COSINE INTEGRALS FOR THE CLAUSEN FUNCTION AND ITS FOURIER SERIES EXPANSION

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Abstract. In a recent work, on taking into account certain finite sums of trigonometric functions I have derived exact closed-form results for some non-trivial integrals, including \( \int_0^\pi \sin(k\theta) \text{Cl}_2(\theta) \, d\theta \), where \( k \) is a positive integer and \( \text{Cl}_2(\theta) \) is the Clausen function. There in that paper, I pointed out that this integral has the form of a Fourier coefficient, which suggest that its cosine version \( \int_0^\pi \cos(k\theta) \text{Cl}_2(\theta) \, d\theta \), \( k \geq 0 \), is worthy of consideration, but I could only present a few conjectures at that time. Here in this note, I derive exact closed-form expressions for this integral and then I show that they can be taken as Fourier coefficients for the series expansion of a periodic extension of \( \text{Cl}_2(\theta) \). This yields new closed-form results for a series involving harmonic numbers and a partial derivative of a generalized hypergeometric function.

1. Introduction

In its more general form, the Fourier series expansion of a periodic real function \( f(x) \) of period \( L \) is conventionally written as (see, e.g., Sec. 4.2 of Ref. [6])

\[
S[f(x)] := \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[ a_k \cos \left( \frac{2\pi k x}{L} \right) + b_k \sin \left( \frac{2\pi k x}{L} \right) \right],
\]

(1.1)

where

\[
a_k = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \cos(2\pi k x/L) \, dx, \quad k \geq 0,
\]

(1.2a)

\[
b_k = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \sin(2\pi k x/L) \, dx, \quad k > 0,
\]

(1.2b)

are the Fourier coefficients and \( x_0 \) is an arbitrary constant (often taken as 0). As is well-known, if \( f(x) \) satisfies the Dirichlet conditions then this series converges to \( f(x) \) at all points of continuity of \( f(x) \) and to the average of \( f(x) \) taken at the lateral limits of \( x \) if it is a point of finite discontinuity. In fact, the periodicity condition is irrelevant for pointwise convergence in the finite domain \([x_0, x_0+L]\), as shown by Connon in Ref. [2], which is important for the Fourier expansion of non-periodic functions using periodic extensions.

In a very recent work, by taking into account certain finite sums involving trigonometric functions at rational multiples of \( \pi \), I have derived exact closed-form expressions for some non-trivial integrals [5]. Among them, I showed in Theorem 6 of Ref. [5] that

\[
\frac{2}{\pi} \int_0^\pi \sin(k\theta) \text{Cl}_2(\theta) \, d\theta = \frac{1}{k^2}
\]

(1.3)

holds for every integer \( k > 0 \). Here, \( \text{Cl}_2(\theta) := \Im \{ \text{Li}_2(e^{i\theta}) \} \) is the Clausen function, \( \text{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad |z| \leq 1 \), being the dilogarithm function [4, Sec. 1.1]. Clausen himself proved in Ref. [1] that \( \text{Cl}_2(\theta) = -\int_0^\theta \ln|2 \sin(t/2)| \, dt \), which is known as the Clausen integral [3, Sec. 4.1]. Since the

Received 26th May, 2017; accepted 29th July, 2017; published 1st September, 2017.
2010 Mathematics Subject Classification. 42A16, 26A42, 26A06.
Key words and phrases. Clausen function; Fourier coefficients; generalized hypergeometric function.

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integral in Eq. (1.3) resembles that of Fourier coefficient $b_k$ in Eq. (1.2b), then a natural follow-up is the investigation of the corresponding cosine integral, i.e.

$$A_k := \frac{2}{\pi} \int_0^\pi \cos(k \theta) \mathrm{Cl}_2(\theta) \, d\theta, \quad k \geq 0. \tag{1.4}$$

However, there in Eqs. (25)–(30) of Ref. [5] I could only conjecture, based upon strong numerical evidence, a few simple results for small values of $k$. They of course suggest a pattern, but there in Ref. [5] I could not find it out.

