(1). \]
Finally, using (3.7),

\[ I_4 = \sum_{n=1}^{\infty} a_{nn}^{1-k-\delta_k} |T_{n,4}| \]

\[ \leq \sum_{n=1}^{\infty} a_{nn}^{1-k-\delta_k} \left\{ a_{nn} |\lambda_n| X_n |s_n| \right\}^k \]

\[ = O(1) \sum_{n=1}^{\infty} a_{nn}^{1-k-\delta_k} \left\{ a_{nn} |\lambda_n| X_n \right\}^k \]

\[ = O(1) \sum_{n=1}^{\infty} (a_{nn})^{-\delta_k} a_{nn} |\lambda|^k X_n^k \]

\[ = O(1) \sum_{n=1}^{\infty} (a_{nn})^{-\delta_k} |\lambda|^k X_n^k \]

\[ = O(1). \]

Thus the proof of the above Lemma is established.

**Proof of the Theorem 3.1.** Since the convergence of the Fourier series at a point is a local property of its generating function \( f(t) \), the theorem follows by formula from chapter II of the book (see details [17]) and from the above Lemma 3.1.

**Applications.** Now we apply the theorem to the weighted mean in which \( A = (a_{nv}) \) is defined as \( a_{nv} = p_v P_n^{-1} \), when \( 0 \leq v \leq n \) where \( P_n = p_0 + p_1 + ... + p_n \); therefore, it is well known that

\[ \bar{a}_{nv} = P_n^{-1}(P_n - P_{v-1}) \]

and

\[ \hat{a}_{n,v+1} = (P_n P_{n-1})^{-1} p_n P_v. \]

One can now easily verify that taking \( \delta = 0 \) the conditions of the theorem reduce to those of Theorem 2.3.

We may now ask whether there are some examples (other than weighted mean methods) of matrices \( A \) that satisfy the hypotheses of the theorem. For this, apply the theorem to the Cesàro method of order \( \alpha \) with \( 0 \leq \alpha \leq 1 \) in which \( A \) is given by [15]

\[ a_{nv} = \frac{A_n^{\alpha-1} P_{n-v}}{A_n^{\alpha}}. \]
It is well known that
\[ \bar{a}_{nv} = A_{n-v}^{\alpha_n} \]
and
\[ \hat{a}_{nv} = v A_{n-1}^{\alpha_n} \]
It is now seen that by taking account of \( A_n^\alpha \approx n^\alpha \Gamma(\alpha+1) \) conditions (3.1)-(3.8) are satisfied. Therefore the above theorem is same as the following result.

**Corollary 3.1.** Let \( k \geq 1 \) and \( 0 \leq \alpha \leq 1 \). If \( (\lambda_n) \) a convex sequence satisfying following conditions:
\[ \sum_{n=1}^{\infty} n^{\alpha k - \alpha - k} \left( |\lambda|^k + |\lambda_{n+1}|^k \right) n^{\delta k} \leq \infty, \]
\[ \sum_{n=1}^{\infty} |\Delta \lambda_n| n^{\delta k} \leq \infty. \]
Then the \( |C,\alpha,(\alpha-1)(1-\frac{1}{k})\delta|_k \) summability of a factored Fourier series \( \sum_{n=1}^{\infty} \lambda_n X_n A_n(t) \) with \( X_n = \frac{A_n^{\alpha_n}}{n} \) at any point \( t = x \) is a local property of the generating function \( f(t) \).

4. **Conclusion**

The result obtained here is more general in the sense that, by substituting \( \delta = 0 \), the \( |A;\delta|_k \)-summability reduces to \( |A|_k \)-summability.

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**References**