In this note, I make use of a well-known series expansion for $\mathrm{Cl}_2(\theta)$ to derive closed-form expressions for $A_k$, one for $k = 0$ and another for $k > 0$. I then use these results to obtain a Fourier series for a suitable periodic extension of $\mathrm{Cl}_2(\theta)$, which yields new closed-form results.

2. Cosine Integrals of Clausen Function

In what follows, we shall make use of a well-known series representation for $\mathrm{Cl}_2(\theta)$.

**Lemma 1** (Clausen series for $\mathrm{Cl}_2(\theta)$). The trigonometric series $\sum_{n=1}^\infty \frac{\sin(n \theta)}{n^2}$ converges to $\mathrm{Cl}_2(\theta)$ for all $\theta \in \mathbb{R}$.

**Proof.** This series representation of $\mathrm{Cl}_2(\theta)$ remounts to Clausen’s original work (1832) [1], but, for completeness, let us present a proof based on Fourier series. In Theorem 3 of Ref. [7], a recent note on Fourier series by Zhang, it is shown that, given a real function $f$ on $[0, L]$ that is integrable on $(0, L)$, then

$$f(x) = \sum_{n=1}^\infty c_{2n} \sin \left( \frac{2n\pi x}{L} \right) \tag{2.1}$$

for all $x \in [-L, L]$ where $f(x)$ is a continuous function. Here,

$$c_{2n} = \frac{4}{L} \int_0^{L/2} f(t) \sin \left( \frac{2n\pi t}{L} \right) \, dt. \tag{2.2}$$

Since $\mathrm{Cl}_2(\theta)$ is an odd function which is continuous (thus integrable) on $(-\pi, \pi)$ and $\mathrm{Cl}_2(\theta) = -\mathrm{Cl}_2(2\pi - \theta)$ [3, Secs. 4.2 and 4.3], then the convergence of $\sum_{n=1}^\infty \sin(n \theta)/n^2$ to $\mathrm{Cl}_2(\theta)$ follows by taking $L = 2\pi$ in Zhang’s theorem and noting that $c_{2n} = 1/n^2$, as seen in Eq. (1.3). Finally, the periodicity of $\mathrm{Cl}_2(\theta)$, as established in Sec. 4.2 of Ref. [3], extends the convergence to all $\theta \in \mathbb{R}$. \hfill \Box

Let us begin our main results with the integral $A_k$ for $k = 0$.

**Theorem 1** (Integral $A_0$). The exact closed-form result

$$A_0 := \frac{2}{\pi} \int_0^\pi \mathrm{Cl}_2(\theta) \, d\theta = \frac{7}{2} \frac{\zeta(3)}{\pi},$$

where $\zeta(3) := \sum_{n=1}^\infty 1/n^3$ is the Apéry’s constant, holds.

**Proof.** From Lemma 1, one has

$$\frac{2}{\pi} \int_0^\pi \mathrm{Cl}_2(\theta) \, d\theta = \frac{2}{\pi} \int_0^\pi \sum_{n=1}^\infty \frac{\sin(n \theta)}{n^2} \, d\theta = \frac{2}{\pi} \sum_{n=1}^\infty \int_0^\pi \frac{\sin(n \theta)}{n^2} \, d\theta$$

$$= -\frac{2}{\pi} \sum_{n=1}^\infty \frac{\cos(n \theta)}{n^3} \bigg|_0^\pi = -\frac{2}{\pi} \sum_{n=1}^\infty \frac{(-1)^n - 1}{n^3} = \frac{4}{\pi} \sum_{\text{odd}} \frac{1}{n^3}, \tag{2.3}$$

where the last sum takes only the odd values of $n$ into account. The interchange of the integral and the series is allowed because this series converges absolutely. Since $\zeta(3) = \sum_{\text{odd}} 1/n^3 = \sum_{\text{even}} 1/(2m)^3 = \frac{1}{8} \zeta(3)$, then $\sum_{\text{odd}} 1/n^3 = \frac{7}{8} \zeta(3)$. \hfill \Box

Now, let us derive a general result valid for all integrals $A_k$, $k > 0$. For this, it will be useful to define $h_n := \sum_{\ell=1}^n 1/(2\ell - 1)$, $n$ being a positive integer, which is the odd analogue of the harmonic number $H_n := \sum_{\ell=1}^n 1/\ell$. Since $h_{\lfloor n/2 \rfloor} = H_n - \frac{1}{2} H_{\lfloor n/2 \rfloor}$, it is easy to rewrite any expression containing $h_n$ in terms of the usual harmonic numbers.
Theorem 2 (Integral $A_k$, $k > 0$). Let $A_k$ be the integral defined in Eq. (1.4). The exact closed-form result

$$A_k = \begin{cases} \frac{2}{\pi} \ln 4 - \frac{2h_{|k/2|} - 1/k}{k^2}, & k \text{ odd} \\ -\frac{4}{\pi} \frac{h_{k/2}}{k^2}, & k \text{ even} \end{cases},$$

holds for all integers $k > 0$.

Proof. From Lemma 1, one has

$$A_k = \frac{2}{\pi} \int_0^\pi \cos (k\theta) \frac{\sin (n\theta)}{n^2} \, d\theta = \frac{2}{\pi} \sum_{n=1}^{\infty} \int_0^\pi \cos (k\theta) \sin (n\theta) \, d\theta = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\int_0^\pi \cos (k\theta) \sin (n\theta) \, d\theta}{n^2},$$

where $k$ is a positive integer. On applying the trigonometric identity $\sin \alpha \cos \beta = \frac{1}{2} \{ \sin (\alpha + \beta) + \sin (\alpha - \beta) \}$ to the last integral, one finds

$$I_{kn} := \int_0^\pi \cos (k\theta) \sin (n\theta) \, d\theta = \frac{1}{2} \int_0^\pi \{ \sin [(n + k)\theta] + \sin [(n - k)\theta] \} \, d\theta.$$

For $n = k$, the above integral reduces to $I_{kn} = \int_0^\pi \cos (n\theta) \sin (n\theta) \, d\theta = \frac{1}{2} \int_0^\pi \sin (2n\theta) \, d\theta = \cos (2n\theta)/(2n) |_0^\pi = 0$. For all $n \neq k$, one has

$$I_{kn} = -\frac{1}{2} \left\{ \frac{\cos [(n + k)\pi]}{n + k} + \frac{\cos [(n - k)\pi]}{n - k} \right\} _0^\pi$$

$$= -\frac{1}{2} \left\{ \frac{\cos [(n + k)\pi] - 1}{n + k} + \frac{\cos [(n - k)\pi] - 1}{n - k} \right\}$$

$$= -\frac{1}{2} \left[ \frac{(-1)^{n+k} - 1}{n + k} + \frac{(-1)^{n-k} - 1}{n - k} \right].$$

Therefore, $A_k = \frac{2}{\pi} \sum_{n=1}^{\infty} I_{kn} / n^2$ expands to

$$A_k = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \frac{1 - (-1)^{n+k}}{n + k} + \frac{1 - (-1)^{n-k}}{n - k} \right],$$

and, since $1 - (-1)^{n+k} = 0$ whenever $n$ and $k$ have the same parity (i.e., when they are both odd or even numbers), whereas $1 - (-1)^{n+k} = 2$ when $n$ and $k$ have opposite parities, then

$$A_k = \frac{1}{\pi} \sum_n \frac{1}{n^2} \left[ \frac{2}{n + k} + \frac{2}{n - k} \right] = \frac{2}{\pi} \sum_n \frac{1}{n^2} \frac{2n}{n^2 - k^2}$$

$$= \frac{4}{\pi} \sum_n \frac{1}{n^2 - k^2},$$

where $\sum'$ means a sum over $n$ values with the opposite parity with respect to $k$. Explicitly,

$$A_k = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m (4m^2 - k^2)}, \quad k \text{ odd},$$

and

$$A_k = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m - 1) [(2m - 1)^2 - k^2]}, \quad k \text{ even}. \quad (2.10)$$

For odd values of $k$, the substitution $k = 2p - 1$, $p > 0$, in Eq. (2.9) yields

$$\frac{\pi}{2} A_{2p-1} = \sum_{m=1}^{\infty} \frac{1}{m [4m^2 - (2p - 1)^2]}.$$

This series can be written in terms of the digamma function $\psi(x) := \frac{d}{dx} \ln \Gamma(x)$, where $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \, dt$ is the classical gamma function. From a well-known series representation for $\psi(x)$,
namely [8, Sec. 8.362]
\[
\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+x}\right),
\]
(2.12)
one finds, after some algebra,
\[
\frac{\pi}{2} A_{2p-1} = -\psi\left(\frac{3}{2} - p\right) + \psi\left(p + 1/2\right) + 2 \gamma,
\]
(2.13)
where \( \gamma := \lim_{n \to \infty} (H_n - \ln n) \) is the Euler’s constant. From Eq. (3) in Ref. [8, Sec. 8.366], one knows that
\[
\psi\left(\frac{1}{2} \pm p\right) = -\gamma - \ln 4 + 2 h_p,
\]
(2.14)
which, together with
\[
\psi\left(\frac{3}{2} - p\right) = \psi\left(\frac{1}{2} - p\right) + \frac{1}{\frac{1}{2} - p},
\]
(2.15)
which promptly follows from \( \psi(x + 1) = \psi(x) + 1/x \) [8, Sec. 8.365], reduces Eq. (2.13) to
\[
\frac{\pi}{2} A_{2p-1} = \ln 4 - \frac{1}{2} - \frac{2 h_p}{(2p-1)^2},
\]
(2.16)
which is equivalent to Eq. (2.9). The special value \( \psi(1/2) = -\gamma - \ln 4, \) as stated in Ref. [8, Sec. 8.366], is required in the derivation of Eq. (2.14).

For even values of \( k \), substitute \( k = 2p \) in Eq. (2.10). This leads to
\[
\frac{\pi}{4} A_{2p} = \sum_{m=1}^{\infty} \frac{1}{(2m-1)\left[(2m-1)^2 - 4p^2\right]}.
\]
(2.17)
The series representation of \( \psi(x) \) given in Eq. (2.12) then leads to
\[
\frac{\pi}{4} A_{2p} = -\frac{\psi(1/2 - p) + \psi(p + 1/2) + 2 \gamma + 2 \ln 4}{16p^2}.
\]
(2.18)
On taking Eq. (2.14) into account, one finds, after some algebra,
\[
\frac{\pi}{4} A_{2p} = -\frac{h_p}{(2p)^2},
\]
(2.19)
which completes the proof. \( \square \)

As expected, this theorem shows that all conjectures stated at the end of Ref. [5] are indeed true.

3. Fourier series for an even periodic extension of Clausen function

Now, let us examine the Fourier cosine series whose coefficients are the \( A_k \) expressions derived above.

**Theorem 3.** The series
\[
\frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos(k \theta),
\]
where \( A_0 \) and \( A_k \) are the coefficients derived in our Theorems 1 and 2, respectively, converges to \( \text{Cl}_2(\theta) \) for all \( \theta \in [0,\pi] \) and to \( -\text{Cl}_2(\theta) \) when \( \theta \in (\pi, 2\pi] \), thus yielding a continuous even function on \( [-2\pi, 2\pi] \). This convergence can be extended to all \( \theta \in \mathbb{R} \).

**Proof.** Let \( g(\theta) \) be a real function defined in the interval \([-2\pi, 2\pi]\) as follows:
\[
g(\theta) := \begin{cases} 
+\text{Cl}_2(\theta), & \theta \in [-2\pi, -\pi) \text{ or } \theta \in [0,\pi] \\
-\text{Cl}_2(\theta), & \theta \in [-\pi, 0) \text{ or } \theta \in (\pi, 2\pi].
\end{cases}
\]
(3.1)
Since \( \text{Cl}_2(\theta) \) is a continuous odd function, it is clear that \( g(\theta) \) is a continuous even function in the interval \([-2\pi, 2\pi] \). In Theorem 4 of Zhang’s paper [7], it is shown that, given a real function \( f(x) \)
In a shortened notation, Eq. (3.4) reads
\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n} \cos \left( \frac{2n\pi x}{L} \right), \] (3.2)
where
\[ a_{2n} = \frac{4}{L} \int_{0}^{L/2} f(t) \cos \left( \frac{2n\pi t}{L} \right) dt, \quad n \geq 0, \] (3.3)
converges to \( f(x) \) for all \( x \in [-L, L] \) where \( f(x) \) is a continuous function. The absence of the term \( a_0/2 \) in Theorem 4 of Ref. [7] is corrected here in our Eq. (3.2). Since the function \( g(\theta) \) defined in Eq. (3.1) is an even function which is continuous (thus integrable) on \((-2\pi, 2\pi)\) and \( g(\theta) = g(2\pi - \theta) \), then the convergence of the series \( A_0/2 + \sum_{k=1}^{\infty} A_k \cos (k \theta) \) to \( g(\theta) \) follows by taking \( L = 2\pi \) in Zhang’s theorem and noting that \( a_{2n} = \frac{2}{\pi} \int_{0}^{\pi} g(\theta) \cos (n\theta) d\theta = \frac{2}{\pi} \int_{0}^{\pi} \text{Cl}_2(\theta) \cos (n\theta) d\theta \) are just the coefficients \( A_0 \) and \( A_n \) derived in our Theorems 1 and 2, respectively. Finally, since this cosine series converges to an even periodic extension of \( \text{Cl}_2(\theta) \), with a period \( 2\pi \), then its convergence to \( g(\theta) \) can be extended to all \( \theta \in \mathbb{R} \).

Interestingly, new closed-form results can be deduced directly from Theorem 3. For instance, on taking \( \theta = 0 \) (or \( \pi \)), one finds

**Corollary 1.** The following closed-form result holds:
\[ \sum_{p=1}^{\infty} \frac{h_{p-1}}{(2p - 1)^2} = \frac{\pi^2}{8} \ln 2 - \frac{7}{16} \zeta(3). \]

On taking \( \theta = \pi/2 \) in Theorem 3, a less obvious expression arises which can be written in terms of the regularized hypergeometric function
\[ _pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; z \right) := \frac{pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; z \right)}{\prod_{j=1}^{q} \Gamma (b_j)}, \] (3.4)
where
\[ _pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; z \right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!} \] (3.5)
is the generalized hypergeometric series. As usual, \( (a)_n := (a + 1) \ldots (a + n - 1) = \Gamma(a + n)/\Gamma(a) \) is the Pochhammer symbol. By convention, \( (a)_0 = 1 \).

**Corollary 2** (A special value for \( \theta = \pi/2 \)). The following closed-form result holds:
\[ _4F_3' \left( \begin{array}{c} 1, 1, 1, 3/2 \\ 2, 2, 3/2 \end{array} ; -1 \right) = \zeta(2) (\gamma + \ln 4) + 7 \zeta(3) - 4\pi \frac{G}{\sqrt{\pi}}, \] (3.6)
where \( \zeta(2) := \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6 \) and \( G := \sum_{n=1}^{\infty} (-1)^n/(2n + 1)^2 \) is the Catalan’s constant. Here, the prime symbol \( (') \) indicates a partial derivative with respect to \( b_3 \).

As shown below, this result can be written in terms of the corresponding generalized hypergeometric function. Interestingly, this yields a nice closed-form result which, to the author knowledge, is not found in literature.

**Corollary 3** (Corresponding generalized hypergeometric function). The following closed-form result holds:
\[ _4F_3' \left( \begin{array}{c} 1, 1, 1, 3/2 \\ 2, 2, 3/2 \end{array} ; -1 \right) = \frac{\pi^2}{6} + \frac{7}{2} \zeta(3) - 2\pi \frac{G}{\sqrt{\pi}}. \] (3.7)

**Proof.** In a shortened notation, Eq. (3.4) reads
\[ _pF_q \left( \begin{array}{c} a \bar{a}, b \bar{b} \\ \bar{a}, \bar{b} \end{array} ; z \right) = \frac{pF_q \left( \begin{array}{c} a, b \bar{a}, \bar{b} \\ \bar{a}, \bar{b} \end{array} ; z \right)}{\prod_{j=1}^{q} \Gamma (b_j)}. \]
where \( \vec{a} \) and \( \vec{b} \) denote the arrays of coefficients \([1, 1, 3/2]\) and \([2, 2, 3/2]\), respectively. This implies that

\[
\frac{\partial}{\partial b_3} \bar{4}F_3((\vec{a}, \vec{b}; -1) = \frac{1}{\prod_{j \neq 3} \Gamma(b_j)} \frac{\partial}{\partial b_3} \left[ \frac{4F_3(\vec{a}, \vec{b}; -1)}{\Gamma(b_3)} \right]
\]

\[
= \frac{1}{\Gamma(b_1) \Gamma(b_2)} \left[ \frac{4F'_3(\vec{a}, \vec{b}; -1)}{\Gamma(b_3)} - \frac{4F_3(\vec{a}, \vec{b}; -1)}{\Gamma'(b_3)} \right]
\]

\[
= \frac{1}{\Gamma^2(2)} \left[ \frac{4F'_3(\vec{a}, \vec{b}; -1)}{\Gamma(3/2)} - \frac{4F_3(\vec{a}, \vec{b}; -1)}{\Gamma'(3/2)} \right]. \tag{3.8}
\]

Since \( \Gamma(1 + x) = x \Gamma(x) \), then \( \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}/2 \), which reduces the last expression, above, to

\[
4\bar{F}'_3(\vec{a}, \vec{b}; -1) = \frac{4F'_3(\vec{a}, \vec{b}; -1)}{\sqrt{\pi}/2} - \frac{4F_3(\vec{a}, \vec{b}; -1)}{\sqrt{\pi}/2} \psi(3/2) \sqrt{\pi}/2
\]

\[
= 2 \frac{4F'_3(\vec{a}, \vec{b}; -1)}{\sqrt{\pi}} - 2 \frac{4F_3(\vec{a}, \vec{b}; -1)}{\sqrt{\pi}} \psi(3/2) \sqrt{\pi}. \tag{3.9}
\]

Note that, for all positive integers \( n \), \( \Gamma(n) = (n-1)! \) (in particular, \( \Gamma(2) = 1! = 1 \)). The proof completes by substituting the result in Corollary 2, together with the special values \( \psi(3/2) = \psi(1/2) + 1/(1/2) = -\gamma - \ln 4 + 2 \) and \( 4F_3(\vec{a}, \vec{b}; -1) = \pi^2/12 \), in Eq. (3.9). \( \square \)

The closed-form result in Corollary 3 has been conjectured by Ancarani and the author in a recent discussion, by following an entirely different approach, but we could not find a formal proof at that time.

Acknowledgments

The author wishes to thank M. R. Javier for checking all closed-form expressions in this work numerically with mathematical software.

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